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Subplanes of $PG\left(2,q^r\right)$, Ruled Varieties V_2^{2r-1} in $PG\left(2r,q\right)$, and Related Codes

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Abstract

In this note we consider ruled varieties V_2^{2r-1} of $PG\left(2r,q\right)$, generalizing some results shown for $r=2,3$ in previous papers. By choosing appropriately two directrix curves, a V_2^{2r-1} represents a non-affine subplane of order q of the projective plane $PG\left(2,q^r\right)$ represented in $PG\left(2r,q\right)$ by a spread of a hyperplane. That proves the conjecture assumed in [1]. Finally, a large family of linear codes dependent on $r\geq 2$ is associated with projective systems defined both by V_2^{2r-1} and by a maximal bundle of such varieties with only an *r*-directrix in common, then are shown their basic parameters.

Keywords

Finite Geometry, Translation Planes, Spreads, Varieties

1. Introduction

It is known that a projective translation plane Π can be represented in a projective space of even order, following the papers of André [2], Bruck and Bose [3]) and Vincenti [1].

A subplane of Π is *affine* and *non-affine* depending on whether it intersects the line at infinity in a *subline* or in one point.

An *affine* subplane of order q is represented by every *transversal* plane to the spread. All that holds also in case Π is the Desarguesian plane $PG\left(2,q^r\right)$ when the spread is a *regular* spread (cf. [2]-[6] for $r=2$, [1] for $r=3$).

Denote $\Pi=PG\left(2,q^r\right)$, $\Sigma=PG\left(2r,q\right)$, $\Sigma'=PG\left(2r-1,q\right)$ and $\mathcal{S}$ is a regular spread of $\left(r-1\right)$-subspaces.

There exist $q^{r-1}+q^{r-2}+\cdots+q^2+q+1$ affine subplanes of $\Pi=PG(2,q^r)$ of order q having the same subline at infinity and through one fixed affine point, while q^{2r-2} affine subplanes having no affine point in common partition the affine points of Π (cf. Proposition 3.6, Theorem 3.7)

A variety V_2^{2r-1} of Σ is a ruled variety of $PG(2r,q)$ with the minimum order directrix a rational curve of order $r-1$ and a maximum order directrix a rational curve of order r, the two curves lying in two complementary spaces of dimension $r-1$ and r, respectively (cf. [7], Capters 13, 8., 9.). The variety can be obtained by joining points of the two directrix curves corresponding via a projectivity.

In Propositions 4.3 - 4.6 and Theorem 4.7 some fundamental incidence properties of V_2^{2r-1} are shown. Such properties allow to prove that V_2^{2r-1} represents a non-affine subplane Π_q of order q of $PG(2,q^r)$ (cf. Theorem 4.8). The properties of Π_q of being a plane, translate into further incidence properties of the affine points of V_2^{2r-1} (cf. Corollary 4.9).

An example is then shown by choosing q and r such that $\gcd(q-1,r)=d\neq 0$ (cf. Paragraph 4.2).

In Theorem 5.2 a maximal bundle $\mathcal{B}$ of varieties V_2^{2r-1} having in common only a curve of order r is constructed.

To conclude, linear codes are associated with the projective systems related both to a variety V_2^{2r-1} and to the bundle $\mathcal{B}$, then their basic parameters are calculated (cf. Proposition 5.1, Theorems 5.3 - 5.5).

Note that a part of Section 3 is necessarily common with previous articles, this representing a generalization as announced in the abstract.

2. Preliminary Notes and Results

Referring to the Section 2 of [1], denote $F=GF(q)$ a finite field, $q=p^s$, p an odd prime, $\bar{F}$ the algebraic closure of the field F, F^{n+1} the $(n+1)$-dimensional vector space over F, $PG(n,q)=PrF^{n+1}$ the *n*-dimensional projective space contraction of F^{n+1} over F. It is considered a sub-geometry of $\overline{PG(n,q)}$, the projective geometry over $\bar{F}$. A subspace of $PG(n,q)$ of dimension h is denoted *h*-space.

For the Definition of a variety V_u^v of dimension u and order v of $PG(n,q)$ see [1], Definition 2.1.

From [7], p. 290, 7., follows the definition of a ruled variety V_2^{n-1} of $PG(n,q)$ (cf. Lemma 2.2 of [1]).

Let Σ be the projective space $PG(2r,q)$, $\Sigma'=PG(2r-1,q)$ a hyperplane of Σ, $\mathcal{S}$ a *spread* of $(r-1)$-spaces of Σ' (for the definition of spread, regulus and regular spread cf. [3] and [1], Definition 2.3 and the representation).

A *transversal line* l to $\mathcal{S}$ is a line of Σ' such that $l\notin S$ for every $S\in\mathcal{S}$. As $\mathcal{S}$ is regular, then the line l meets $q+1$ subspaces of $\mathcal{S}$ consisting of a regulus (cf. [1], Definition 2.3).

For the following definitions and results, see [8] and [9].

Definition 2.1 *A linear* $[n,k]_q$ *-code* C *of length* n *is a k-dimensional subspace of the vector space* F^n *. The dual code of* C *is the* $(n-k)$ *-dimensional subspace* $C^{\perp}$ *of* F^n *and it is an* $[n,n-k]_q$ *-code.*

For $t \geq 1$ *the t-th* higher weight *of* C *is defined by*

$$d_t = d_t(C) = \min\{\|D\| \text{ for all } D < C, \dim D = t\},$$

where $\|D\|$ *is the number of indices* i *such that there exists* $v \in D$ *with* $v_i \neq 0$.

Note that $d_1 = d_1(C)$ is the classical minimum distance of C , the *Hamming distance*.

An $[n,k]_q$ -code C of minimum distance d is also denoted $[n,k,d]_q$ -code.

Definition 2.2 *An* $[n,k]_q$ *-projective system* $\mathcal{X}$ *of the projective space* $PG(k-1,q)$ *is a collection of* n *not necessarily distinct points. It is called non-degenerate if these* n *points are not contained in any hyperplane.*

Assume that $\mathcal{X}$ consists of n distinct points having rank k .

For each point of X choose a generating vector. Denote by M the matrix having as rows such n vectors and let $C_{\mathcal{X}}$ be the linear code having M^t as a generator matrix. The code $C_{\mathcal{X}}$ is the *k*-dimensional subspace of F^n which is the image of the mapping from the dual *k*-dimensional space $(F^k)^*$ onto F^n that calculates every linear form over the points of X .

Hence the length n of codeword of $C_{\mathcal{X}}$ is the cardinality of X , the dimension of $C_{\mathcal{X}}$ being just k .

There exists a natural [1-1] correspondence between the equivalence classes of a non-degenerate $[n,k]_q$ -projective system X and a non-degenerate $[n,k]_q$ -code $C_{\mathcal{X}}$ such that if X is an $[n,k]_q$ -projective system and $C_{\mathcal{X}}$ is the corresponding code, then the non-zero codewords of $C_{\mathcal{X}}$ correspond to hyperplanes of $PG(k-1,q)$, up to a non-zero factor, the correspondence preserving the parameters n,k,d_t .

Generally, subcodes D of $C_{\mathcal{X}}$ of dimension r correspond to (projective) subspaces of codimension r of $PG(k-1,q)$, therefore
$d_1 = d_1(C_{\mathcal{X}}) = n - \max\{|X \cap H| : H < PG(k-1,q), \operatorname{codim} H = 1\}$.

If d is the minimum weight of a linear code $C_{\mathcal{X}}$ then $C_{\mathcal{X}}$ is an *s*-error-correcting code for all $s \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. We call $\left\lfloor \frac{d-1}{2} \right\rfloor$ the *error-correcting capability of* $C_{\mathcal{X}}$.

3. Affine Subplanes of Order q of $PG(2,q^r)$

From now on denote $\Sigma = PG(2r,q)$ the 2*r*-dimensional geometry over the field $F = GF(q)$, $\Sigma' = PG(2r-1,q)$ a hyperplane of Σ, $\mathcal{S}$ a regular spread of $(r-1)$ -spaces of Σ' . Clearly $|\mathcal{S}| = q^r + 1$.

Let $\Pi = PG(2,q^r)$ be the Desarguesian plane over the field $GF(q^r)$. Denote l_∞ the line at infinity of Π. Represent Π in $\Sigma = PG(2r,q)$ by the spread $\mathcal{S}$.

Define the following incidence structure $\Pi' = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ (points, lines, incidence, respectively) where

$$\mathcal{P} = \{P \in \Sigma \setminus \Sigma'\} \cup \{S_{r-1} \in \mathcal{S}\},$$

$$\mathcal{L} = \{\mathcal{L}_0 = \{S_r \subset \Sigma \setminus \Sigma' \mid S_r \cap \Sigma' \in \mathcal{S}\}\} \cup \{l_\infty = \mathcal{S}\},$$

$\mathcal{I}$ is defined as follows

If $P \in \Sigma \setminus \Sigma', l \in \mathcal{L}_0$ then $P\mathcal{I}l \Leftrightarrow P \in l$, no point of $\Sigma \setminus \Sigma'$ is incident l_∞, $S_{r-1}\mathcal{I}l_\infty$ for all $S_{r-1} \in \mathcal{S}$, $S_{r-1}\mathcal{I}l$ where $l \in \mathcal{L}_0 \Leftrightarrow l \cap \Sigma' = S_{r-1}$.

Lemma 3.1 $\Pi' \cong \Pi$.

Proof. See [3].

From [1] and [3], Definitions 2.7, 2.8, Propositions 2.9, referred to the current dimension, follows that the affine points of Π are represented by the q^{2r} affine points of $\Sigma \setminus \Sigma'$, the points at infinity by the $q^r + 1$ subspaces of $\mathcal{S}$. The affine lines of Π are represented by the *r*-spaces S_r of $\Sigma \setminus \Sigma'$ such that the subspaces $S_{r-1} = S_r \cap \Sigma'$ belong to $\mathcal{S}$, the line at infinity l_∞ by the spread $\mathcal{S}$.

Definition 3.2 *A subplane* $\pi' = (P', L', I')$ *of a plane* $\pi = (P, L, I)$ *is a subgeometry of* π, *that is, an incidence substructure for which* $P' \subset P$, *for each line* $l' \in L'$ *there exists a line* $l \in L$ *such that* $l' \subset L$ *and* $I' = I$.

Definition 3.3 *A subplane of* $\Pi = PG(2, q^r)$ *of order* q *is affine if it meets the line* l_∞ *of* Π *in a subline consisting of* $q+1$ *points, it is non-affine if it meets the line* l_∞ *in one point.*

Let t be any *transversal line* to $\mathcal{S}$, that is a line meeting $q+1$ $(r-1)$-spaces of $\mathcal{S}$. As $\mathcal{S}$ is regular, these $q+1$ elements form a regulus $\mathcal{R} \subset \mathcal{S}$ (cf. [3], Lemma 12.2). Choose and fix a plane $\alpha \subset \Sigma \setminus \Sigma'$ through the line t, that is, a *transversal plane.*

As in Proposition 2.9 of [1], one can easily prove

Proposition 3.4 *The plane* α *is isomorphic to a subplane* $\pi \cong PG(2, q)$ *of* Π *whose points at infinity are represented by the* $q+1$ $(r-1)$*-spaces of* $\mathcal{R}$, *the lines of* π *being represented by the sublines intersections of* α *with the r-spaces of* Σ *through the* $(r-1)$*-spaces of* $\mathcal{R}$. *As the line at infinity of* π *is a subline of the infinity line of* Π, *then* π *is an affine subplane.*

For the construction of transversal lines to $\mathcal{R}$ in Σ' the procedure is similar to the one used for the dimension 5 (cf. [1], Proposition 2.10).

Proposition 3.5 *The set of the transversal lines to* $\mathcal{R}$ *has cardinality* $q^{r-1} + q^{r-2} + \cdots + q^2 + q + 1$, *that is, they are as many as the points of an* $(r-1)$*-subspace of* $\mathcal{R}$.

Proof. Denote S_1, S_2, S_3 three $(r-1)$-subspaces of the regulus $\mathcal{R}$. Fix a point $P \in S_1$ and denote $S = \langle P, S_2 \rangle$ the *r*-space of Σ' direct sum of P and S_2 and $S' = \langle P, S_3 \rangle$ the *r*-space of Σ' direct sum of P and S_3. Lying in a $(2r-1)$-dimensional subspace, then $S \cap S' = t$ is a line. As a line of S, t meets S_2 in a point, as a line of S', t meets S_3 in a point. Therefore t is a transversal line to the subspaces S_1, S_2, S_3. As S_1, S_2, S_3 belong to the regulus $\mathcal{R}$, the line t meets each of the $q+1$ elements of $\mathcal{R}$. In such a way one can construct a transversal line for every point P chosen in S_1, that is, $q^{r-1}+q^{r-2}+\cdots+q^2+q+1$.

Proposition 3.6 *The cardinality of the affine subplanes of* Π *isomorphic to* $PG(2,q)$ *having the same subline of* $q+1$ *points at infinity and containing one affine point is* $q^{r-1}+q^{r-2}+\cdots+q^2+q+1$.

Proof. Let r_0 be a transversal line to $\mathcal{R}$, $\alpha_0 \supset r_0$ a transversal plane, O an affine point of α_0. Denote $\{t_i \mid i=0,\cdots,q^{r-1}+q^{r-2}+\cdots+q^2+q\}$ the transversal lines to the regulus $\mathcal{R}$. Each of the $q^{r-1}+q^{r-2}+\cdots+q^2+q+1$ planes $\alpha_i=\langle O,r_i\rangle$ represents an affine subplane π_i of Π, $\pi_i \cong PG(2,q)$ (cf. Proposition 3.4).

Choose and fix a transversal line t. Consider the bundle (t) of the planes of $\Sigma\setminus\Sigma'$ having the line t as axis. Each plane $\alpha\in(t)$ is isomorphic to $PG(2,q)$ (cf. Proposition 3.4) and it is an affine subplane of Π having a same subline of $q+1$ points on the line at infinity.

Theorem 3.7 *The planes of* (t) *are* q^{2r-2} *and partition the* q^{2r} *affine points of* Π.

Proof. The planes of (t) are *parallel* to each other, therefore they have no affine point in common otherwise they would coincide. Each such a plane contains q^2 affine points.

Let $h=|(t)|$. As a line and an independent point define a plane, fixed the line t, there are q^{2r} choices for a point in $\Sigma\setminus\Sigma'$ to get the plane $\langle t,P\rangle\subset\Sigma\setminus\Sigma'$, this number to be divided by q^2, which equals the choices of an affine point on a same plane, hence $h=q^{2r-2}$.

4. A Ruled Variety V_2^{2r-1} of $PG(2r,q)$

In $\Sigma=PG(2r,q)$ consider two normal rational curves $\mathcal{C}^m$ and $\mathcal{C}^{2r-m-1}$ of order m and $2r-m-1$, respectively in two complementary subspaces S_m and S_{2r-m-1} of Σ. Each of them consists of $q+1$ points (cf. [10], Theorem 21.1.1). They are projectively equivalent. From Lemma 1 follows that a ruled rational surface V_2^{2r-1} of order $2r-1$ of $PG(2r,q)$ is generated by connecting corresponding points of the two directrices $\mathcal{C}^m$ and $\mathcal{C}^{2r-m-1}$ of V_2^{2r-1}, (cf. [7], p. 290, 7.). The variety consists of $q+1$ skew lines and $(q+1)^2$ points.

Choose and fix $m=r-1$ so that $2r-m-1=r$.

For our purpose to choose appropriately a directrix $\mathcal{C}^r$ in an *r*-dimensional subspace of Σ, some considerations have to be made.

It is well known that a rational normal curve $\mathcal{C}^r$ of order r of an *r*-dimensional geometry $PG(r,q)$ can be defined by $r+1$ independent binary forms of order r, $g_i(s_0,s_1)\in F[s_0,s_1]$, $i=0,1,\cdots,r$, or by $r+1$ functions $f_i(s):=g_i(1,s)$ where at least one of f_i has degree r. Moreover it must be $q\geq r$ (cf. [10], p. 229).

A hyperplane of the geometry $PG(r,q)$ meets $\mathcal{C}^r$ in at most r points, corresponding to the solutions of an equation of degree r over $F=GF(q)$.

The orbits of the hyperplanes under the action of the group of the projectivities of $PG(r,q)$ fixing $\mathcal{C}^r$, correspond just to such possibilities (for $r=3$ cf. [10], pp. 229-230, and p. 234, Corollary 4, $\mathcal{N}_5$).

For our construction, we need an *r*-curve having no point in the hyperplane of $PG(r,q)$ chosen as hyperplane at infinity. Therefore it needs to find irreducible polynomials over F of degree r. Two ways are indicated in [11] and in [12].

However we show an example of what is written above.

Let us introduce coordinates $(x_0,x_1,\cdots,x_r)$ in $PG(r,q)$ so that a curve $\mathcal{C}^r$ of order r can be expressed as follows $\mathcal{C}^r=\left\{\left(1,s,s^2,\cdots,s^{(r-1)},f_r(s)\right)\mid s\in F^+\right\}$, where $f_r(s)$ is an irreducible polynomial of degree r (for the symbology see [10], p. 229).

Example 4.1 *The curve* $\mathcal{C}^r=\left\{\left(1,s,s^2,\cdots,s^{(r-1)},f_r(s)\right)\mid s\in F^+\right\}$ *and the hyperplane* $x_r=0$ *have no point in common.*

Another way to find irreducible polynomials of a given degree, is obtained by considering the problem of searching in $F^*=F\setminus\{0\}$ the elements that are not *r*-th powers.

Given $q=p^n$, p a prime, a field $F=GF(q)$ and a positive integer r, denote $d=\gcd(r,q-1)$, the great common divisor.

Lemma 4.2 *The subset* $N_r\subset F^*$ *of the non-r-th powers has cardinality* $\frac{(q-1)(d-1)}{d}$*, so that if* $d\neq 1$ *each polynomial* $x^r-h\in F[x]$ *with* $h\in N_r$ *is irreducible over* F.

Proof. Denote $\varphi:F^*\to F^*$ the mapping $\varphi:x\mapsto x^r$. Set $\ker\varphi=\left\{x\in F^*\mid x^r=1\right\}$, the subset of the *r*-th roots of unity. Then $|\ker\varphi|=\gcd(r,q-1)=d$. Hence $\varphi(F^*)\cong\frac{F^*}{\ker\varphi}$ so that in F^* there are $\frac{|F^*|}{|\ker\varphi|}=\frac{q-1}{d}$ elements that are *r*-th powers. If $d\neq 1$, then the complementary set $N_r=F^*\setminus\varphi(F^*)$ of the elements that are not *r*-th powers has cardinality $\frac{(q-1)(d-1)}{d}$, hence every polynomial x^r-h with $h\in N_r$ is irreducible over F.

NOTE 1—A rational normal curve $\mathcal{C}^r$ of an *r*-space consists of $q+1$ points ($q\geq r$) no $r+1$ of which in a hyperplane S_{r-1} (that is, a hyperplane meets $\mathcal{C}^r$ in at most r points, cf. [10], p. 229, Theorem 21.1.1, (iv)). Hence r points lie in no S_{r-2}, $r-1$ points in no S_{r-3}.

Choose and fix an $(r-1)$-space $S_{r-1}^\infty\in\mathcal{S}$ and a rational normal curve $\mathcal{C}_\infty^{r-1}\subset S_{r-1}^\infty$ of order $r-1$. Let S_r^0 be an *r*-dimensional subspace of $\Sigma\setminus\Sigma'$ such that $S_r^0\cap\Sigma'=S_{r-1}^0\in\mathcal{S}\setminus S_{r-1}^\infty$ and $\mathcal{C}_0^r\subset S_r^0$ a rational normal curve of it of order r with $\mathcal{C}_0^r\cap S_{r-1}^0=\varnothing$.

Let $\Lambda:\mathcal{C}_\infty^{r-1}\to\mathcal{C}_0^r$ be a projectivity. Represent $\mathcal{C}_\infty^{r-1}=\left\{G_{i_\infty}\mid i=1,\cdots,q+1\right\}$, $\mathcal{C}_0^r=\left\{G_i=\Lambda G_{i_\infty}\mid i=1,\cdots,q+1\right\}$. Denote V_2^{2r-1} the variety arising by connecting corresponding points of $\mathcal{C}_\infty^{r-1}$ and $\mathcal{C}_0^r$ via Λ (cf. [7], p. 291). The curves $\mathcal{C}_\infty^{r-1}$

and $\mathcal{C}_0^r$ are directrix curves of V_2^{2r-1}, the set $\mathcal{G}=\{g_i=G_iG_{i_\infty}\mid i=1,\cdots,q+1\}$ is the set of the generatrix lines of V_2^{2r-1}. The set $\mathcal{G}$ partitions the variety.

Let H be any hyperplane. In a suitable complexification of Σ, $H\cap V_2^{2r-1}$ is a curve of order $2r-1$ (cf. [7], p. 288, 5.).

Proposition 4.3 *The variety* V_2^{2r-1} *consists of* $q+1$ *mutually skew affine generatrix lines and of* q^2+q *affine points.*

a) *A directrix curve* $\mathcal{C}\neq\mathcal{C}_\infty^{r-1}$ *cut by a hyperplane on* V_2^{2r-1} *cannot lie in an* $(r-1)$ *-space. The curve* $\mathcal{C}_\infty^{r-1}$ *is the unique minimum order* $(r-1)$ *directrix.*

If a space S *contains* r *points of* $\mathcal{C}_\infty^{r-1}$, *then* $S\supset\mathcal{C}_\infty^{r-1}$.

Moreover $k\le r$ *generatrix lines are independent and belong to a* $(2k-1)$ *-space.*

b) *An r-space containing* S_{r-1}^∞ *contains at most one generatrix line.*

c) *The r-space joining one generatrix line and the* $(r-1)$ *-space* S_{r-1}^∞ *meets no other generatrix in an affine point.*

d) r *generatrices* $\{g_i\mid i=1,\cdots,r\}$ *are joint by a hyperplane* H *that contains the* $(r-1)$ *-space* S_{r-1}^∞, *so that* $H\cap V_2^{2r-1}=\{g_i\mid i=1,\cdots,r\}\cup\mathcal{C}_\infty^{r-1}$.

e) *A hyperplane contains neither a fixed directrix, nor a fixed generatrix.*

Proof. The proof of the first statement is analogous to that of Proposition 3.1 of [1].

a) Assume a hyperplane H meets V_2^{2r-1} in a directrix curve $\mathcal{C}^{r-1}\neq\mathcal{C}_\infty^{r-1}$ lying in a $(r-1)$-space S. Then V_2^{2r-1} is contained at most in the $(2r-1)$-space generated by S and S_{r-1}^∞ and the variety generated by the two curves would have order at most $2r-2$, a contradiction. Hence the curve $\mathcal{C}_\infty^{r-1}$ is the unique minimum order $r-1$ directrix.

For the proof of the last two statements see [7], 5., 6., pp. 288-289.

b) Assume S is an *r*-space containing S_{r-1}^∞ and two generatrix lines g_1,g_2. Denote $G_i=g_i\cap\mathcal{C}_0^r$, $i=1,2$ then the line G_1G_2 belongs to both S_r^0 and S so that the point $G=G_1G_2\cap\Sigma'$ is a common point of S_{r-1}^∞ and S_{r-1}^0, a contradiction.

c) Denote $S=\langle g,S_{r-1}^\infty\rangle$ with $g\in\mathcal{G}$, an *r*-space. Assume that for $g'\in\mathcal{G}\setminus\{g\}$ with $g'\cap\mathcal{C}_\infty^{r-1}=G'_\infty$ is $S\cap g'\setminus G'_\infty\neq\varnothing$. Then $g'\subset S$, so that S contains two generatrix lines and the $(r-1)$-space S_{r-1}^∞, a contradiction to b).

d) Assume r generatrices $\{g_i\mid i=1,\cdots,r\}$ are joint by a $(2r-2)$-space S'. As S' contains the r independent points $G_{i\infty}=g_i\cap\mathcal{C}_\infty^{r-1}$, $i=1,\cdots,r$, $G_{i_\infty}\in S_{r-1}^\infty$, then $S_{r-1}^\infty\subset S'$ and $\mathcal{C}_\infty^{r-1}\subset S'$. As S' cannot contain V_2^{2r-1}, a hyperplane $H\supset S'$ and through a further point $P\in V_2^{2r-1}\setminus S'$ should contain also the generatrix g_P through P. Hence H would meet V_2^{2r-1} in $(r+1)$ generatrix lines and in a curve of order $r-1$, that is, in a curve of order $2r$, a contradiction (cf. [7], p. 288, 5.). Hence $H\cap V_2^{2r-1}=\{g_i\mid i=1,\cdots,r\}\cup\mathcal{C}_\infty^{r-1}$, that is, a curve of order $2r-1$ (and H contains no further point of V_2^{2r-1}).

e) Let $\mathcal{G}_{r-1}=\{g_i\mid i=1,\cdots,r-1\}$ be a subset of $r-1$ generatrices of V_2^{2r-1}. Denote $S=S_{2(r-1)-1}=S_{2r-3}$ the subspace containing $\mathcal{G}_{r-1}$ (cf. [7], 6., p. 289). Let H be a hyperplane with $H\supset S$ and assume H contains a residual and fix directrix curve $\mathcal{C}$ of order r. Let P be a point of V_2^{2r-1}, $P \in H \setminus \mathcal{C}$. Denote $S'' = \langle S, P \rangle$. Then every hyperplane containing S' and S' itself, would contain the generatrix g_P through P, so that $\mathcal{G}_{r-1} \cup \{g_P\} \subset S'$, a contradiction to d).

An analogous contradiction is reached if we assumed a generic hyperplane H with $g, g' \subset H$ contained a fix generatrix (cf. [7], 6., pp. 289-290).

From [7], pp. 287-290 follows

Proposition 4.4 *A hyperplane* H *containing* $r-1$ *generatrices contains a residual curve* $\mathcal{C}$ *of order* r *of an r-space* $S \subset H$. *Moreover* S *is skew to* S_{r-1}^{∞}, $\mathcal{C}$ *is irreducible and is a directrix.*

Proof. A hyperplane H meets V_2^{2r-1} in a rational normal curve of order $2r-1$ or in a curve of order $m < 2r-1$ met by all generatrices and in $2r-1-m$ generatrices. In the current case $m = r$ or $m = r-1$ can happen.

If a hyperplane meets V_2^{2r-1} in $\mathcal{C}_{\infty}^{r-1}$, the unique directrix curve of order $r-1$ (see *a*), Proposition 4.3), then it contains $2r-1-(r-1) = r$ generatrix lines and viceversa (see d), Proposition 4.3). If a hyperplane contains $2r-1-r = r-1$ generatrices, then it meets V_2^{2r-1} in a residual curve $\mathcal{C}$ of order r.

Assume $\mathcal{C}$ irreducible and contained in an μ-space S_{μ}, with $\mu < r$. Let H be the hyperplane containing S_{μ} and $2r-1-\mu$ points P_i of V_2^{2r-1} and then also the $2r-1-\mu$ generatrix lines g_{P_i}. In such a case H would meet V_2^{2r-1} in a curve of order $r + 2r - 1 - \mu > 2r-1$, a contradiction. Hence each curve $\mathcal{C}$ irreducible of order r, lives in an *r*-space S and it is a directrix curve, that is, meets each generatrix in one point (cf. [7], 3. p. 287). If such an *r*-space S met S_{r-1}^{∞}, then a hyperplane $H \supseteq \langle S, S_{r-1}^{\infty} \rangle$ would contain V_2^{2r-1}, a contradiction. Hence $S \cap S_{r-1}^{\infty} = \varnothing$.

Assume $\mathcal{C}$ is reducible. The unique possibility is that it consists of r generatrix lines. Let H and H' be two hyperplanes. Assume H has in common with V_2^{2r-1} r generatrices and H' has in common with V_2^{2r-1} other r generatrices. Denote $S_{2r-2} = H \cap H'$. By varying the hyperplanes in the bundle (S_{2r-2}) of hyperplanes, both each hyperplane and the space S_{2r-2} itself would have in common with V_2^{2r-1} the locus of all these points. Such a locus would be a directrix contained in all the hyperplanes of the bundle. Therefore such a directrix curve should exist in all the hyperplanes of the bundle, a contradiction to Proposition 4.3, e) (cf. [7], 6. p. 290).

Proposition 4.5 a) *Each directrix curve of order* r *is obtained by cutting* V_2^{2r-1} *with the hyperplanes through any* $r-1$ *generatrices.*

b) *The cardinalities of the intersections of hyperplanes* H *with* V_2^{2r-1} *are* $q+1, rq+1, (r+1)q+1$. *It is* $\max\{|H \cap V_2^{2r-1}| : H \text{ hyperplane}\} = (r+1)q+1$.

c) *The cardinalities of the intersections of hyperplanes* H *with* $V_2^{2r-1} \setminus \mathcal{C}_{\infty}^{r-1}$ *are* $\leq q+1, r(q-1)+2, rq$. *It is* $\max\{|H \cap V_2^{2r-1} \setminus \mathcal{C}_{\infty}^{r-1}| : H \text{ hyperplane}\} = rq$.

Proof. a) An irreducible curve $\mathcal{C}$ of order r is a rational normal curve, that is, it lies in an r-space (cf. Proposition 4.4).

Let $\mathcal{C} \subset V_2^{2r-1}$ be a directrix curve of order r and $H \supset \mathcal{C}$ a hyperplane. As H cannot contain $\mathcal{C}_{\infty}^{r-1}$ otherwise $H \supset V_2^{2r-1}$, then H must contain $r-1$

generatrix lines.

b) Let H be a hyperplane. If $H\cap V_2^{2r-1}$ is an irreducible curve of order $2r-1$, then $\left|H\cap V_2^{2r-1}\right|=q+1$.

If $H\cap V_2^{2r-1}=\left\{g_1,\cdots,g_{r-1},\mathcal{C}^r\right\}$, $g_i\in\mathcal{G}$ (cf. a)), then $\left|H\cap V_2^{2r-1}\right|=rq+1$.

If $H\cap V_2^{2r-1}=\left\{g_1,\cdots,g_r,\mathcal{C}_\infty^{r-1}\right\}$, $g_i\in\mathcal{G}$ (cf. d), Proposition 4.3), then $\left|H\cap V_2^{2r-1}\right|=(r+1)q+1$.

That is we get the following possibilities: $q+1, rq+1, (r+1)q+1$.

It is easy to prove that $(r+1)q+1=\max\left\{\left|H\cap V_2^{2r-1}\right|: H \text{ hyperplane}\right\}$.

c) Let H be a hyperplane. If $H\cap V_2^{2r-1}$ is an irreducible curve of order $2r-1$, then $\left|H\cap V_2^{2r-1}\setminus\mathcal{C}_\infty^{r-1}\right|\le q+1$, depending on whether it has or does have not points on Σ'.

If $H\cap V_2^{2r-1}=\left\{g_1,\cdots,g_{r-1},\mathcal{C}^r\right\}$, $g_i\in\mathcal{G}$ (cf. a)), then $\left|H\cap V_2^{2r-1}\setminus\mathcal{C}_\infty^{r-1}\right|=(q-1)(r-1)+q+1=r(q-1)+2$.

If $H\cap V_2^{2r-1}=\left\{g_1,\cdots,g_r,\mathcal{C}_\infty^{r-1}\right\}$, $g_i\in\mathcal{G}$ (cf. d), Proposition 4.3), then $\left|H\cap V_2^{2r-1}\setminus\mathcal{C}_\infty^{r-1}\right|=rq$.

That is, we get the following possibilities: $\le q+1, r(q-1)+2, rq$.

It is easy to prove that $rq=\max\left\{\left|H\cap V_2^{2r-1}\setminus\mathcal{C}_\infty^{r-1}\right|: H \text{ hyperplane}\right\}$.

Proposition 4.6 a) *No two directrix curves* $\mathcal{C}$ *and* $\mathcal{C}'$ *of order* r *belong to a same r-space.*

b) *Two directrix curves of order* r *meet in one point.*

Proof. a) Assume $\mathcal{C}$ and $\mathcal{C}'$ belong to a same *r*-space S. Then a hyperplane $H\supset S$ would meet V_2^{2r-1} in a curve of order at least $2r$, a contradiction.

b) Let $\mathcal{C}$ and $\mathcal{C}'$ contained in two different *r*-spaces, S and S', respectively. Assume $\mathcal{C}\cap\mathcal{C}'$ contains two different points, P and Q. Then $S\cap S'\supset PQ$ so that the hyperplane $H=\langle S,S'\rangle$ meets V_2^{2r-1} in a curve of order $2r$, a contradiction. If $\mathcal{C}\cap\mathcal{C}'=\varnothing$, then by connecting corresponding points, V_2^{2r-1} would contain a variety of order $2r$, a contradiction.

4.1. Bundles of Curves of Order r on V_2^{2r-1} and a Non-Affine Subplane

Choose two $(r-1)$-spaces $S_{r-1}^\infty, S_{r-1}^0\in\mathcal{S}$ and an *r*-space S_r^0 of $\Sigma\setminus\Sigma'$ through S_{r-1}^0.

Fix the minimum order directrix $\mathcal{C}_\infty^{r-1}$ in S_{r-1}^∞ and in S_r^0 the curve $\mathcal{C}_0^r$ as an *r*-directrix so that $\mathcal{C}_0^r\cap S_{r-1}^0=\varnothing$.

Represent $\mathcal{C}_\infty^{r-1}=\left\{O',G_{i_\infty}\mid i=1,\cdots,q\right\}$, $\mathcal{C}_0^r=\left\{O,G_i\mid i=1,\cdots,q\right\}$. The two curves are referred through the projectivity $\Lambda:\mathcal{C}_\infty^{r-1}\to\mathcal{C}_0^r$ such that $\Lambda(O')=O$, $\Lambda\left(G_{i_\infty}\right)=G_i$.

The variety V_2^{2r-1} arises by connecting the points of $\mathcal{C}_\infty^{r-1}$ and $\mathcal{C}_0^r$ that correspond through Λ (cf. [7], p. 291). Denote g_0 the generatrix line OO' where $O=g_0\cap\mathcal{C}_0^r$, $O'=g_0\cap\mathcal{C}_\infty^{r-1}$.

The set $\mathcal{G}=\left\{g_0,g_i=G_iG_{i_\infty}\mid i=1,\cdots,q\right\}$ of the generatrix lines of V_2^{2r-1} partitions the variety.

In a suitable complexification of Σ, each hyperplane H meets V_2^{2r-1} in a curve of order $2r-1$ (cf. [7], p. 288, 5.).

Choose a generatrix $g_1 \neq g_0$ and a point $P_1 \in g_1$, $P_1 \neq \{\mathcal{C}_\infty^{r-1} \cap g_1, \mathcal{C}_0^r \cap g_1\}$.

Set $OP_1 \cap \Sigma' = P_1^\infty$. Denote S_{r-1}^1 the $(r-1)$-space of the spread $\mathcal{S}$ to which P_1^∞ belongs. For the choices we made follows $S_{r-1}^1 \neq S_{r-1}^0$ so that if $S_r^1 = \langle OP_1, S_{r-1}^1 \rangle$, then $S_r^1 \neq S_r^0$. If we project from O the line g_1 by constructing the $q-1$ lines $\{OP_1 \mid P_1 \in g_1 \setminus \{g_1 \cap \mathcal{C}_0^r, g_1 \cap \mathcal{C}_\infty^{r-1}\}\}$, we get the plane $\pi = \langle O, g_1 \rangle$ such that $t = \pi \cap \Sigma'$ is a transversal to the three subspaces $S_{r-1}^\infty, S_{r-1}^0, S_{r-1}^1$.

Therefore the line t is a transversal line to the whole regulus $\mathcal{R} \subset \mathcal{S}$ defined by $\{S_{r-1}^\infty, S_{r-1}^0, S_{r-1}^1\}$.

By varying the point $P_1 \in g_1$ a set of $q-1$ r-spaces S_r^i through the point O are generated in addition to S_r^0. Represent such a bundle $\mathcal{B}_r^O = \{S_r^{O_i} \mid i = 0, \cdots, q-1\}$.

Moreover, for each $P \in g_0 \setminus \{O\}$, we can repeat the same procedure to obtain a bundle $\mathcal{B}_r^P = \{S_r^{P_i} \mid i = 0, \cdots, q-1\}$.

Generality is not loss if we start by choosing two generatrix lines $\{g_0', g_1'\} \neq \{g_0, g_1\}$.

Theorem 4.7 a) *Through each point* $P \in g_0 = OO' \setminus \{O'\}$ *there exists a bundle* C_P *of* q *curves of order* r *on* V_2^{2r-1} *having the point* P *in common, each curve of* C_P *lying in one r-space intersecting an* $(r-1)$*-space of* $\mathcal{R} \setminus S_{r-1}^\infty$. *Each bundle covers the* q^2 *points of* $V_2^{2r-1} \setminus OO'$.

b) *The cardinality of the set* $\mathbf{C} = \{\mathrm{C}_P \mid P \in g_0 = OO' \setminus \{O'\}\}$ *is* q^2.

c) $\mathbf{C}$ *is the whole set of the directrix curves of order* r *of* V_2^{2r-1}.

Proof. a) For each r-space $S_r^{O_i} \in \mathcal{B}_r^O$, denote H_i the hyperplane containing $S_r^{O_i}$ and a set R_i of $r-1$ generatrix lines with $g_0, g_1 \in R_i$. From Propositions 4.4 and 4.5 follows that $S_r^{O_i}$ must contain a directrix curve $\mathcal{C}_i^r$ of order r.

For construction each curve $\mathcal{C}_i^r$ has no points in $S_{r-1}^i = S_r^i \cap \Sigma'$. Denote C_O the bundle of all $\mathcal{C}_i^r$. From Proposition 4.6, a) follows that such q pairwise curves have only the point O in common.

The bundle C_O consists of q curves, each curve collecting q points of $V_2^{2r-1} \setminus \{O\}$ hence C_O covers $q \cdot q = q^2$ points of $V_2^{2r-1} \setminus OO'$.

In a completely similar way for each $P \in OO' \setminus \{O\}$ we get the same result for C_P.

b) The cardinality of $\{\mathrm{C}_P \mid P \in OO' \setminus \{O'\}\}$ is q^2 as for each point $P \in OO' \setminus \{O'\}$ it is $|\mathrm{C}_P| = q$ and the points of $OO' \setminus \{O'\}$ are q.

c) Let $\mathcal{C}$ be a directrix curve of order r. As $\mathcal{C}$ meets each generatrix line, if $P = \mathcal{C} \cap g_0$ then $\mathcal{C} \in \mathrm{C}_P$.

Denote $\mathcal{V}'$ the set of the affine points of V_2^{2r-1}. Represent $\Pi = PG(2, q^r)$ as in Section 3, Lemma 3.1.

Let $\Pi_q = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ be the incidence substructure of Π defined as follows:

$$\mathcal{P}' = \{P \in \mathcal{V}'\} \cup \{S_{r-1}^\infty\},$$

$$\mathcal{L}' = \{\mathcal{C} \in \mathrm{C}_P \mid P \in OO' \setminus \{O'\}\} \cup \mathcal{G},$$

$\mathcal{I}'$ is defined as follows

$\mathcal{I}' = \mathcal{I}$ restricted to the affine points and lines, $S_{r-1}^{\infty}\mathcal{I}'g$ for all $g \in \mathcal{G}$.

Theorem 4.8 Π_q *is a non-affine subplane of* Π *of order* q.

Proof. It is known (cf. [13] [14] pp. 160-161 and [5] pp. 40-41) that if in an incidence structure the following four properties hold

$$\begin{pmatrix} 1 & 3 & - \\ 2 & - & 6^2 \end{pmatrix}$$

where

1—the number of the points is q^2+q+1,

2—the number of the lines is q^2+q+1,

3—each line contains $q+1$ points,

6^2—two lines meet in at most one point,

then the structure is a projective plane of order q.

From Proposition 4.3 follows that the cardinality of the affine points of $\mathcal{V}_2^{2r-1}$ is q^2+q to which the *point at infinity* S_{r-1}^{∞} has to be added. Hence $|\mathcal{P}'| = q^2+q+1$, that is, 1 - holds.

From Theorem 4.7 follows $\mathbf{C} = \{\mathrm{C}_P \mid P \in OO' \setminus \{O'\}\}$ is q^2. As $|\mathcal{G}| = q+1$ then $|\mathcal{L}'| = q^2+q+1$, that is, 2 - holds.

Each curve of C_P has as many points as $\mathcal{C}_0^r$ has, that is $q+1$. Each generatrix line $g \in \mathcal{G}$ has q affine points and the point ad infinity S_{r-1}^{∞}, hence 3 - holds.

From Proposition 4.6 follows that two curves of order r meet in one point. Each such a curve is a directrix so that meets each generatrix line in one point. Two generatrix lines meet only in the point S_{r-1}^{∞}. Hence 6^2, really 6 holds.

To verify that Π_q is a subgeometry of Π (cf. Definition 3.2), note first that its set of points is clearly a subset of the points of Π. Moreover, every line $g \in \mathcal{G}$ is contained in a unique 3-space $S = \langle g, \pi_{\infty} \rangle$ which meets no other generatrix (cf. Proposition 4.3, (c)) and every cubic of C_P lies in a unique *r*-space (cf. Proposition 4.6, (a)) meeting Σ' in an $(r-1)$-space of $\mathcal{R}$ (cf. Theorem 4.7, (a)).

The properties of Π_q of being a plane can be translated into further incidence properties of V_2^{2r-1}.

Corollary 4.9 *Let* P,Q *be two points of* V_2^{2r-1}. *If* $P \in \mathcal{V}'$ *and* $Q = S_{r-1}^{\infty}$ *then the line* PQ *of* Π_q *is the generatrix* g_P, *if* $P,Q \in \mathcal{V}'$ *then* P,Q *belong to one directrix curve of order* r *of an r-space* S *with* $S \cap \Sigma' \in \mathcal{R} \setminus S_{r-1}^{\infty}$.

4.2. An Example

Denote $F = GF(q)$, $\Sigma = PG(2r,q)$. Let $\Sigma' = PG(2r-1,q)$ be a hyperplane of Σ, $\mathcal{S}$ a regular spread of Σ'.

Let $(\mathbf{x},\mathbf{y},t) = (x_1,\cdots,x_r,y_1,\cdots,y_r,t)$ in Σ be a coordinate system so that $t=0$ represents Σ', $(\mathbf{x},\mathbf{y})$ are *internal coordinates* for Σ' and for a point

$P \in \Sigma \setminus \Sigma'$, $P \approx (\mathbf{x},\mathbf{y},t) = (x_1,\cdots,x_r,y_1,\cdots,y_r,t) = F^*(\mathbf{x},\mathbf{y},t)$, $F^* = F \setminus \{0\}$.

Represent the spread $\mathcal{S}$ as follows

$$\mathcal{S} = \left\{S_{r-1}^{\infty} = (\mathbf{0},\mathbf{y}) \mid \mathbf{y} \in F^r\right\} \cup \left\{S_{r-1}^{\mathbf{m}} = (\mathbf{x},\mathbf{xm}) \mid \mathbf{x},\mathbf{m} \in F^r\right\}$$

where $\mathbf{y} = \mathbf{xm}$ is the multiplication in the field F^r, $\mathbf{xm} = \mathbf{x}M$ with M a $r \times r$ matrix over F. The set $\mathcal{M} = \{M \mid xM = \boldsymbol{xm}\}$ is a field isomorphic to $(F^r)^2$, strictly transitive over F^r.

Denote $\mathcal{R} = \left\{S_{r-1}^{\infty}, S_{r-1}^{k} \mid k \in F\right\}$ the regulus of $\mathcal{S}$ represented by the scalar matrices kI.

Let $f_r(x) \in GF(q)[x]$ be an irreducible polynomial of degree r (cf. [11] [12]). For instance, more explicitly, choose q and r such that $d = gcd(q-1,r) \neq 1$. From Lemma 4.1 follows that in $F = GF(q)$ there is a subset N_r of $\dfrac{(q-1)(d-1)}{d}$ *non-r-th powers* elements so that the polynomials $x^r - s$ are irreducible whenever $s \in N_r$.

Choose and fix the irreducible curve $\mathcal{C}_{\infty}^{r-1}$ of order $r-1$ in the space S_{r-1}^{∞} and the irreducible curve $\mathcal{C}_0^r$ of order r in the *r*-space S_r^0 of $\Sigma \setminus \Sigma'$ through S_{r-1}^0 so that $\mathcal{C}_0^r \cap S_{r-1}^0 = \varnothing$, $\mathcal{C}_{\infty}^{r-1}$ and $\mathcal{C}_0^r$ represented as follows

$$\mathcal{C}_{\infty}^{r-1} = \left\{\left(0,\cdots,0,1,\lambda,\cdots,\lambda^{r-1},0\right) \mid \lambda \in GF(q)\right\} \cup \left\{O' = (0,\cdots,0,0,\cdots,1,0)\right\}$$

$$\mathcal{C}_0^r = \left\{\left(1,\lambda,\cdots,\lambda^{r-1},0,\cdots,0,f_r(\lambda)\right) \mid \lambda \in GF(q)\right\} \cup \left\{O = (0,\cdots,0,0,\cdots,0,1)\right\},$$

where $f_r(\lambda)$ is an irreducible polynomial of degree r.

The two curves are referred through a projectivity $\Lambda : \mathcal{C}_{\infty}^{r-1} \to \mathcal{C}_0^r$ represented by having inserted the same parameter λ for which it is agreed that the points are considered corresponding to each other, plus $\Lambda(O') = O$.

If $\mathcal{C}_{\infty}^{r-1} = \left\{O', G_{i_\infty} \mid i = 1,\cdots,q\right\}$ then $\mathcal{C}_0^r = \left\{O, G_i = \Lambda G_{i_\infty} \mid i = 1,\cdots,q\right\}$. The variety V_2^{2r-1} arises by connecting the corresponding points of $\mathcal{C}_{\infty}^{r-1}$ and $\mathcal{C}_0^r$ (cf. [7], p. 291). The curves $\mathcal{C}_{\infty}^{r-1}$ and $\mathcal{C}_0^r$ are directrix curves of V_2^{2r-1}, the set $\mathcal{G} = \left\{g_0 = OO', g_i = G_i G_{i_\infty} \mid i = 1,\cdots,q\right\}$ of the generatrix lines of V_2^{2r-1} partitions the variety.

Consider the following $2r \times 2r$ matrix in $r \times r$ blocks

$$M_{\varphi} = \left(\begin{array}{c|c} I & kI \\ \hline 0 & I \end{array}\right).$$

Denote φ the affinity of Σ represented by M_{φ}

The extended projectivity $\bar{\varphi}$ is represented by the $(2r+1)\times(2r+1)$ matrix $M_{\bar{\varphi}}$ obtained from M_{φ} by adding the vector $(0,\cdots,0,0,\cdots,0,1)$ as the $(2r+1)$ th column and the $(2r+1)$ th row.

Theorem 4.10 a) *Through each point* $P \in OO' \setminus \{O'\}$ *there exists a bundle* C_P *of* q *curves of order* r *on* V_2^{2r-1} *having the point* P *in common, each curve lying in one r-space intersecting an* $(r-1)$ *-space of* $\mathcal{R} \setminus S_{r-1}^{\infty}$. *Each bundle cover the* q^2 *points of* $V_2^{2r-1} \setminus OO'$.

b) *The cardinality of the set* $\mathbf{C} = \{C_P \mid P \in OO' \setminus \{O'\}\}$ *is* q^2.

c) $\mathbf{C}$ *is the whole set of the directrix curves of order* r *of* V_2^{2r-1}.

Proof. a) For each point $A=(\mathbf{0},\mathbf{a},0)=(0,\cdots,0,a_1,\cdots,a_r,0)\in S_{r-1}^{\infty}$ it is $\bar{\varphi}(A):=(A)M_{\bar{\varphi}}=A$, that is, S_{r-1}^{∞} is pointwise fixed. For each point $B=(\mathbf{b},\mathbf{0},0)=(b_1,\cdots,b_r,0,\cdots,0,0)\in S_{r-1}^{0}$ it is $\bar{\varphi}(B):=(B)M_{\bar{\varphi}}=(\mathbf{b},k\mathbf{b},0)$, that is, $\bar{\varphi}(S_{r-1}^{0})=S_{r-1}^{k}$, and $\bar{\varphi}(O)=O$. Hence $\bar{\varphi}(S_{r-1}^{0})$ is an *r*-space S_r^k through O with $S_r^k\cap\Sigma'=S_{r-1}^{k}$. The curve $\mathcal{C}_0^r\subset S_r^0$ of order r is mapped onto an *r*-curve $\mathcal{C}_k^r\subset S_r^k$ with $O\in\mathcal{C}_k^r$ and $\mathcal{C}_k^r\cap S_{r-1}^{k}=\varnothing$. Therefore there exists a bundle C_0 of q curves of order r through O collecting the q^2 points of $V_2^{2r-1}\setminus OO'$.

Let $P=(0,\cdots,0,0,\cdots,h,1)$ be a point of $OO'\setminus\{O,O'\}$ and denote τ_h the associated translation. Therefore $\tau_h(O)=P$ and $\tau_h(\mathrm{C}_0)=\mathrm{C}_P$.

b) The cardinality of $\{\mathrm{C}_P\mid P\in OO'\setminus\{O'\}\}$ is q^2 as for each point $P\in OO'\setminus\{O'\}$ it is $|\mathrm{C}_P|=q$ and the points of $OO'\setminus\{O'\}$ are q.

c) For the proof see (c), Theorem 4.7.

Corollary 4.11 *Chosen and fixed* $S_{r-1}^{\infty},S_{r-1}^{0}\in\mathcal{S}$, *the variety* V_2^{2r-1} *selects in the spread* $\mathcal{S}$ *a regulus to which* S_{r-1}^{∞} *and* S_{r-1}^{0} *belong.*

Let $\Pi=PG(2,q^r)$ be the projective plane over $GF(q^r)$. Represent Π in Σ, $\Pi=(\mathcal{P},\mathcal{L},\mathcal{I})$ as in Lemma 3.1.

Let $\mathcal{V}'$ be the set of the q^2 affine points of V_2^{2r-1}. Define $\Pi_q=(\mathcal{P}',\mathcal{L}',\mathcal{I}')$ as in the previous Section 4.1.

It is immediate to prove the following results, analogous to Theorem 4.8 and Corollary 4.9, respectively.

RESULT 1— Π_q is a non-affine subplane of Π of order q.

RESULT 2—Let P,Q be two points of V_2^{2r-1}. If $P\in\mathcal{V}'$ and $Q=S_{r-1}^{\infty}$ then the line PQ of Π_q is the generatrix g_P, if $P,Q\in\mathcal{V}'$ then P,Q belong to one directrix curve of order r of an *r*-space S with $S\cap\Sigma'\in\mathcal{R}\setminus S_{r-1}^{\infty}$.

5. Codes Related to V_2^{2r-1}

To construct linear codes starting from V_2^{2r-1} and bundles of them, first we must associate projective systems and calculate their number of points. Then it needs to calculate the cardinalities of intersection with the hyperplanes to find the distance and the error-correcting capability.

Let $\mathcal{X}=V_2^{2r-1}$, $\mathcal{X}'=V_2^{2r-1}\setminus\mathcal{C}_{\infty}^{r-1}$ be projective systems defined by V_2^{2r-1}. It is $|\mathcal{X}|=q^2+2q+1$ and $|\mathcal{X}'|=q^2+q$. Denote $C_{\mathcal{X}}$ and $C'_{\mathcal{X}}$ the codes associated to them.

From Definitions 2.3, 2.4 and Proposition 4.5 follows

Proposition 5.1 $C_{\mathcal{X}}$ *is an* $[n,k,d]_q$ *-code with* $n=q^2+2q+1$, $k=2r+1$, $d=q^2-(r-1)q$.

$C'_{\mathcal{X}}$ *is an* $[n',k,d']_q$ *-code with* $n'=q^2+q$, $k=2r+1$, $d'=q^2-(r-1)q$.

Proof.

The distance of a code related to a projective system equals the number of the points of the system minus its max intersection with hyperplanes. Hence, from Proposition 4.5 follows that the minimum distance for $C_{\mathcal{X}}$ equals

$q^2+2q+1-((r+1)q+1)=q^2-(r-1)q$, for $C'_{\mathcal{X}}$ equals $q^2+q-rq=q^2-(r-1)q$. Then both codes have same dimension and minimum distance which is the better the smaller r is. In any case the code $C_{\mathcal{X}'}$ seems to be better than $C_{\mathcal{X}}$ because $n>n'$.

In Section 4.1 is shown that the variety V_2^{2r-1} selects the regulus $\mathcal{R}$ to which both $S_{r-1}^{\infty}, S_{r-1}^{0}$ belong. Denote $S_{r-1}^{\infty}:=S_{1_\infty}$, $\mathcal{C}_{\infty}^{r-1}:=\mathcal{C}_{1_\infty}$, $\mathcal{R}:=\mathcal{R}_1$. Fix the directrix curve $\mathcal{C}_0^r$ of order r in S_r^0.

Theorem 5.2 *There exists a bundle* $\mathcal{B}$ *of varieties* V_2^{2r-1} *with the curve* $\mathcal{C}_0^r$ *as directrix,* $|\mathcal{B}|=q^{r-1}$, *any two varieties having in common no element of the spread* $\mathcal{S}$.

Proof. It involves choosing step by step an $(r-1)$-space of the spread $\mathcal{S}$ outside the regulus identified by the variety of the previous step, and, in this, a directrix curve of order $r-1$.

Step 1—Construct the variety $\mathcal{V}_1=V_2^{2r-1}$ starting from the curve $\mathcal{C}_{1_\infty}^{r-1}$ and the curve $\mathcal{C}_0^r\subset S_0$. In $\mathcal{S}\setminus\mathcal{R}_1$ there are q^r-q possible choices for the next step.

Step 2—Choose an $(r-1)$-space $S_{2_\infty}\in\mathcal{S}\setminus\mathcal{R}_1$. Fix a curve $\mathcal{C}_{2_\infty}^{r-1}$ in it and construct the variety $\mathcal{V}_2=V_2^{2r-1}$ starting from $\mathcal{C}_{2_\infty}^{r-1}$ and the curve $\mathcal{C}_0^r\subset S_r^0$. Let $\mathcal{R}_2$ be the regulus of $\mathcal{S}$ to which S_{2_∞} and S_{r-1}^0 belong. In $\mathcal{S}\setminus\{\mathcal{R}_1,\mathcal{R}_2\}$ there are q^r-2q possible choices for the next step.

Step 3—Choose an $(r-1)$-space $S_{3_\infty}\in\mathcal{S}\setminus\{\mathcal{R}_1,\mathcal{R}_2\}$. Fix a curve $\mathcal{C}_{3_\infty}^{r-1}$ in it of order $r-1$ and construct the variety $\mathcal{V}_3=V_2^{2r-1}$ starting from $\mathcal{C}_{3_\infty}^{r-1}$ and the curve $\mathcal{C}_0^r\subset S_r^0$. Let $\mathcal{R}_3$ be the regulus of $\mathcal{S}$ to which S_{3_∞} and S_{r-1}^0 belong. In $\mathcal{S}\setminus\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3\}$ there are q^r-3q possible choices for the next step. And so on.

The procedure ends evidently at the q^{r-1}-th step. Therefore $\mathcal{B}=\{\mathcal{V}_i \mid i=1,2,\cdots,q^{r-1}\}$ and $|\mathcal{B}|=q^{r-1}$.

Each variety of $\mathcal{B}$ represents a non-affine subplane of $\Pi=PG(2,q^r)$ and by construction follows that two such subplanes have in common the subline represented by $\mathcal{C}_0^r$ and no infinite point.

Denote $\mathcal{V}_{\mathcal{B}}=\{P\in\bigcup\mathcal{V}_i \mid \mathcal{V}_i\in\mathcal{B}\}$, $\mathcal{V}_i'=\mathcal{V}_i\setminus\mathcal{C}_{i\infty}^{r-1}$ and $\mathcal{V}_{\mathcal{B}}'=\{P\in\bigcup\mathcal{V}_i' \mid \mathcal{V}_i\in\mathcal{B}\}$, $i=1,2,\cdots,q^{r-1}$.

Theorem 5.3 *Any two varieties of* $\mathcal{B}$ *have in common only* $\mathcal{C}_0^r$.

The set $\mathcal{V}_{\mathcal{B}}$ *has cardinality* $q^{r+1}+q^r+q+1$.

The set $\mathcal{V}_{\mathcal{B}}'$ *has cardinality* $q^{r+1}-q^{r-1}+q+1$.

Proof.

Assume two varieties $\mathcal{V}_1,\mathcal{V}_2\in\mathcal{B}$, have in common, in addition to $\mathcal{C}_0^r$, a point $P\notin\mathcal{C}_0^r$. Among the $q+1$ lines $\{PC \mid C\in C_0^r\}$ there are the two generatrix lines, $g_i\in\mathcal{V}_i$, $i=1,2$, $PC_1=g_1$, $PC_2=g_2$, g_1 defining the $(r-1)$-space S_{1_∞} and g_2 the $(r-1)$-space S_{2_∞} of the spread $\mathcal{S}$.

Choose a point $C\in C_0^r\setminus\{C_1,C_2\}$. From Corollary 4.9 follows that through the points P and C is defined one directrix curve of order r of an *r*-space S_1 of the variety $\mathcal{V}_1$ with $S_1\cap\Sigma'\in\mathcal{S}$ and one directrix curve of order r of an *r*-space S_2 of the variety $\mathcal{V}_2$ with $S_2\cap\Sigma'\in\mathcal{S}$.

On the other hand, by considering the points P,C as points of Π, the line PC selects in Σ' an $(r-1)$-space $S'\in\mathcal{S}$ so that the r-space $\langle PC,S'\rangle$ represents the line of Π through P and C. This implies that $S_1\cap\Sigma'=S_2\cap\Sigma'=S'$, that is, the subplanes represented by $\mathcal{V}_1$ and $\mathcal{V}_2$ would have in common the infinite point represented by S', a contradiction to Theorem 5.2.

For each variety $\mathcal{V}_i\in\mathcal{V}_{\mathcal{B}}$, $\mathcal{V}_i\setminus\mathcal{C}_0^r$ consists of $(q+1)^2-(q+1)=q^2+q$ points so that $|\mathcal{V}_{\mathcal{B}}|=q^{r-1}(q^2+q)+q+1=q^{r+1}+q^r+q+1$.

For each variety $\mathcal{V}_i'\in\mathcal{V}_{\mathcal{B}}'$, $|\mathcal{V}_i'|=q^2+2q+1-(q+1)=q^2+q$ so that $\mathcal{V}_i'\setminus\mathcal{C}_0^r$ consists of $q^2+q-(q+1)=q^2-1$ points and
$|\mathcal{V}_{\mathcal{B}}'|=q^{r-1}(q^2-1)+(q+1)=q^{r+1}-q^{r-1}+q+1$.

Theorem 5.4 *The cardinalities of the intersections of hyperplanes with* $\mathcal{V}_{\mathcal{B}}$ *are*:

a) $=q^r+q^{r-1}$,

b$_1$) $\le(r-1)q^r+2q+1$,

b$_2$) $\le(r-1)q^r+r$,

b$_3$) $=(r-1)q^r+q+1$

and $\max\{|H\cap\mathcal{V}_{\mathcal{B}}|:H\text{ hyperplane}\}\le(r-1)q^r+2q+1$.

The cardinalities of the intersections of hyperplanes with $\mathcal{V}_{\mathcal{B}}'$ *are*:

b$_1'$) $\le(r-1)q^r+q$,

b$_2'$) $\le(r-1)(q-1)q^{r-1}+r$,

b$_3'$) $=(r-1)(q-1)(q^{r-1}-1)+q+1$

and $\max\{|H\cap\mathcal{V}_{\mathcal{B}}'|:H\text{ hyperplane}\}=(r-1)(q-1)(q^{r-1}-1)+q+1$.

Proof.

a) Assume $H=\Sigma'$. By construction, H contains the q^{r-1} subspaces $\mathcal{S}_{i\infty}\in\mathcal{S}$, each of them containing the directrix curve $\mathcal{C}_{i_\infty}^{r-1}$, one for each of the q^{r-1} varieties of $\mathcal{B}$. Then $|H\cap\mathcal{V}_{\mathcal{B}}|=q^{r-1}(q+1)=q^r+q^{r-1}$.

b) Let H be a hyperplane $H\ne\Sigma'$.

b$_1$) Assume H contains an $(r-1)$-space $\mathcal{S}_{i_\infty}$ for some i so that it contains $\mathcal{C}_{i_\infty}^{r-1}$. Of course H cannot contain any other $(r-1)$-space of the spread being $H\ne\Sigma'$. From Proposition 4.3, *d*), follows that H contains also a set of r generatrix lines meeting $\mathcal{C}_0^r$ in a subset I of r points.

Hence H contains at most these $(q+1)+qr$ points.

As $H\cap\Sigma'=S_{2r-2}$, then H meets each of the $q^{r-1}-1$ subspaces $S_{j_\infty}\in\mathcal{S}\setminus\mathcal{S}_{i_\infty}$ (with directrix curves $\mathcal{C}_{j_\infty}^{r-1}$ of $\mathcal{V}_j\in\mathcal{B}$ for every $j\ne i$) in an S_{r-2}. Such a space can meet each curve $\mathcal{C}_{j_\infty}^{r-1}$, $j\ne i$, in at most $r-1$ points (cf. NOTE 1), that is, in total $(r-1)(q^{r-1}-1)$ points. The hyperplane H could contain $r-1$ generatrix lines through those points for each of the $(q^{r-1}-1)$ varieties $\mathcal{V}_j\ne\mathcal{V}_i$, cutting the directrix $\mathcal{C}_0^r$ in subsets of I otherwise H would contain the whole variety $\mathcal{V}_i$. That is we must add at most further $(r-1)q(q^{r-1}-1)$ points. Then H contains at most
$(r-1)(q^{r-1}-1)+(r-1)q(q^{r-1}-1)=(r-1)(q+1)(q^{r-1}-1)$ points.

Summarizing, as
$(q+1)+qr+(r-1)q(q^{r-1}-1)=(r-1)(q^r-q)+(r+1)q+1=(r-1)q^r+2q+1$,

then we get $|H \cap \mathcal{V}_{\mathcal{B}}| \le (r-1)q^r + 2q + 1$.

b$_2$) Assume H contains no $(r-1)$-space $\mathcal{S}_{i_\infty}$ for every $i = 1,\cdots,q^{r-1}$. As $H \cap \Sigma'$ is a subspace $S = S_{2r-2}$, then $S \cap \mathcal{S}_{i_\infty}$ for every i is an $(r-2)$-space $S_i = S_{r-2}$ which meets $\mathcal{S}_{i_\infty}$ in at most $r-1$ points (cf. NOTE 1). These points are at most $(r-1)q^{r-1}$.

Apart of the points on $\mathcal{C}_0^r$, for each $\mathcal{S}_{i_\infty}$ the $r-1$ generatrix lines contain $(r-1)(q-1)$ points, that is, in total $(r-1)(q-1)q^{r-1}$. To this number at most r points of $\mathcal{C}_0^r$ have to be added.

Summarizing, as $(r-1)q^{r-1} + (r-1)(q-1)q^{r-1} + r = (r-1)q^r + r$, then $|H \cap \mathcal{V}_{\mathcal{B}}| \le (r-1)q^r + r$.

b$_3$) Assume H contains S_r^0 and therefore $\mathcal{C}_0^r$, that is $q+1$ points of $\mathcal{V}_{\mathcal{B}}$. In such a case H contains $r-1$ generatrices for every variety $\mathcal{V}_i$, that is $(r-1)q \cdot q^{r-1} = (r-1)q^r$. Summarizing we get $|H \cap \mathcal{V}_{\mathcal{B}}| = (r-1)q^r + q + 1$.

It is easy to prove the following inequalities hold:
$(r-1)q^r + 2q + 1 > (r-1)q^r + q + 1 > (r-1)q^r + r$ as $q \ge r$ (cf. NOTE 1), moreover $q^r + q^{r-1} < (r-1)q^r + 2q + 1$, that is,
$\max\{|H \cap \mathcal{V}_{\mathcal{B}}| : H \text{ hyperplane}\} \le (r-1)q^r + 2q + 1 =$.

To calculate the intersections of hyperplanes with $\mathcal{V}'_{\mathcal{B}}$, all those relating to Σ' must be subtracted from the cardinalities calculated for $\mathcal{V}_{\mathcal{B}}$.

Let H be a hyperplane $H \ne \Sigma'$.

b$'_1$) From b$_1$) we get $|H \cap \mathcal{V}'_{\mathcal{B}}| \le (r-1)q^r + 2q + 1 - (q+1) = (r-1)q^r + q$.

b$'_2$) From b$_2$) we get
$|H \cap \mathcal{V}'_{\mathcal{B}}| \le (r-1)q^{r-1} + (r-1)(q-1)q^{r-1} + r - ((r-1)q^{r-1}) = (r-1)(q-1)q^{r-1} + r$.

b$'_3$) In b$_3$) the hyperplane H contains $r-1$ generatrix lines for each variety $\mathcal{V}_i$, in this case equivalent to $(r-1)(q-1)(q^{r-1}-1)$ points. So that by adding the points of $\mathcal{C}_0^r$ we get $|H \cap \mathcal{V}'_{\mathcal{B}}| = (r-1)(q-1)(q^{r-1}-1) + q + 1$.

By comparing the three inequalities: 1) $(r-1)q^r + q$, 2) $(r-1)(q-1)q^{r-1} + r$, 3) $(r-1)(q-1)(q^{r-1}-1) + q + 1$ we get 1) > 2), 1) < 3), 3) > 2) we can say $\max\{|H \cap \mathcal{V}'_{\mathcal{B}}| : H \text{ hyperplane}\} = (r-1)(q-1)(q^{r-1}-1) + q + 1$.

Let $\mathcal{X} = \mathcal{V}_{\mathcal{B}}$, $\mathcal{X}' = \mathcal{V}'_{\mathcal{B}}$ be the projective systems defined by $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}'_{\mathcal{B}}$, respectively. It is $|\mathcal{X}| = q^{r+1} + q^r + q + 1$ and $|\mathcal{X}'| = q^{r+1} - q^r - q^{r-1} + q + 1$. Denote $C_{\mathcal{X}}$ and $C'_{\mathcal{X}}$ the codes associated to them.

From Theorem 5.3 and 5.4 follows

Theorem 5.5 $C_{\mathcal{X}}$ *is an* $[n,k,d]_q$ *-code with* $n = q^{r+1} + q^r + q + 1$, $k = 2r+1$, $d \ge q^{r+1} - (r-2)q^r - q$.

$C'_{\mathcal{X}}$ *is an* $[n',k,d']_q$ *-code with* $n' = q^{r+1} - q^{r-1} + q + 1$, $k = 2r+1$, $d' = q^{r+1} - (r-1)q^r - (r-2)q^{r-1} + (r-1)q - (r-1)$.

Proof. The distance of a code related to a projective system equals the number of the points of the system minus its max intersection with hyperplanes, so that we get $d \ge q^{r+1} + q^r + q + 1 - ((r-1)q^r + 2q + 1) = q^{r+1} - (r-2)q^r - q$ and

$$\begin{aligned} d' &= q^{r+1} - q^{r-1} + q + 1 - ((r-1)(q-1)(q^{r-1}-1) + q + 1) \\ &= q^{r+1} - (r-1)q^r - (r-2)q^{r-1} + (r-1)q - (r-1) \end{aligned}.$$

Given the same dimension $2r+1$, the code $C_{\mathcal{X}}$ has both greater length of codeword and greater distance than the code $C'_{\mathcal{X}}$, hence $C_{\mathcal{X}}$ is better than $C'_{\mathcal{X}}$, despite $C'_{\mathcal{X}}$ has a precise distance.

Example 5.6 *For minimum* $r=2$, *the code* $C_{\mathcal{X}}$ *is an* $[n,k,d]_q$ *-code with* $n=q^3+q^2+q+1$, $k=5$, $d \ge q^3-q$.

The code $C'_{\mathcal{X}}$ *is an* $[n',k,d']_q$ *-code with* $n'=q^3+1$, $k=5$, $d'=q^3-q^2+q-1$.

By comparing these two codes with those of Proposition 5.1 for $r=2$ *it is clear that the codes of Theorem 5.5 are better.*

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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