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Where Is Phase Velocity in Minkowski Space?

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Abstract

In the special theory of relativity, massive particles can travel at neither the speed of light c nor faster. Meanwhile, since the photon was quantized, many have thought of it as a point particle. How pointed? The idea could be a mathematical device or physical simplification. By contrast, the preceding notion of wave-group duality has two velocities: a group velocity v_g and a phase velocity v_p. In light $v_p = v_g = c$; but it follows from special relativity that, in massive particles, $v_p > c$. The phase velocity is the product of the two best measured variables, and so their product constitutes internal motion that travels, verifiably, faster than light. How does v_p then appear in Minkowski space? For *light*, the spatio-temporal Lorentz invariant metric is $s^2 = c^2t^2 - x^2 - y^2 - z^2$, the same in whatever frame it is viewed. The space is divided into 3 parts: firstly a cone, symmetric about the vertical axis $ct > 0$ that represents the world line of a stationary particle while the conical surface at $s = 0$ represents the locus for light rays that travel at the speed of light c. Since no real thing travels faster than the speed of light c, the surface is also a horizon for what can be seen by an observer starting from the origin at time $t = 0$. Secondly, an inverted cone represents, equivalently, time past. Thirdly, outside the cones, inaccessible space. The phase velocity v_p, group velocity v_g and speed of light are all equal in free space, $v_p = v_g = c$, constant. By contrast, for *particles,* where causality is due to *particle* interactions having rest mass $m_o > 0$, we have to employ the Klein-Gordon equation with $s^2 = c^2t^2 - x^2 - y^2 - z^2 + m_o^2c^2$. Now special relativity requires a complication: $v_p.v_g = c^2$ where $v_g < c$ and therefore $v_p > c$. In the volume outside the cones, causality due to light interactions cannot extend beyond the cones. However, since $v_p > c$ and even $v_p >> c$ when wavelength λ is long, extreme phase velocities are then limited in their causal effects by the particle uncertainty σ, i.e. to $v_g t \pm \sigma/\omega$, where ω is the particle angular frequency. This is the first time the phase range has been described for a massive particle.

Keywords

Event Horizon, Scattering Range, Wave Packet, Phase Velocity, Group Velocity, Dispersion Dynamics, Quantum Physics

1. Introduction

The extraordinary facts, that emerge within the properly explained diffraction in quasicrystals, support a redefinition of the quantum: it is the consequence of resonant harmonies in emission, scattering and measurement [1]-[3]. These special harmonies are geometric in long range, while being simultaneously linearly periodic in short range. The diffraction reveals a new Quasi-Bragg law. Its novel features have been used to explain quantum collapse [1] that is sometimes called the greatest outstanding problem in physics [4]. Partly in view of large effort currently directed to instrumentation for quantum computing and for elementary particle physics, the proper time has arrived to re-examine not only the quantum, but effects of internal motion.

Quantum mechanics has aroused dispute since its inception [5] [6]. For example, internal motion in the electron has always been neglected: Dirac gives it only the merest mention in his *Principles of Quantum Mechanics* [7] and it has been axiomatized out of existence in the well-known versions that follow Schrodinger or Heisenberg [8]. This neglect adds to ambiguities in the notion of the "point-particle" in atomic data and of "Uncertainty" in wave-particle duality. However, the particle that became evident in the early 20th century need not be the "point particle" of mathematical theories; the opposite view of the wave-group is closer to experimental fact, including everyday physical understanding of transmission and scanning electron microscopies [9] and indeed of optical physics [10].

Typically, our event horizon is wider in phase space than in real Minkowski space because $v_p > c$. However, the horizon relates to other phenomena that have been mentioned before but which are still intriguing: how can the photon demonstrate momentum without rest mass? Why is the momentum of a wave-particle not proportional to its wavelength, as is its wave velocity? How small is a "point particle"? What is an "uncertainty limit"? What physical measurement is instantaneous and discontinuous? How is a macroscopically dispersed wave-packet (as in Young's slits) reduced to atomic scales during measurement? These are a few of many weird features adopted without explanation in mathematical quantum theory. We have, in previous papers, explored *physical* answers to all of them.

In truth the "'point-particle" is computational. Mathematics chooses its axioms; Physics falsifies them; Chemistry employs them. Computation is free to follow no particular discipline. For this reason, the calculations are neither undeniably true nor properly beautiful.

Heisenberg's point particle dispersed into his Uncertainty, though Huygens, Fraunhofer and Fresnel had preceded him more accurately a hundred years earlier. We take another point: the center of the wave group, and we illustrate it with experimental detail.

Whether quantum theory is mathematics or physics, it is the duty of science to know. Where it is computational, it is free to adapt and progress without discipline. In Stephen Hawking's Department of Applied Mathematics and Theoretical Physics, we were once invited to a seminar series. At one point our host interrupted my question, "Let me understand: you do experiments as well as theory!" I was the only researcher present that enjoyed the advantage, designing, building, and measuring with instrumentation for light optics and electron optics—including of course, the wave packet. I take it that Popper's schism is outdated: "scientists" will do what they individually can in all their variety.

2. The Photonic Quantum

After Planck had interpreted the spectrum of black body radiation by supposing that the ultraviolet collapse in electromagnetic radiation is quantized, Einstein reached the same conclusion in 1905 in the photoelectric effect: when light is absorbed into a metallic surface, electrons are emitted. The energy states E_n of these electrons depends directly, not on the incident light intensities as had been previously supposed[1], but on the frequency of the light, $E_n = \hbar\omega_n$, *i.e.* on angular frequency ω in units of Planck's reduced constant. The simplest illustration of a free quantum, that is consistent with wave-particle duality, is the wave group (**Figure 1**),

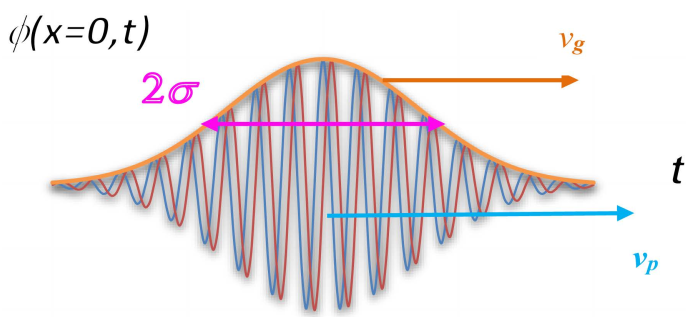


Figure 1. Normal wave packet including conservative function (orange) enveloping infinite, responsive, elastic, complex wave (red and blue), with uncertainty σ (pink double arrow) at time $x = 0$. The Fourier transform of a Gaussian is Gaussian, so t may represent any of the four variables x, k_x, t, ω, or t. In massive particles, the group velocity $v_g < c$ (orange, distance travelled per unit time); the phase velocity $v_p > c$ (blue).

specifically the wave function:

$$\varphi = A \cdot \exp\left(\frac{X^2}{2\sigma^2} + iX'\right)$$

with imaginary:

$$X = i\left(\bar{\omega}t - \bar{k}x\right); \quad X' = i\left(\omega t - kx\right) \tag{1}$$

where A is a normalizing constant; σ is proportional to Uncertainty [1]; and t, k and x represent respectively time, wave vector and position, in one dimension for simplicity[2]. The imaginary factor X contains mean values that stabilize the normal envelope function in free space and X' represents the variables in a plane wave.

The raw energy of this packet was described in footnote 1, but the mechanics of

[1] $E = \int\left(\boldsymbol{E}^2/2\varepsilon_o + \boldsymbol{B}^2/2\mu_o\right)\mathrm{d}\tau$ with electric field $\boldsymbol{E}$ vertical in **Figure 1** and magnetic intensity $\boldsymbol{B}$ horizontal. With quantization, $E_n = \hbar\omega_n$, owing to harmonic resonance to be described below.

[2] $\Delta t = 2\sigma/\bar{\omega}$; $\Delta\omega = 4\bar{\omega}/\sigma$; $\Delta x = 2\sigma/\bar{k}_x$; $\Delta k_x = 4\bar{k}_x/\sigma$; $\Delta\omega \cdot \Delta t = 8$ etc.

resonant quantization and resonant collapse is now evident in the facts of quasi-crystal diffraction [1]. With quantization, the packets are counted by integral numeration, and this gave rise, a hundred years ago, to the theory of quantum mechanics, including Einstein's photelectric effect in 1905. Meanwhile, we have to compare special relativity and the version of wave-particle duality that is often called uncertainty by those who consider the quantized wave as if it must be a point-particle.

In the same year, Einstein discovered that physical laws are invariant in all inertial reference frames. This includes the universal speed of light in free space, $c = (e_o.\mu_o)^{-1/2}$, the inverse square root of its electric permittivity times its magnetic permeability. From this fact he derived the equivalence between energy E and relativistic mass, $E = m'c^2 = \gamma m_o c^2$, *i.e.* including kinetic energy and the Lorentz factor $\gamma = \left(1 - v_g / c^2\right)^{-1/2}$, and furthermore $E^2 = \boldsymbol{p}^2 c^2 + m_o^2 c^4$, where $\boldsymbol{p}$ is the 3-dimensional momentum of a particle, having rest mass m_o. After applying Planck's law, $E = \hbar\omega$, and the de Broglie hypothesis in terms of wavevector, $p = \hbar k$, the equation yields [1]:

$$\omega^2 = k^2 c^2 + m_o^2 c^4 / \hbar^2 \tag{2}$$

in one spatial dimension for simplicity. Notice that m_o is a wave-particle since $m_o^2 = \omega^2 - k^2 = (\omega - k)(\omega + k)$ -the first part oscillating and the second conserving. By differentiation:

$$\frac{\omega}{k} \cdot \frac{\mathrm{d}\omega}{\mathrm{d}k} = c^2, \tag{3}$$

the product of phase velocity (frequency x wavelength or angular-frequency/wavevector) $v_p = \omega / k$ with group velocity $v_g = d\omega / dk$, is constant c^2 [1]. If the y, z motion has cylindrical symmetry normal to the x propagation direction (supposed in equation 2) and if the motion is represented in 4-dimensional space-time, the 2-dimensional transverse plane is conveniently normal to the direction of propagation, the representation becomes conveniently Euclidean.

3. Uncertainty in Particulate Waves

Our free wave packet is a closer description of ionization beams from discharge tubes (typical of experiments described in the 19th Century) or of electrons from a modern microscope gun, than are Schrödinger's bound states in a hydrogen atom or Heisenberg's operators $[x, p] > i\hbar/2$. Our description has the further advantage of being more detailed and more accurate: for example, the uncertainty in time Δt that is evident in **Figure 1** is the full width half maximum and given in footnote 2. After Fourier transform, the corresponding uncertainty in angular frequency $\Delta\omega$ is found, and the dual uncertainty, $\Delta\omega . \Delta t = 8$, is sixteen times larger than Heisenberg's "limit". Besides, our dual uncertainty is experimentally verified in **Figure 2**, where the critical condition in X-ray lithography corresponds to the extreme width of Cornu's theoretical spiral [10] [12] [13] that is used to calculate Fresnel diffraction (along with uncertainties [14]) in both photon and electron optics.

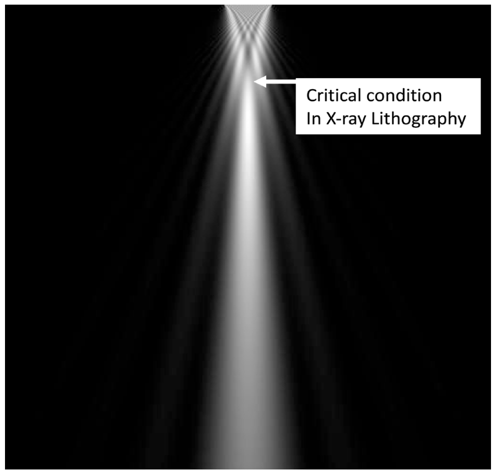


Figure 2. Simulation of 100 keV electron beam transmitted by a narrow slit. Notice the variations from near field to far field imaging in Fresnel diffraction: Δx is minimum at the critical condition, with Δp_x passing from negative (converging) to positive (diverging). At the experimental critical condition, the dual uncertainty is 4× larger than at Heisenberg's "limit" and as much as 20× larger in far field. The critical condition corresponds to the extremum width in Cornu's spiral [9]-[12]. Simulation due to C.B. Boothroyd.

4. Dispersion Dynamics

From special relativity, including equations 2 and 3, the derived functions of relativistic free particles are illustrated in **Figure 3**. The values are consistent with a hundred years of electron microscope usage and are also easily calculated, as below. Meanwhile, by using natural units $\hbar = 1 = c$, notice firstly the rest mass m_o, at the extreme left of relativistic mass plotted against k, *i.e.* $m' = \gamma m_o^3$. Secondly as $k \to 0$, the group velocity is consistent with Newtonian kinetic energy, $KE \approx m_o v_g^2 / 2$ [4]; but as k increases, v_g limits to c by special relativity as is well known[5]. The normalized phase velocity is its inverse by equation 3, $v_p/c = c/v_g$.

These values and derived properties are simply calculated (**Figure 4**) using Pythagoras' formula. They can often be found in electron microscope manuals [9]. The energy is the sum of electron rest mass energy with the gun energy Ve, measured and known. This addition gives us ω by Plank's law. The momentum is the vector difference between energy and rest mass. Corresponding values are plotted in **Figure 5**, where, owing to longstanding consistency, they are virtually experimental. These values have moreover been used in many applications starting with collision impact factors in elemental microanalysis [15]. Most recently they are used to redefine the quantum owing to its multiple properties that have been

[3] $m'/m = \gamma m_o/m_o = 1\big/\sqrt{v_g^2 - c^2} = \sqrt{c^2/\left(c^2 - v_g^2\right)}$; $m' \to m_o$ as $v_g \to 0$; $v_g \to kc$ as $v_g \to c$ *per* eq (1).

[4] $KE = E - m_o c^2 = (\gamma - 1) m_o c^2 \approx \left(1 + \frac{1}{2} v_g^2/c^2 - 1\right) m_o c^2 = \frac{1}{2} m_o v_g^2$, as $k, v_g \to 0$.

[5] From $pc = \hbar kc = \left(\left(1 - v^2/c^2\right)^{-1/2} - 1\right) m_o c^2$, rearrange $c^2 - v^2 = \left(\hbar^2 k^2 + m_o^2 c^2\right)^{-1}$ and $v_g = c - \left(\hbar^2 k^2 + m_o{}^2 c^2\right)^{-1}$.

systematically demonstrated in quasicrystal diffraction. In particular, whereas the claim for its original data was, “A metallic phase with long range order but no translational symmetry” [1] [3], in fact quasi-Bloch waves have been observed [16] and their diffraction proves to have simultaneous dual translational symmetry, *i.e.* in both linear and logarithmic space [1] [17]. It is not rational to ascribe the multiple, extraordinary properties to chance occurrence; they must be attributed, in any serious understanding, to the fact that [1] [3]:

All quanta are resonances to emission
or scattering stimuli and measurement.

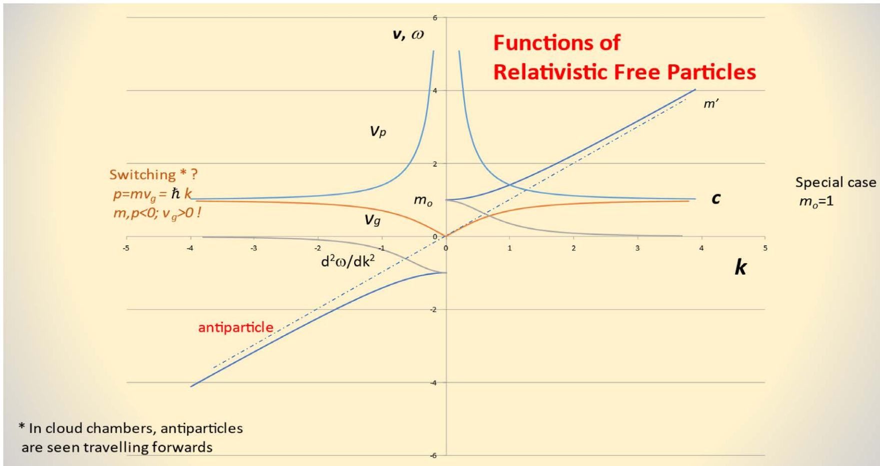


Figure 3. Functions of relativistic free particles (equations 2 and 3). To avoid unphysical singularities when $k = m_o c$, the antiparticle is ascribed negative mass, positive velocity and negative momentum. The antiparticle travels forward in time, as in cloud chambers and bubble chambers. The dynamics of the photon, with $m_o = 0$, are represented by the dash-dot line, $\omega = kc$.

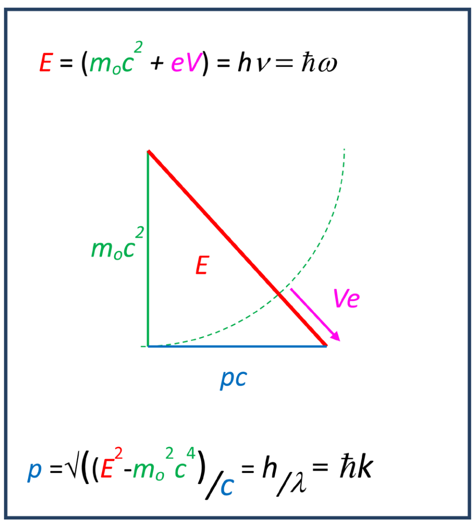


Figure 4. The frequency *n* and wavelength *λ* of an electron microscope probe are related by Pythagoras’ theorem as in relativity (eq. 2). The probe energy is the electron rest mass energy plus the accelerating energy of the gun, *Ve.*

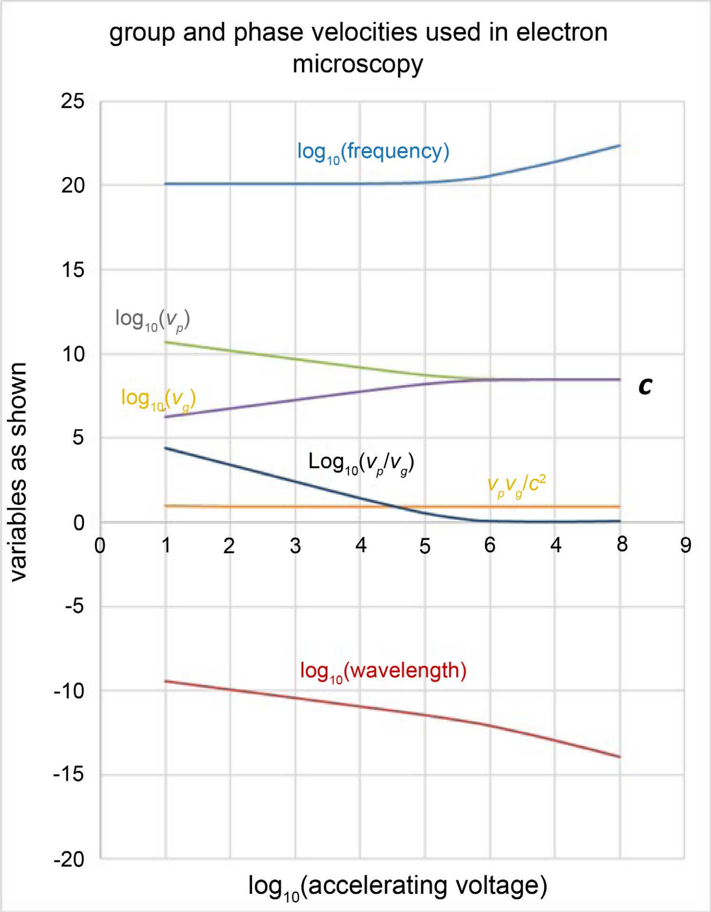
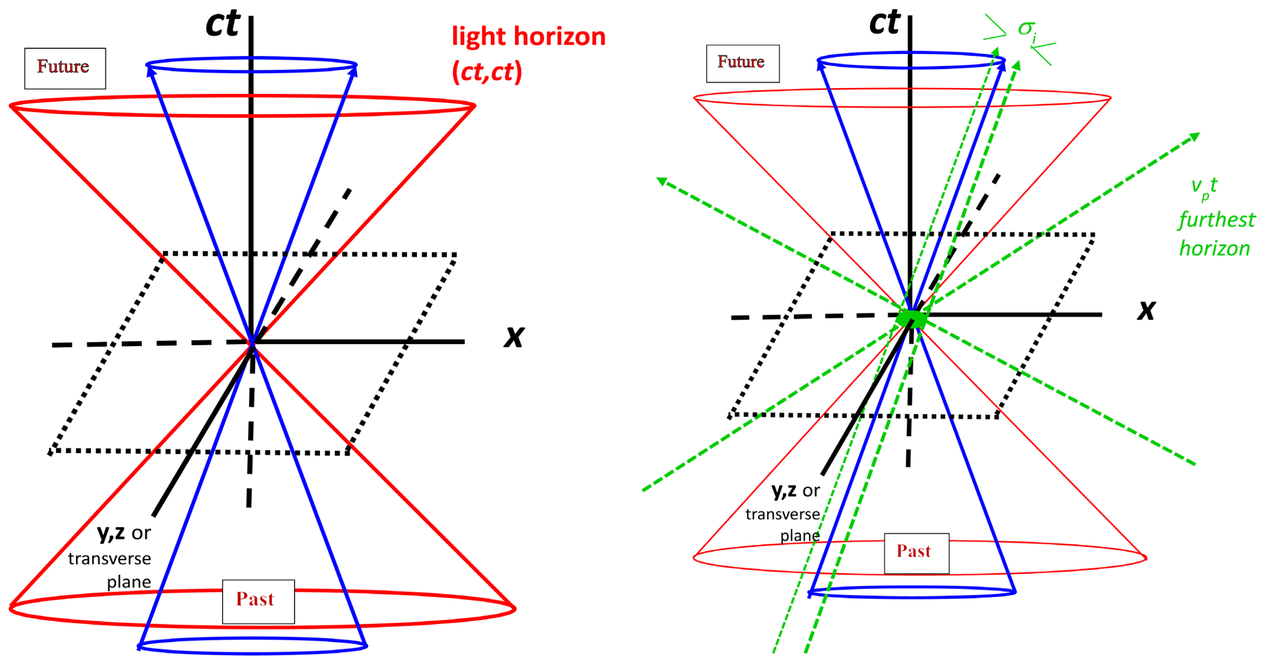


Figure 5. Plots of values calculated for parameters in various electron probes against accelerating at voltage/*keV*, including from top down: the frequency **(blue)** in SI units; the phase velocity v_p **(green)**, the group velocity v_g **(purple)**; the ratio of phase/group velocities **(navy blue)**; the product of phase with group velocities **(yellow)**; and the wavelength **(red)**. Notice the systematic relativistic changes when $Ve \simeq m_o c^2 \simeq 0.5$ MeV, excepting only the constant product $v_p.v_g$.

The physical quanta have sensible reality that supports their corresponding, numerical equivalent in mathematical physics, namely of integers and irrationals. The explanation of the quasicrystal diffraction shows also how wave functions collapse during measurement [1]. Collapse is fundamental in wave-particle duality, particularly in diffraction from Young's slits. Dispersion Dynamics have been variously applied [18]. With this general, affirmative background, we now progress to consider the horizon and range of phases travelling internally in the electron, faster than c.

5. Phase Velocity in Minkowski Space

Theoretically, Minkowski space is 4-dimensional and represents the spacetime of special relativity with inertial reference frames. The dimensions are time in the vertical direction in units of ct; space in the direction of particle propagation; and two other spatial dimensions in the transverse plane. Since the latter two dimensions are typically similar and comparatively simple, they are most conveniently represented together as one dimension of a 3-dimensional diagram (Figures 6(a) and Figures 6(b)). The defining property of the space is clear by comparison with Euclidean space which has the metric $s^2 = x^2 + y^2 + z^2$. In Minkowski space, Lorentz invariance provides, $s^2 = ct^2 - x^2 - y^2 - z^2$. The origin represents present time: particle dynamics of position and time are graphed on any plane through the origin, for any particular reference frame. The worldline of a stationary particle is the vertical axis through the origin. The speed of light is the slope of a massless particle that follows a straight line at 45˚ to the vertical. This is shown as two red cones in Figure 6(a), with apices at the origin, future time being positive; past time negative. No massive particle can travel faster than light. The surfaces of the red cones represent horizons for causal interaction, having present time at $t = 0$.



(a) (b)

Figure 6. (a) In Minkowski space, Red cones represent dynamics of *light photons* from the origin at time $t = 0$. The conical surface with apex semi-angle 45˚ is its horizon limit for causal interaction. Blue cones represent dynamics of massive particles with $pc \approx m_o c^2$. The semi angle in the blue cones, where $v_g \approx c/2$, is given by $\tan^{-1}(\theta) = v_g t/ct \approx 1/2$. When particle propagation proceeds on the horizontal line x, the third dimension, half forwards, represents motion in the transverse plane y, z, when cylindrically symmetric. Large, black, dashed lines represent negative axes. Small, black,dashed, lines represent the horizontal plane at time $t = 0$. Positive ct values lie in the future; negative values lie in the past. (b) Additional to Figure 6(a), green dashed lines represent phase velocities. The 'angle at furthest reach' is, in this case, $\tan^{-1}(\theta) = v_p t/ct \approx 3ct/2$; but is not reached by the particle outside of its group that is determined by the envelope part of equation 1. The phase velocities are only realized within a wavegroup as uncertainties σ_i about both sides of the blue wavegroup spacetime trajectory, i.e. as displacements about the blue wavegroup center. At time $t = 0$, the range corresponds to the green ellipse at the origin corresponding to the uncertainty and adjacent range in green-dashed, parallel ranges of phase motions (see text).

The blue cone represents the path in time and space of a particle travelling at about half the speed of light from the position of the observer at $t = 0$.

How should we represent phase velocity in Minkowski space? The phase causes interference in waves, but it allows for conservation of selected properties, m_o, e, ω, k, σ, *etc.* that are realized in complex calculations $\varphi^*.\varphi$. The phase is as real as its components, ω and λ, as in **Figure 5**. Besides causing uncertainty, phase requires representation on the Minkowski diagram in spite of its high velocity. **Figure 6(b)** shows an imaginary "furthest horizon" that is illustrated by single green lines at an angle $\tan^{-1}(\theta) = v_p t/ct \approx 3ct/2$. This corresponds to a higher phase velocity than on the light horizon on the surfaces of the red cones. However, the "horizon" of the phase of the massive particle is restricted by the center of the wave group because of the envelope on equation 1. The group is scarcely observable where it is far from the group center except by restrictive tunneling that is reminiscent of radioactive decay in particles. Short of tunneling, more significant is the uncertainty width σ between two dashed green lines that attach to the center of the blue group velocity. This green uncertainty is attached to the blue group velocity at its angle $\tan^{-1}(\theta) = v_g t/ct \approx 1/2$ on a particle traveling at half the speed of light, *i.e.* at half the angle taken by that light horizon on the surface of the red cone. The uncertainty corresponds to the envelope function in equation 1 that limits the range of the phase to a region close to the particle center of mass, as illustrated in **Figure 1**.

6. Discussion

The phase velocity of light is the ratio of two of its best-known constants of motion, ω and k. This ratio is the speed of light, a fundamental constant. In matter, since $m_o > 0$, the corresponding constants of motion, in free and inertial objects, have an extra degree of freedom (**Figures 4** and **Figures 5**). The constants are studied and known in electron microscopy, in X-ray lithography and in other disciplines. The phase velocity is the ratio E/p (equals σ/k); the group velocity is inverted, pc^2/E, following equation 3. Whereas it is well known in relativity, that the group velocities of *massive particles*, $v_g < c$; it follows their phase velocities are faster, $v_p > c$. This fact is contrary to common opinion in science, presumably because of prior influence made by optical physics which is different.

In near stationary particles, the wavelengths are long, corresponding to short wavevectors; angular frequencies are then mostly due to mass and are hardly motive. Conversely, large momenta accompany large wavevectors, *i.e.* with short wavelengths, and the momenta are principally dynamic. Time dilation, $\Delta t' = \Delta t \big/ \sqrt{1 - v_g^2 / c^2}$, and space contraction $\Delta x' = \Delta x \sqrt{1 - v_g^2 / c^2}$ are also asymmetric. Here, the primes represent measurements made in frame 2; compared with others made in the unprimed frame 1. In the direction of propagation, $v_g = v_x$, and in the relativistic region, $pc > m_o$ in **Figure 5**, the spatio-temporal measurements are more strongly dependent on v_g; but they are weaker in the transverse planes when v_y and v_z and comparatively small, and so less relativistic. This is partly why we converge the two axes in the transverse plane into the single axis for visual convenience in **Figure 6**.

That the phase velocity of a free and massive particle is greater than the speed of light, for reasons that are theoretical and established experimentally, is especially true in particles with low momentum. However, this fact does not imply that the phase velocity is an effective agent for physical properties, including momentum and energy that are governed by group properties; in contrast, the phase determines interference effects in the wave-particle, including diffraction and collapse [1]. In matter, the transport properties depend on the combined phase and group, though their individual motions are driven by different velocities.

It should not be surprising on reflection, that the "furthest horizon" of the phase velocity in a massive particle velocity seems strange. It is faster than the speed of the light *in vacuo* which is limited by the red cone in **Figure 6**; but nothing is so strange as the alternative, "instantaneous" transformations. The phase velocity is due to two sources: at *low momenta*, principally to rest mass with long wavelength and slow motion ($m_o c \gg p$); while at *high momenta* the phase velocity is due to short wavelengths and high, dynamic frequencies ($p \gg m_o c$). These opposites are features we should expect and become used to. In particular, the angles that world lines lines make with the vertical axes in **Figures 6**, signify not particle positions, but speeds: the positions are signified in the horizontal plane.

More generally, Physics that is not mathematical fantasy, is a variation of two themes. These are wave and particle: phase and group. They form opposites: mass and light; charge and polarization; scattering and superposition; gravitational force and metric; conserved properties and annihilations, Fermions and Bosons, *etc.* The duality also forms complementarities: spin; Pauli-exclusions and condensates; with extraordinary similarities in electron and optical microscopies; and in electron and optical spectroscopies. The hidden variables that may have not yet been measured in a particular experiment (in phase for example) may be temporarily probable, but the spaceless point is neither wave nor group.

The early point particle did not explain interference properties in matter. The application of harmonic emission and scattering to the wave-group satisfies the various requirements of interference. What remains is the core property of quantization. By subtraction, it is the same as matter in Newton's first law of motion: every "quantum" continues in its state of rest or uniform motion in a straight line, except in so far as it is compelled by applied external forces to change that state. It differs from transmission signals in undersea cables that are indeed distorted by external forces. It is remarkable that Newton's law applies to internal motion as well as external motion.

7. Conclusions

In massless particles modeled on the free Maxwellian *photon,* group velocity equals the phase velocity: both equal the speed of light, $v_g = c = v_p$.

Because photon optics are principally the same as electron optics (excepting polarization in the photon versus mass and charge in the electron), special relativity shows that the free, *massive particle* has limited group velocity, less than the speed of light $v_g < c$. Consequently, causality about the particle group lies, conventionally, within the Minkowski light cone as in Figure 6(a). The phase velocity, that is calculated and measured to be faster than the speed of light, $v_p > c$, does not escape the Minkowski light cones because the envelope (equation 1) on the *free particle wave function, pins the phase velocity to the center of the grou*p. Uncertainty ($\sigma / \bar{\omega}$) restricts the wave-group envelope to the region about the group center (where $\omega t = kx$) that lies on the group velocity boundary ($d\omega / dk = 1/ v_p$ in natural units). The restriction limits causality to the outer parallel phase boundary within the Minkowski cones.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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