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Covariant Newtonian Dynamics and the Principle of Material Frame-Indifference

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Abstract

As all natural laws, Newtonian dynamics should be governed by Einstein's Covariance Principle; *i.e.*, being covariant under all coordinate transformations, even time-dependent transformations. But Newton's Second Law, as it is generally understood, is unchanged only under Galilean transformations, which do not include time-dependent coordinate transformations. To achieve the covariant formulation of Newton's Second Law, a distinction must be made between frames and coordinate systems, as advanced by the Principle of Material Frame-Indifference, and furthermore, the ordinary time derivative must be replaced by the rotational time derivative. Elevating Newton's Second Law to covariancy has born many fruits in flight dynamics from the theoretical underpinning of unsteady flight maneuvers to the practical modeling of complex flight engagements in tensors, followed by efficient programming with matrices.

Keywords

Einstein, Galileo Galilei, Newton's Second Law, Tensor Flight Dynamics, Computer Programming

1. Introduction

Newtonian Dynamics has ruled classical dynamics over three centuries and even today is still the dominant theory for engineering and day-to-day applications. The simple f=ma equation may be the best-known relationship in classical physics. But then came Einstein, not to abolish Newton, but to expand his laws into realms where movements approach the speed of light and matter shapes the very fabric of space.

Einstein's special and general theories of relativity converge to Newton's dynamic and gravitational laws for everyday speeds and matter. In addition, Einstein also introduced the universal *Principle of Covariance*, that holds equally well for classical and relativistic dynamics, quote: "All natural laws must be covariant with respect to arbitrary continuous transformations of the coordinates." [1]

To be precise, let's distinguish between *invariant* and *covariant*, using the following definitions:

Invariant: Any physical quantity is invariant when its *value* remains unchanged under coordinate transformations. Examples are physical quantities represented by scalars, such as temperature, pressure, etc., whose values remain the same.

Covariant: The term covariant is used when the *equations* of physical systems are unchanged under coordinate transformations. These equations are represented by tensors and formulated in tensorial form.

In high school and university [2], we are taught how to deal with Newton's Second Law. Its simple form $f = ma$ is valid for all inertial coordinate systems, but when the coordinate system is rotating, special care has to be taken by including another term to compensate for the rotation. In other words, Newton's law as taught is not covariant under all coordinate transformations.

This flies in the face of Einstein's Covariance Principle.

In this paper, I will show how to formulate Newton's Second Law in a covariant form. I will summarize in Section 2, Newton's classical view of the world with its absolute space and time, and the mystifying *inertial frame*, which escapes rational pinpointing. We can state that Newton's Second Law, in its classical formulation, is unchanged under Galilean transformations, which, however, does not include time-dependent transformations. In Section 3, I will give Einstein the floor with his Covariance Principle, and we will see that the Lorentz transformation of Special Relativity converges to the Galilean transformation for everyday speeds. Then in Section 4, I will switch to continuum dynamics as portrayed by Walter Noll, whose clear distinction between frames and coordinate systems made a great impression on me. His Principle of Material Frame-Indifference is closely related to Einstein's Covariance Principle, though he does not make that connection, like others have done. Finally, in Section 5, I will retrace the steps from my dissertation to my latest publications, as I have developed the covariant formulation of Newton's Second Law in Cartesian tensors, and its implications for flight dynamics and compact computer programming.

2. Newton and His Dynamics—Absolute Space and Time

What better way is there than to start by quoting Newton's (1643-1727) three laws from his publication in 1687 *Philosophiae Naturalis Principia Mathematica*, as translated by the University of California at Berkeley [3]:

1) Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

2) The rate-of-change of linear momentum equals the impressed force and is in the direction in which the force acts.

3) To every action there is always opposed an equal reaction.

(Newton used the word *motion* instead of *linear momentum* to define the second law, but the meaning is the same.)

The first law is validated by our experience. We do not notice our own linear momentum unless a wall stops us. The wall exerts the force that kills us (second law). Newton's third law is important in mechanics, because it assures us that internal forces cancel amongst a collection of particles. Newton did not specify a frame in formulating his second law. Later, others attempted to affix what was called the *luminiferous ether* to his law, until Michelson and Morley in 1887 disproved the concept.

Today, in engineering and physics, the application determines the proper frame, which we call the *inertial frame*. From the First Law, we know that any nonaccelerating frame qualifies equally well. But does it exist? Is it the frame formed by the so-called fixed stars, or the ecliptic of our sun? Yet we know that our solar system is located in the spiral arms of the Milky Way and therefore accelerating. Other theories suggest that all galaxies and their stars are fleeting with increasing speed. Where is this inertial frame? It probably does not exist in absolute terms. The inertial frame is dictated by the application. Interplanetary travel requires the heliocentric frame; Earth satellites use the ecliptic frame, which most commonly is called the inertial frame; and Earth-bound, low-speed flights can use the Earth frame. Whatever the accuracy requirement is, this will determine the choice of the inertial frame.

Nevertheless, Newton believed there exists absolute space and absolute time. But he was not the first one. Rather we have to go way back to Aristotle (384-322 BC). He posited that the cosmos is the absolute frame. Now Newton's reasoning was not shaped as much by Aristotle as by Galileo (1564-1642), who died just one year before Newton was born. Galileo believed in absolute space and absolute time and was the first one to realize experimentally that dynamical laws are precisely the same when referred to any uniformly moving frame. This led him to formulate a transformation law, which we now call the *Galilean Transformation Law*

$$\boldsymbol{x}' = \boldsymbol{T}\left(\boldsymbol{x} - \boldsymbol{v} \cdot t\right), \quad t' = t; \ \boldsymbol{T}\, const, \ \boldsymbol{v}\, const$$

with $\boldsymbol{x}$ position vector, $\boldsymbol{T}$ spatial constant transformation matrix, $\boldsymbol{v}$ constant linear velocity, and t universal time.

Newton's Second Law, as generally perceived, does not change under Galilean transformations. Let's have a look and take the time derivative of Galileo's transformation, but pre-multiply $\boldsymbol{x}$ by the constant mass m

$$m\boldsymbol{x}' = \boldsymbol{T}\left(m\boldsymbol{x} - \boldsymbol{v} \cdot t\right)$$

Taking the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{x}' = \boldsymbol{T}\left(m\boldsymbol{x} - \boldsymbol{v} \cdot t\right)\right) \rightarrow m\dot{\boldsymbol{x}}' = \boldsymbol{T}\left(m\dot{\boldsymbol{x}} - \boldsymbol{v}\right)$$

and again

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\boldsymbol{x}}' = \boldsymbol{T}\left(m\dot{\boldsymbol{x}} - \boldsymbol{v}\right)\right) \to m\ddot{\boldsymbol{x}}' = \boldsymbol{T}\left(m\ddot{\boldsymbol{x}}\right)$$

Now Newton's Second Law is simply

$$m\ddot{\boldsymbol{x}} = \boldsymbol{f}$$

and substituted

$$m\ddot{\boldsymbol{x}}' = \boldsymbol{Tf}$$

Comparing the last two equations shows that Galileo's transformation does not change the inertial force, as long as $\boldsymbol{T}$ is not a function of time. Only the external force has to be adjusted to the new attitude.

3. Einstein's View of Newtonian Dynamics—Covariance

Einstein (1879-1955), in his first major publication [4], questioned the entrenched Galilean postulate that all velocities are additive. His study of Maxwells electrodynamics led him to hypothesize that the speed of light is constant. If you move the light source, you cannot make the photons move faster. He was led to this postulate by realizing that in Maxwell's equations the permittivity ε_0 and permeability μ_0 are constant numbers in vacuum. If combined in this relationship $c^2 = 1/(\varepsilon_0\mu_0)$, we get the square of the speed of light, as it was known then and now. If ε_0 and μ_0 are universal constants, so should the speed of light be. With this statement, Einstein revolutionized the world of dynamics. The absolute space-time of Galileo and Newton has become only an approximation for velocities that are a small fraction of the speed of light.

Einstein realized that Galileo's transformation had to be expanded and embedded into the Minkowski (1864-1909) spacetime, with time now relegated as fourth dimension. His Special Relativity was born, and the new transformation was called the *Lorentz Transformation*.

Let's have a look at the Lorentz transformation. To keep it simple, I use only the x-direction

$$x' = \gamma\left(x - v\cdot t\right),\ t' = \gamma\left(t - \frac{v\cdot x}{c^2}\right),\ \gamma = \left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1}$$

The new γ variable affects now position *and* time. It is a function of the inertial velocity v and the constant speed of light c.

Buried in the Lorentz transformation is still the Galilean transformation of our daily experience:

$$\text{for}\quad v \ll c : \gamma \approx 1,\ t' \approx t\ \to\ x' = \left(x - v\cdot t\right), t' = t$$

However, Einstein was not done yet. He wanted to include even time-dependent transformations. Thus, he formulated the *Covariance Principle* in his famous paper on General Relativity [1]: "All natural laws must be covariant with respect to arbitrary continuous transformations of the coordinates." Much later, he also penned an account for Dover Publications [5] to make his theories of relativity accessible to a broader audience, while reiterating the same Covariance Principle.

His reasoning may have been that coordinate systems are solely a mathematical artifice to conduct numerical computations and have nothing to do with physical phenomena. So why should the formulation of the physical phenomena be dependent on the time-dependency of the coordinate transformation? Such *Gedankenexperimente* may have led him to formulate the Covariant Principle, though no such reference could be found.

If the Covariance Principle applies to all natural laws, then it must also apply to Newton's Second Law. In other words, Newton's Second Law must be covariant under any coordinate transformations, even time-dependent transformations. The ramification of this statement has major implications for classical dynamics, as we shall see in Section 5.

4. Noll's View of Newtonian Dynamics—Principle of Material Frame-Indifference

Now we switch to *continuum dynamics*, another branch of classical dynamics, which was shaped in the 1950-1970 foremost by Noll and Truesdell [6]. Like all classical dynamics, continuum dynamics is also based on Newton's and the associated Euler's laws, which are called here the *constitutive laws*.

Walter Noll, a student of Cliff Truesdell at Carnegie Mellon University, promoted in many of his publications the principle of material indifference, which he later renamed the Principle of Material Frame-Indifference: The constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external reference coordinates are used to describe them. Here is a quote from Walter Noll [7]:

> Although the ideas of my thesis were entirely mine, Clifford Truesdell helped me very much with putting them into clear English. He also taught me respect for good writing, something that is sadly lacking among too many mathematicians. At first he was uncomfortable with my *coordinate-free, conceptual way of mathematical reasoning* (my emphasis), and he persuaded me to include versions with coordinates of all equations in my thesis. (A few years later, he himself wrote papers in which no coordinates were used.)

The constitutive laws consist of the three *Balance Laws [8]*

$$\dot{\rho} = \rho\, div\, \dot{\boldsymbol{x}} = 0$$

$$\rho\, \ddot{\boldsymbol{x}} - div\, T = \rho\, \boldsymbol{b}$$

$$\rho\, \dot{\varepsilon} + div\, \boldsymbol{q} - T\; grad\; \dot{\boldsymbol{x}} = \rho\, r$$

consisting of mass, linear momentum, and energy equations, with density ρ, force $\boldsymbol{b}$, Cauchy stress T, heat flux $\boldsymbol{q}$ and internal energy ε, and the *Navier-Stokes Equation* [9]

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot grad\; \boldsymbol{u} = -\frac{grad\; p}{\rho} + \nu\; grad^2 \boldsymbol{u}$$

with $\boldsymbol{u}$ velocity, p pressure, ρ density, ν viscosity.

They are valid independently of coordinate systems and are unchanged under the Galilean transformation. However, Lumley states in the Journal of Applied Mechanics [10]: "The principle may not be used in turbulence modelling." This statement may refer to the partial time derivative $\partial \boldsymbol{u} / \partial t$ of the Navier-Stokes equation.

At first glance, the Covariance Principle and the Principle of Material Frame-Indifference seem to express the same physical reality, as Kempers [11] writes in the paper's summary:

> It is shown that the principle of material frame indifference follows from the covariance principle in the non-relativistic limit when inertia is considered to be absent. As a result, the principle of material frame indifference has received further justification, but it cannot retain its status of a fundamental principle.

The difference lies in the physical perspective. The Covariance Principle is an absolute statement of physical reality, while the Principle of Material Frame-Indifference refers to the coordinate system of an observer. To quote I-Shih Liu [12]: "The response of a material is the same for all observers".

Noll also reiterated his opinion on Newton's absolute reference frames in a 1995 research report [13]:

> Newton's law is valid only in certain preferred frames, which we will call inertial frames. It seems that Newton dealt with this issue by postulating a particular preferred frame, which he called "absolute space"… However, I believe that Newton's absolute space is a chimera.

Though Newton's laws require the concept of an inertial frame, this frame is not an absolute frame.

5. My View of Newtonian Dynamics—Covariance

As an aerospace engineer, Newtonian dynamics is the foundation of my theoretical and practical endeavors. While wrestling with my PhD thesis [14], I came across some of Walter Noll's publications. Just as he was attempting to formulate his findings in a "coordinate-free, conceptual way of mathematical reasoning" in continuum mechanics (see above [7]), so was I trying to express the flight dynamics of Magnus Rotors in an invariant, coordinate free formulation. To do this, I had to start with the basics, namely Newton's Second Law.

We have seen that Newton's Second Law holds only true in an inertial frame, though we were not able to pinpoint its physical character, except to say that if there is an inertial frame, any other co-moving frame with constant velocity is also an inertial frame. What piqued my interest was the distinction Noll made between frames and coordinate systems, though I was used to consider them as synonyms, as taught in my flight dynamics class. What if they are completely different entities?

In 1995, Noll stated [13]:

Some people confuse the concept of a frame [*of reference*] *with that of a coordinate system.* It makes no sense to talk about a coordinate system unless a frame-space (or at least some kind of manifold) is given first. One can consider many different coordinate systems on one and the same frame-space. Using coordinate systems in conceptual considerations is an impediment to insight; they have a legitimate place only in the context of very specific situations. (My brackets [ ]).

In my dissertation [14], I wrote:

A *frame* is an unbounded continuous set of elements over the Euclidean three-space, whose distances are time-invariant, and which possesses, as a subset, at least three non-collinear points.
A *coordinate system* is an abstract entity embedded in Euclidean three-space that establishes a one-to-one correspondence between the elements of a *frame* and the ordered triple of algebraic numbers.

To fashion an invariant formulation of Newton's Second Law, it was clear to me that frames and coordinate systems had to be treated as different entities and not as synonyms, though there was and still is much confusion. I knew how to handle Newton's Law in coordinate systems. But how do I express it in a covariant form with an inertial frame?

This was the same struggle Einstein faced when formulating his General Theory of Relativity, as Earman and Glymour stated in their paper [15] entitled: *Lost in the Tensors: Einstein's Struggles with Covariance Principles*. It was Einstein's neighbor Marcel Grossmann who introduced and taught him how to apply the recently published tensor calculus of Ricci, G., Levi-Civita [16] for a covariant formulation of the General Theory of Relativity.

Now my realm is classical dynamics, but as Einstein stated [3]: "All natural laws must be covariant with respect to arbitrary continuous transformations of the coordinates". That certainly includes Newton's Second Law.

Since tensors are defined by their transformations, I had to make a choice, whether to express them in their indexed form $v'_k = t_{ki} v_i$, $V'_{kl} = t_{ki} V_{ij} t_{jl}$, or symbolic form, $\boldsymbol{v}' = \boldsymbol{T}\boldsymbol{v}$, $\boldsymbol{V}' = \boldsymbol{T}\boldsymbol{V}\boldsymbol{T}^{-1}$. There are two ways to interpret them. In the indexed form, a tensor is the aggregate of its expression in all admissible coordinate systems, while in the symbolic form, all the coordinate systems are hidden in the abstract boldface fonts, which take on an absolute meaning. It seemed to me that the symbolic form is better suited to express Newton's Law in a covariant form.

In symbolic tensor form, Newton' Second Law is written $m\boldsymbol{a} = \boldsymbol{f}$. Expressed in inertial coordinates $m[a]^I = [f]^I$, where I am using my matrix notation of brackets with superscript to indicate that the tensors are evaluated in one particular coordinate system, namely here in an inertial coordinate system *I*. The tensor relationship has become a matrix equation, and the covariant tensor law has found a special coordinated realization.

As we saw in Section 2, Newton's Second Law is unchanged under Galilean transformations. But what happens if the coordinate transformation $\boldsymbol{T}$ is time-dependent? This is often the case in flight dynamics, just think of the transformation between the aircraft coordinates and inertial coordinates. How can the covariance of Newton's Second Law be maintained even under time-dependent transformations?

In flight dynamics, the linear momentum $\vec{p}$ is used in Newton's law $\left.\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right|_I = \vec{f}$, with $\vec{p} = m\vec{v}$ and $\vec{v}$ the velocity in inertial coordinates, while the time derivative is taken relative to the inertial coordinates *I*. Now if the time derivative of the linear momentum is taken relative to non-inertial coordinates, say the time-dependent aircraft coordinates *B*, then as everybody knows, we have to adjust Newton's law accordingly: $\left.\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right|_B + \vec{\omega}\times\vec{p} = \vec{f}$, with $\vec{\omega}$ the angular velocity of the airframe coordinates relative to the inertial coordinates. However, the covariance of the formulation has been broken by the additional term $\vec{\omega}\times\vec{p}$!

The ordinary time derivative is incapable of maintaining the tensor form if we change to a non-inertial coordinate system. That led me to search for a time operator that did not destroy the covariance of Newton's Second Law.

I found the relevant starting point in a paper by Wundheiler [17]. In the introduction, he says: "In some type of investigations the inclusion of a time-dependent coordinate system is indicated, and even mandatory because of the nature of the problem, just as it is the case with the inclusion of curvilinear coordinates" (my translation from German).

The total differential form δv^i of vector v^i for a rheonomic (time-dependent) space is according to Wundheiler

$$\delta v^i = \mathrm{d}v^i + \Gamma^i_{jk} v^j \mathrm{d}x^k + \Lambda^i_j v^j \mathrm{d}t$$

where $\Gamma^i_{jk} v^j \mathrm{d}x^k$ is the spatial transplantation contribution, with Γ^i_{jk} the *connection matrix*; and $\Lambda^i_j v^j \mathrm{d}t$ the term that Wundheiler added to account for the time dependency, where Λ^i_j is called the *rotation matrix* and is of the form $\Lambda^i_j = \frac{\partial x^i}{\partial \overline{x}^h}\frac{\partial^2 \overline{x}^h}{\partial x^j \partial t}$.

Now we switch from Riemannian space to Euclidean space. Because of its flatness, the connection matrix vanishes, but not the rotation matrix

$$\delta v^i = \mathrm{d}v^i + \Lambda^i_j v^j \mathrm{d}t$$

Introducing new nomenclature for the transformations

$$t^i_h = \frac{\partial x^i}{\partial \overline{x}^h};\ \overline{t}^h_j = \frac{\partial \overline{x}^h}{\partial x^j}$$

the rotation matrix becomes

$$\Lambda^i_j = t^i_h \frac{\partial \overline{t}^h_j}{\partial t} = t^i_h \frac{\mathrm{d}\overline{t}^h_j}{\mathrm{d}t}$$

where the partial derivative of time has become the total time derivative, because of the flatness of Euclidean space. The rheonomic transformation is now, after dividing by *dt*

$$\frac{\delta v^i}{\delta t} = \frac{\mathrm{d}v^i}{\mathrm{d}t} + t_h^i \frac{\mathrm{d}\overline{t}_j^{\,h}}{\mathrm{d}t} v^j$$

For Newtonian applications, we can further simplify the transformation by using the Cartesian space, in which case the transformation t_{ih} is orthogonal and its inverse equals its transpose

$$\frac{\delta v_i}{\delta t} = \frac{\mathrm{d}v_i}{\mathrm{d}t} + t_{ih} \frac{\mathrm{d}t_{jh}}{\mathrm{d}t} v_j$$

On the left, we have the total time differential $\delta v_i / \delta t$ of vector v_i, on the right, the time derivative $\mathrm{d}v_i / \mathrm{d}t$ related to the particular coordinate system; and, if the coordinate transformation is time-dependent, there appears the additional term $t_{ih}\left(\mathrm{d}t_{jh} / \mathrm{d}t\right) v_j$. The total time differential $\delta v_i / \delta t$ I called the *rotational time derivative*.

I only sketched an abbreviated derivation of Wundheiler's rheonomic transformation from Riemannian space to Euclidean and then to Cartesian space. The detailed derivation is given in the Annex *D* of my textbook *Modeling and Simulation of Aerospace Vehicle Dynamics* [18].

In flight dynamics, where coordinate transformations abound, identifying coordinate systems by over bars or other symbols is cumbersome. The matrix notation is preferred, with the coordinate system indicated by a capital superscripted letter; e.g., for two coordinate systems *A* and *B*, the transformation of *B* with respect to *A* is $[T]^{BA}$; and the transformation of tensor $\boldsymbol{v}$ of rank one is

$$[v]^B = [T]^{BA} [v]^A$$

Applying this nomenclature to the rotational time derivative, and replacing the total time differential $\delta v_i / \delta t$ with the expression $\left[D^A v\right]^B$ provides the operational form

$$\left[D^A v\right]^B \equiv \left[\frac{\mathrm{d}v}{\mathrm{d}t}\right]^B + [T]^{BA} \overline{\left[\frac{\mathrm{d}T}{\mathrm{d}t}\right]}^{BA} [v]^B$$

The indexed form of the transformation matrix t_{ih} has become $[T]^{BA}$. The *i* coordinate system (now *B*) of Wundheiler's reduced equation $\delta v^i = \mathrm{d}v^i + \Lambda_j^i v^j \mathrm{d}t$ is the new coordinate system, which is time-variant relative to the original coordinate system *j* (now *A*). Coordinates $]^A$ are embedded in frame $\boldsymbol{A}$. The rotational time derivative $\left[D^A v\right]^B$ determines how the vector $\boldsymbol{v}$, originally expressed in *A* coordinates, is changing in time when subject to the time-dependent coordinate transformation. Here we see the paramount importance of distinguishing between frames and coordinate systems. The superscript *A* of *D* refers to the original frame $\boldsymbol{A}$, while the superscript *B* of $]^B$ refers to the new coordinate system *B*.

Let's apply the rotational time derivative to Newton's Second Law and show that it can be formulated in a covariant tensor form valid for all time-dependent

Cartesian coordinate transformations. The reference frame of Newton's law is always the inertial Frame ***I***. So, Newton's law is written in tensor form

$$D^I \boldsymbol{p} = \boldsymbol{f}$$

If expressed in inertial coordinates $]^I$, $\left[D^I p\right]^I = [f]^I$, where the left side is

$$\left[D^I p\right]^I \equiv \left[\frac{\mathrm{d}p}{\mathrm{d}t}\right]^I + [T]^{II} \overline{\left[\frac{\mathrm{d}T}{\mathrm{d}t}\right]}^{II} [v]^I = \left[\frac{\mathrm{d}p}{\mathrm{d}t}\right]^I, \text{ because } \overline{\left[\frac{\mathrm{d}T}{\mathrm{d}t}\right]}^{II} = [0]$$

Thus, Newton's law expressed in inertial coordinates, using the rotational time derivative, is what you expect $[\mathrm{d}p/\mathrm{d}t]^I = [f]^I$. Now let's use a non-inertial coordinate system $]^B$, whose coordinate transformation $[T]^{BI}$ is time-dependent: $\left[D^I p\right]^B = [f]^B$

$$\left[D^I p\right]^B \equiv \left[\frac{\mathrm{d}p}{\mathrm{d}t}\right]^B + [T]^{BI} \overline{\left[\frac{\mathrm{d}T}{\mathrm{d}t}\right]}^{BI} = [f]^B$$

But notice in each case we had the same form using the rotational time derivative $\left[D^I p\right]^I = [f]^I$ and $\left[D^I p\right]^B = [f]^B$ with no extra term as we encountered earlier using the ordinary time derivative $\left.\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right|_I = \vec{f}$ and $\left.\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right|_B + \vec{\omega} \times \vec{p} = \vec{f}$ (The extra term is hidden in $\left[D^I p\right]^B$). Without the extra term, we find that the rotational time derivative maintains the covariance of Newton's Second Law and therefore can be expressed in the tensor form $D^I \boldsymbol{p} = \boldsymbol{f}$ valid in all Cartesian coordinate systems even those related by time-variant transformations.

What I demonstrated here is not a proof, but in Annex D, Section 3 [18], I provide the rigorous proof that the rotational time derivative maintains the covariant tensor form of first-order and second-order tensors for all Cartesian transformations, even those with time-dependent elements.

It has been 54 years since I introduced the covariant treatment of Newton's Second Law [14]. Its benefits were especially important for flight dynamics. In flight dynamics a myriad of coordinate systems is used to model the dynamics of aircraft, drones, spacecraft, missiles, and rockets. Particularly in engagements where multiple vehicles interact with each other, it is better to first model the physical engagement with tensors, and only after the scenario has been properly shaped, coordinate systems are introduced, and the tensors have become matrices, suitable for computer programming; therefore, my motto: *From tensor modeling to matrix coding*.

Besides the practical benefit of untangling physics from computation, the covariancy of Newton's Second Law also led to some theoretical advances in flight dynamics. In an AIAA paper [19], I presented the general perturbation equations of unsteady reference flight, which was credited at the 2012 AIAA Atmospheric Flight Conference, as the most influential paper on flight dynamics of the 1970s. Here are the equations in tensor form:

$$m D^{Bp} \varepsilon \boldsymbol{v}_B^I + m \boldsymbol{R}^{BpBr} \boldsymbol{\Omega}^{BrI} \overline{\boldsymbol{R}^{BpBr}} \varepsilon \boldsymbol{v}_B^I + m \varepsilon \boldsymbol{\Omega}^{BI} \boldsymbol{R}^{BpBr} \boldsymbol{v}_{Br}^I = \varepsilon \boldsymbol{f}_a + \varepsilon \boldsymbol{f}_t + \left(\boldsymbol{E} - \boldsymbol{R}^{BpBr}\right) \boldsymbol{f}_{gr}$$

$$\boldsymbol{I}_{Bp}^{Bp} D^{Bp} \varepsilon \boldsymbol{\omega}^{BI} + \boldsymbol{R}^{BpBr} \boldsymbol{\Omega}^{BrI} \overline{\boldsymbol{R}^{BpBr}} \boldsymbol{I}_{Bp}^{Bp} \varepsilon \boldsymbol{\omega}^{BI} + \varepsilon \boldsymbol{\Omega}^{BI} \boldsymbol{R}^{BpBr} \boldsymbol{I}_{Br}^{Br} \boldsymbol{\omega}^{BrI} = \varepsilon \boldsymbol{m}_a + \varepsilon \boldsymbol{m}_t$$

Some explanations: the first equation determines the perturbation of the velocity of the center of mass B of the aircraft relative to an inertial frame ***I***, $\varepsilon \boldsymbol{v}_B^I$; and the second equation is the perturbation of angular velocity of the airframe ***B*** relative to the inertial frame ***I***, $\varepsilon \boldsymbol{\omega}^{BI}$. The unsteady effects are caused by the term $\varepsilon \boldsymbol{\Omega}^{BI} \boldsymbol{R}^{BpBr} \boldsymbol{I}_{Br}^{Br} \boldsymbol{\omega}^{BrI}$. It models the perturbations of an aircraft in maneuvering flight, given by $\boldsymbol{\omega}^{BrI}$, which is the angular velocity of pitch-up or pitch-down, as well as turning flight. For more details go to my textbook [18], Chapter 6.

By introducing the rotational time derivative and distinguishing carefully between frames and coordinate systems, I was able to formulate Newtons Second Law in a covariant tensor formulation, valid for all Cartesian coordinate systems. Since all my theoretical and practical applications are based on Newton's Second Law, I took advantage of the new paradigm in my technical publications, starting with my textbook *Modeling and Simulation of Aerospace Vehicle Dynamics* [18], a product of my 30 years of graduate teaching at the University of Florida. For undergraduate instructions, I wrote *Introduction to Tensor Flight Dynamics* [20], which can be taught in a one semester course, requiring only minimum prerequisites.

With modeling and simulation becoming such an important part of aerospace engineering, the approach of modeling the physics with tensors and programming in matrices has streamlined the simulation task for greater effectiveness. Accompanying my textbook are the three CD Roms [21]-[23] of my graduate courses, published by AIAA, which teach how to model the dynamics of missiles, aircraft, and hypersonic vehicles in tensors, followed by programming their trajectories in C++. For the introductory course, I have provided over 40 problems that challenge the student using MATLAB® or PYTHON to obtain numerical solutions for aerospace applications that have been formulated by tensors.

More recently, I have distilled my short courses, which I have given to industry and academia, in five workbooks [24]-[28]. They summarize my experience gained over five decades with modeling in tensors and programming with matrices, foremost in C++. All these workbooks are also the basis of my seven online courses at UDEMY.

6. Conclusions

More than five decades have passed since I introduced a covariant formulation of Newton's Second Law. But the foundations had been laid much earlier by Einstein's postulate that *all natural laws must be covariant with respect to arbitrary continuous transformations of the coordinates*, and later Noll's Principle of Material Frame-Indifference with its clear distinction between frames and coordinate systems. Wundheiler's paper provided the theoretical underpinning for deriving the *rotational time derivative*. Without replacing the ordinary time derivative with the rotational time derivative, the covariant tensor formulation of Newton's law would not have been possible.

My dissertation has borne much fruit particularly in flight dynamics, which deals with many coordinate systems. The two-phased approach, which involves formulating the physics in tensors and then converting them to matrices by introducing coordinate systems for coding, is now called *Tensor Flight Dynamics*. It has enabled me and others to model complex flight engagements and develop sophisticated simulations, as documented in my AIAA publications, AMAZON workbooks, and UDEMY online courses.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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