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The Goldbach Conjecture Is True

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Abstract

Let $2m>2$, $m\in\mathbb{Z}$, be the given even number of the Strong Goldbach Conjecture Problem. Then, m can be called the median of the problem. So, all Goldbach partitions (p,q) exist a relationship, $p=m-d$ and $q=m+d$, where $p\le q$ and d is the distance from m to either p or q. Now we denote the finite feasible solutions of the problem as $S(2m)=\{(2,2m-2),(3,2m-3),\cdots,(m,m)\}$. If we utilize the Eratosthenes sieve principle to efface those false objects from set $S(2m)$ in p_i stages, where $p_i\in P$, $p_i\le\sqrt{2m}$, then all optimal solutions should be found. The Strong Goldbach Conjecture is true since we proved that at least one optimal solution must exist to the problem. The Weak Goldbach Conjecture is true since it is a special case of the Strong Goldbach Conjecture. Therefore, the Goldbach Conjecture is true.

Keywords

Median, The Feasible Goldbach Partitions, The Optimal Goldbach Partitions, Congruences, Effacement

1. Introduction

The Goldbach Conjecture is one of the oldest unsolved problems in number theory. The conjecture has been shown to hold for all integers less than 4×10^{18} [1], but remains unproven. The Goldbach Conjecture has two research directions. One is the Strong Goldbach Conjecture and another is the Weak Goldbach Conjecture. The paper starts to prove that the Strong Goldbach Conjecture is true, and then we prove that the Weak Goldbach Conjecture is true since the Weak Goldbach Conjecture is a special case of the Strong Goldbach Conjecture. The Strong Goldbach Conjecture states:

Definition 1. Every even integer greater than 2 can be expressed as the sum of

two primes [2].

First, the Strong Goldbach Conjecture problem can be written as Equation (1), and then we can deduce new equations from Equation (1).

$$p+q=2m \tag{1}$$

where $2m$ is given even integer. Assume p is small prime and q is large prime. Let $2d=q-p$, where, $d\in 0\cup\mathbb{Z}^+$. Then, we take $q=p+2d$ into the Equation (1) and get

$$2\times p+2\times d=2m$$

Dividing this equation by 2, then we simplify the equation and obtain

$$p=m-d;q=m+d \tag{2}$$

or rewrite as:

$$m=p+d;m=q-d \tag{3}$$

Now, we should make a few definitions as below.

Definition 2. Let the small prime be p, $0\le p\le m$, and the large prime be q, $m\le q\le 2m$. Let $2m$ be the given even integer. m is called the median of the problem. Also, d, $0\le d\le m$, is the distance from median m to either p or q. Hence, any pair numbers (p,q) and $p+q=2m$ is called a feasible Goldbach partition of the problem. If any feasible Goldbach partitions (p,q) contains both primes, then (p,q) is called an optimal Goldbach partition of the problem.

From the Equation (3), all Goldbach partitions shall exist the following property.

Definition 3. Let $p_i\in P$ and $p_i\le\sqrt{2m}$ and $p_{i+1}>\sqrt{2m}$. Then give an analysis of the property of the Goldbach partitions.

1) Median m is a prime

If m is a prime, them $2m=m+m$. It is a solution. Hence, assume that m is not a prime.

2) $p_i\,|\,m$

If $p_i\,|\,m$, then both $m-k\times p_i\equiv 0(\mathrm{mod}\,p_i)$, and $m+k\times p_i\equiv 0(\mathrm{mod}\,p_i)$, where $k\in N$. We can efface all these feasible Goldbach partitions $d=k\times p_i$ in sequence.

3) $p_i\nmid m$

If $p_i\nmid m$, then $p_i\nmid(m-k\times p_i)$, and $p_i\nmid(m+k\times p_i)$, where $k\in N$. Since we can not partition the number p_i-2 integers in half equally, the distance d from m to each p_i composite number in the small number side and large number side are different. Hence, they should be effaced in two distinct arithmetic progressions respectively.

4) Repeat from 2 to 3 for $p_i\le\sqrt{2m}$

By using the Eratosthenes sieves principle, $2\le p_i\le\sqrt{2m}$, all failure pairs inside $[0,2m]$ can be effaced by p_i in stages. The remaining feasible Goldbach partitions should be the optimal Goldbach partitions.

By utilizing the above property of the Goldbach partitions, we can efface all

failure pairs from all feasible Goldbach partitions and all optimal Goldbach partitions are remaining. Since we can prove that at least one optimal Goldbach partition exists consistently in all cases, then the Strong Goldbach Conjecture is true.

Now we introduce the chapter of the paper in detail. We review some existing published related documents to analyze their drawbacks. Then, we prove that the Strong Goldbach Conjecture is true. Furthermore, shows an example of the Strong Goldbach Conjecture. Fourth, we demonstrate that the Weak Goldbach Conjecture is true. Finally, we conclude that the Goldbach Conjecture is true and give a conclusion to the paper.

2. Existing Solutions and Their Drawbacks

On 7 June 1742, in a letter, the Russian mathematician Christian Goldbach was the first to state his conjecture to Leonhard Euler [3]. On 30 June 1742, in a reply, Euler proposed the following conjecture "Every positive even integer can be written as the sum of two primes." and "Every integer greater than two can be written as the sum of three primes." [4]. At that time, 1 was considered to be a prime number.

Finally, Goldbach remarked on his conjecture as below: "Every even integer greater than two can be written as the sum of two primes." Now, Goldbach's conjecture has been split into two halves:

1) The Weak Goldbach Conjecture asserts that all odd numbers greater than 7 are the sum of three odd primes.

2) The Strong Goldbach Conjecture asserts that every even integer greater than two can be written as the sum of two primes.

Until the 19th century, by the analytic number theory, people created a breakthrough path in studying the Strong Goldbach Conjecture. Brun, in 1919 devised his new method using the "sieve of Eratosthenes" as an asymptotic formula for the number of representations of n as the sum of two numbers, neither divisible by any fixed number of primes [5]. After that, many mathematicians improved Brun's idea published. Following, Chinese mathematician Jingrun Chen proved that every sufficiently large even integer is a sum of a prime and a product of at most 2 primes [6]. This result shall be the best improvement now. However, people understand that the sieve method is impossible to solve the Strong Goldbach Conjecture by some modifications of the current strategies. Hence, many people thought that another new path should be found out to solve the Strong Goldbach Conjecture problem. The Weak Goldbach Conjecture appeared to have been proved in 2013 [7] [8], but the strong Goldbach Conjecture remains unproven.

Harald Helfgott published a proof of Goldbach's weak conjecture in 2013, but it has not been published formally.

The present research mainly uses the analytic number theory on the distribution of prime numbers. For any searching of the Strong Goldbach Conjecture, the probability of a Goldbach partition inside among two closed intervals $[3, n/2]$ and $[n/2, n-3]$ simultaneously are primed separately, where n is the given even

number and $n \in \mathbb{N}$. This strategy separated a pair of the Goldbach partition into two isolated numbers. Hence, the complexity of this strategy is

$$\sum_{k=3}^{n/2} \frac{1}{\ln(k)} \frac{1}{\ln(n-k)} \approx \frac{n}{2(\ln(n))^2}$$

where k selected at random has a $\frac{1}{\ln(k)}$ chance of being prime

(https://en.wikipedia.org/wiki/Goldbach conjecture), and its complexity is difficult to be reduced. If people could find a new relationship in these pairs, the problem might obtain the next breakthrough. Now we notice that the two integers of any pair of Goldbach partitions have the same distance from the median point.

3. Analyse the Strong Goldbach Conjecture

3.1. Congruences

A natural number list $n \in \mathbb{Z}$ can use modulo 2 and the congruence principle to be partitioned into two disjoint subsets, $\langle 0 \rangle$, and $\langle 1 \rangle$. If $\langle 1 \rangle$ to modulo 3, we can partition into $\langle 0 \rangle$, $\langle 1 \rangle$, and $\langle 2 \rangle$, and so on. Now let us use the congruence principle to partition the list $\langle 1 \rangle$ to analyze their results.

Theorem 1. *Let* p_i *be a prime number, and let b be the remainder of modulo* p_i, *and* $a = p_1 \times p_2 \times \cdots \times p_{i-1}$, *where* $p_1, p_2, \cdots, p_{i-1}$ *are distinct primes starting from* $2,3,5,7,\cdots$, *in sequence. Then the numbers*

$$b+a, b+2a, b+3a, \cdots, b+(p_i-1)a \pmod{p_i}$$

are the same as the numbers

$$1,2,3,\cdots,(p_i-1) \pmod{p_i}$$

although they may be in a different order.

Proof of Theorem 3. If $b=0$, and $a = p_1 \times p_2 \times \cdots \times p_{i-1}$, $a \neq 0 \pmod{p_i}$, then the numbers

$$a, 2a, 3a, \cdots, (p_i-1)a \pmod{p_i}$$

are the same as the numbers

$$1,2,3,\cdots,(p_i-1) \pmod{p_i}$$

although they may be in a different order.

If $b \neq 0$, then add b to each element of the above list and obtain

$$1+b, 2+b, 3+b, \cdots, (p_i-1)+b \pmod{p_i}$$

then the numbers are the same as the numbers

$$1,2,3,\cdots,(p_i-1) \pmod{p_i}$$

although they may be in a different order.

If we can show that the numbers in the first list are distinct modulo p_i, it will follow that the two lists are the same. The list $1+b, 2+b, 3+b, \cdots, (p_i-1)+b$ contains $p_i - 1$ numbers, and clearly none of them are divisible by p_i. Suppose

that we take two numbers $j+b$ and $k+b$ in this list, and suppose that they are happen to be congruent,

$$j+b=k+b\left(\mathrm{mod}\ p_i\right).$$

Then $p_i \mid (j+b-k-b)$, so $p_i \mid (j-k)$, since we are assuming that p_i does not divide b. Notice that if a prime divides a product then it divides one of the factors. On the other hand, we know that $1 \le j,k \le p_i - 1$, so $|j-k| < p_i - 1$. There is only one number with absolute value less than $p_i - 1$ that is divisible by p_i and that number is zero. Hence, $j=k$. This shows that different $1+b, 2+b, 3+b, \cdots, (p_i-1)+b\left(\mathrm{mod}\ p_i\right)$ in the list are distinct modulo p_i.

So we know that the list $b+a, b+2a, b+3a, \cdots, b+(p_i-1)a$ contains $p_i - 1$ distinct nonzero values modulo p_i. But there are only $p_i - 1$ distinct nonzero values modulo p_i, that is, the numbers $1,2,3,\cdots,(p_i-1)$. Hence, the list $b+a, b+2a, b+3a, \cdots, b+(p_i-1)a$ and the list $1,2,3,\cdots,(p_i-1)$ must contain the same numbers modulo p_i, although the numbers may appear in a different order. Our next task is to look at the numbers between the odd integers with congruence properties. Let p be an odd prime and let list $A = p, (p+2), \cdots, (p+2n)$, where $n \in \mathbb{N}$ and p is odd prime number. Then the minimum distance between p and $3p$ in list A equals $2p$, and the numbers between p and $3p$, two composite numbers with factor p, equals $p-1$.

In list A, all the composite numbers x with divisor factor p, $p \mid x$, must belong to the arithmetic progression $B = p, 3p, \cdots, (2n+1)p$. Then,

$p, 3p, \cdots, (2n+1)p\left(\mathrm{mod}\ p\right) \equiv 0\left(\mathrm{mod}\ p\right)$, so $(A-B)\left(\mathrm{mod}\ p\right) \neq 0\left(\mathrm{mod}\ p\right)$.

Hence, $2p$ is the minimum spacing of the composite numbers with divisor factor p in A.

Then, let us count the numbers between the distance $2p$. If we have an odd integers list

$$\begin{aligned}&(p+2), (p+4), \cdots, \left(p+2(p-1)\right)\left(\mathrm{mod}\ p\right)\\ &\equiv 2, 4, \cdots, 2(p-1)\left(\mathrm{mod}\ p\right)\\ &\equiv 2\times\left(1, 2, \cdots, (p-1)\right)\left(\mathrm{mod}\ p\right)\end{aligned}$$

We know that $p \nmid (p-1)!$ and $gcd(2,p)=1$. Therefore, there are $p-1$ distinct odd integers between two composite numbers with the common divisor factor p in A.

3.2. Utilize the Property of the Composite Number

In the number theory, finding each prime number is a challenging job. However, it is an easy task using the Eratosthenes sieve principle to search composite numbers, after that, those remaining integers should be primes. Now, in the Strong Goldbach problem, we utilize the strategy to explore the optimal Goldbach partitions from all feasible Goldbach partitions.

1) Special cases and effacement even numbers

If m is a prime, then $2m = m+m$, it is an optimal Goldbach partition. For

example, $4=2+2$ and $6=3+3$. So, effacing all even integers since even numbers can not be prime except 2. Also, assume that m is not a prime number.

2) Creates a feasible Goldbach partition list

Let $2m$ be the given integer for the Strong Goldbach Problem, and all feasible Goldbach partitions should be $(i,2m-i)$, where $2\le i\le m$. Such that, we can obtain all feasible Goldbach partitions saved in a list

$$S(2m)=\{(2,2m-2),(3,2m-3),\cdots,(m,m)\}. \tag{4}$$

3) Find optimal Goldbach partitions

One should set the feasible Goldbach partitions to false if any object has at least one composite number. So, $S(2m)$ passing the Eratosthenes sieving in stages by p_i, and $p_i\le\sqrt{2m}$, all false objects had been effaced out from $S(2m)$. The termination condition is $p_i>\sqrt{2m}$. Then, the remaining objects must be optimal Goldbach partitions for the problem.

3.3. Analyse the Successive Goldbach Partition List

Since given even number $2m>4$ is composite numbers, we can efface those false feasible Goldbach partitions from $S(2m)$. To analyze the Goldbach partition problem, let any odd prime p, and $3\le p\le\sqrt{2m}$, also $2m-p=m+m-p=m+d$, where $d=m-p$. Then, any one of Goldbach partitions is an feasible Goldbach partitions containing a pair of integers $(m-d,m+d)$, or $(p,2m-p)$, such that they sum to $2m$. We compute a Goldbach list modulo p, clearly leading to the following four cases.

Case 1. $p\nmid(m-d)$ and $p\nmid(m+d)$.

Exist either $p\nmid m$ or $p\mid m$.

Case 2. $p\nmid(m-d)$ and $p\mid(m+d)$.

Only $p\nmid m$, then exist $(m+d)\equiv 0(\bmod p)$, and $(m-d)\neq 0(\bmod p)$.

Case 3. $p\mid(m-d)$ and $p\nmid(m+d)$.

Only $p\nmid m$, then exist $(m-d)\equiv 0(\bmod p)$, and $(m+d)\neq 0(\bmod p)$.

Case 4. $p\mid(m-d)$ and $p\mid(m+d)$.

Only $p\mid m$, then exists $(m-d),(m+d)\equiv 0(\bmod p)$.

We now analyze the above cases in detail.

1) If $p\mid m$, there are either $p\nmid d$ or $p\mid d$, which lead to Case 1 or Case 4, respectively. In each $2p$ distance region containing p successive feasible Goldbach partitions, Case 4 appears once in which $p\mid d$, and Case 1 appears $p-1$ times in which $p\nmid d$.

2) If $p\nmid m$, three cases appear, Case 1, Case 2, and Case 3. Because inside an odd numbers arithmetic progression, any $2p$ distance must have $p-1$ objects that can not be divided by p. So, in the median region, the median number must be one of the numbers, so there are $p-2$ numbers remaining. This situation implies that we can not partition $p-2$ in half equally. Hence, Case 2 must appear once in each $2p$ distance in which $p\mid(m+d)$ and $p\nmid(m-d)$. Case 3 must appear once in each $2p$ distance in which $p\mid(m-d)$ and $p\nmid(m+d)$. The

remaining events are Case 1 situations in which $p \nmid (m-d)$ and $p \nmid (m+d)$.

Following, we take an odd number object list. Each odd prime p_i, and $3 \le p_i \le \sqrt{2m}$.

$$\begin{aligned} S(2m) = \{&(p_i, 2m-p_i),(p_i+2, 2m-p_i-2),(p_i+4, 2m-p_i-4),\\ &(p_i+6, 2m-p_i-6),\cdots,(m,m)\} \end{aligned} \tag{5}$$

and consider what $S(2m) \pmod{p_i}$ can be.

First, we partition $S(2m)$ into three lists by its distinct congruence modulo p_i.

1) If any odd prime $p_i \mid m$, then

$$\begin{aligned} &(p_i, 2m-p_i),(3p_i, 2m-3p_i),\cdots,((2k+1)p_i, 2m-(2k+1)p_i),\cdots \pmod{p_i}\\ &\equiv (0,0) \pmod{p_i} \end{aligned},$$

where $k \in \mathbb{Z}^+$.

Otherwise,

$$(a_1, 2m-a_1),(a_2, 2m-a_2),\cdots,(a_j, 2m-a_j),\cdots \pmod{p_i} \equiv (x,y) \pmod{p_i},$$

where $p_i \nmid a_j$, and $0 < x, y < p_i$.

Hence, each $2p_i$ distance of the list, there is an object with zero modulo p_i.

2) If $p_i \nmid m$, then $p_i \nmid 2m$. Hence, there are three cases.

First case,

$$\begin{aligned} &(p_i, 2m-p_i),(3p_i, 2m-3p_i),\cdots,((2k+1)p_i, 2m-(2k+1)p_i),\cdots \pmod{p_i}\\ &\equiv (0,y) \pmod{p_i} \end{aligned},$$

where $y \ne 0$ and $0 < y < p_i$.

Second case,

$$(a_1, 2m-a_1),(a_2, 2m-a_2),\cdots,(a_j, 2m-a_j),\cdots \pmod{p_i} \equiv (x,0) \pmod{p_i},$$

where $x \ne 0$ and $0 < x < p_i$. Since, $2m-a_j \equiv 0 \pmod{p_i}$, and $p_i \nmid a_j$. So, $2m-a_j = kp_i$.

The remaining cases are,

$$(b_1, 2m-b_1),(b_2, 2m-b_2),\cdots,(b_j, 2m-b_j),\cdots \pmod{p_i} \equiv (x,y) \pmod{p_i},$$

where $p_i \nmid b_j$, and $p_i \nmid (2m-b_j)$. Hence, $x \ne 0$ and $y \ne 0$, and $0 < x, y < p_i$.

Now, we summary the case as below.

Inside each $2p_i$ distance of the list $S(2m)$, if we ignore p_i itself as a prime, then there are only two objects with zero modulo p_i.

Above cases should happen in p_i and $3 \le p_i \le \sqrt{2m}$. Continuing in this fashion until $p_i > \sqrt{2m}$ stopped, then we must obtain the optimal solutions. Following, we define the number of the biprimes of the Strong Goldbach partition (BIP).

Definition 4. Let set $A = [2, m]$ and set $B = [m, 2m-2]$, where, $m \in \mathbb{Z}^+$ is the median between $[2, 2m]$. Let $q_{m-d_i} = m - d_i$ and $q_{m+d_i} = m + d_i$ be a pair

of Goldbach partition, such that $q_{m-d_i}+q_{m+d_i}=m-d_i+m+d_i=2m$, where d is the distance between any integer to m, and $0\le d_i\le m$. Hence, if both q_{m-d_i} and q_{m+d_i} are primes, then $\left(q_{m-d_i},q_{m+d_i}\right)$ is an optimal solution of the Strong Goldbach problem.

Furthermore, let

$$BIP(m)=\left\{d_i:\text{prime } q_{m-d_i}=m-d_i,\text{ and prime } q_{m+d_i}=m+d_i;\text{ with } 0\le d_i\le m\right\}.$$

Also, let $biu(m,d_i)=\begin{cases}1, & m-d_i \text{ and } m+d_i \text{ both are primes}\\ 0, & \text{otherwise}\end{cases}$.

Such that, $BIP(m)=\sum_{0\le d_i\le m} biu(m,d_i)$.

We use Eratosthenes's pair sieve method to find all prime pairs sum to a given natural even number $2m$. This sieve method allows us to determine whether any pair numbers are both primes. We now explain how the pair sieve of the Eratosthenes method can be used to find all prime pair numbers that sum to the given even natural number $2m$. If a number a is composite, then from 2 to m has at least one prime factor of a less than or equal to $\sqrt{2m}$. Otherwise, a is a prime.

We proceed as follows.

- Create a pair object list of consecutive integers from $(1,2m-1)$, $(2,2m-2)$, $\cdots$, (m,m), such that each pair sum to $2m$. There are m pair numbers.
- $(1,2m-1)$ can be removed. Then small numbers are from 2 to m, and big numbers are from $2m-2$ to m.
- Initially, let p equal 2, the smallest prime number. Remove all even numbers because they are composite. Hence, the remaining numbers have a half of the pair list, $\frac{1}{2}\times m$. We can simply remove every second number starting from 2.
- The next integer on the list that is 3, and ignore 3 is prime. There are two cases to be removed. One is $(3c,x)$ and another is $(y,3d)$ in each three consecutive pairs, where $3\nmid x$, $3\nmid y$ and c,d belong to natural numbers. Hence, the remaining numbers are one-third of the remaining list, and obtain $\frac{1}{2}\times\frac{1}{3}\times m$.
- The next integer on the list that is removed is 5, and ignore 5 is prime. Remove all multiples of 5. They are $(5c,x)$ and $(y,5d)$ in each five consecutive pairs, where $5\nmid x$ and $5\nmid y$ and c,d belong to natural numbers. Hence, the remaining numbers are $\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times m$ from the remaining list.
- Continue in this way until the number $p_i>\sqrt{2m}$. We ignore p_{i-1} these primes. We still have pairs in the remaining list.

$$\left\{\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\cdots\times\frac{p_k-2}{p_k}\times m\right\},$$

where $p_k\le\sqrt{2m}$ and $p_{k+1}>\sqrt{2m}$, where we ignore that p_k is itself prime.

- The remaining integer pairs should be both primes solution. By using the sieve of Eratosthenes, any integer j, where $2m\ge j>\sqrt{2m}$. If $p_i\nmid j$ where p_i is

prime and $p_i \le \sqrt{2m}$, then j must be a prime.

- According to Theorem 1, any arithmetic sequence can be shown the same as the numbers

$$0,1,2,3,\cdots,(p_i-1)(\mathrm{mod}\ p_i)$$

although they may be in a different order. While we consider both the big numbers and the small numbers, there are two pairs included zero number, $\{p_i c, x\}$ or $\{y, dp_i\}$, $p_i \nmid x$ and $p_i \nmid y$ and c,d belong to natural numbers. Hence, we have $\left(\frac{p_i-2}{p_i}\right)$ to remove the two zero included pair composite numbers each time.

- To obtain prime pairs, these process continue until $p_i > \sqrt{2m}$ and then stop.

For example for number 3, the remainders are 0, 1, and 2. In the worst case, there are one-third list to remain. The three cases are $(x,3c),(3d,y),(x,y)$, where $3 \nmid x$ and $3 \nmid y$ and c,d belong to natural numbers.

Hence,

$$BIP(m) \ge \left\{\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\cdots\times\frac{p_k-2}{p_k}\times m\right\} \tag{6}$$

where $p_k \le \sqrt{2m}$ and $p_{k+1} > \sqrt{2m}$, where we ignore that p_k is itself prime.

3.4. Prove the Strong Goldbach Conjecture

Following, let us prove that if $m \ge 2$, then $BIP(m) \ge 1$.

Theorem 2. *Every even integer* $2m > 2$ *can be expressed as the sum of two primes.*

Proof of Theorem 3. If m is a prime, then $2m = m + m$, then we are done. So, suppose that m itself is not prime. We have factored m as a product of primes, say

$$m = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r},$$

where $p_1, p_2, \cdots, p_r$ are all different primes. By the Eratosthenes sieves principle, any composite numbers $x < 2m$ must exist at least a prime factor $p_i \mid x$ and $p_i \le \sqrt{2m}$, where, $0 \le i \le r$. Then the following conditions is true:

1) Efface All Even Numbers

If $2m > 4$, then all even numbers can not be prime. So, we efface all of them. The remaining objects equal $\frac{m}{2}$.

2) Search Stages

The search stages about equal the number of primes, $\pi\left(\sqrt{2m}\right)$. Let $p_k \le \sqrt{2m}$, and $p_{k+1} > \sqrt{2m}$. Then we have

$$BIP(m) = \sum_{2 \le p_i \le \sqrt{2m}} biu(m, m-p_i) = \sum_{i=1}^{k} biu(m, m-p_i).$$

3) Analyze the Case A

If $m = 2^n$, $n \ge 2$ and $n \in \mathbb{Z}^+$, then $p_i \nmid m$ and $i > 2$ or $p_i \mid m$ and

$p_i > \sqrt{2m}$, such that, there are two arithmetic progressions of the composite numbers for each p_i and $i > 2$ respectively, one is the small side numbers and another is the large side numbers. Hence, we should efface failure pairs

$$\frac{m}{2} \times \prod_{i=2}^{k} \left(\frac{p_i - 2}{p_i} \right), \text{ where } p_i \in P, \text{ and } 3 \le p_i \le \sqrt{2m}.$$

Ensures at least one optimal solution exists in the case

$$\begin{aligned} BIP(m) &\ge \frac{1}{2} \times \frac{1}{3} \times \frac{3}{5} \times \cdots \times \frac{p_k - 2}{p_k} \times m \\ &= \frac{1}{2} \times \frac{1}{3} \times \frac{3}{5} \times \frac{5}{7} \times \frac{9}{11} \times \frac{11}{13} \times \frac{15}{17} \times \frac{17}{19} \times \cdots \times \frac{p_k - 2}{p_k} \times \frac{2m}{2} \\ &\text{reduce the numerator and keep the denominator beyond} \\ &\ge \frac{1}{4} \times \frac{1}{3} \times \frac{3}{5} \times \frac{5}{7} \times \frac{7}{11} \times \frac{11}{13} \times \frac{13}{17} \times \frac{17}{19} \times \cdots \times \frac{p_k - 2}{p_k} \times \sqrt{2m} \times \sqrt{2m} \\ &\ge \frac{1}{4} \times \frac{1}{3} \times \frac{3}{5} \times \frac{5}{7} \times \frac{7}{11} \times \frac{11}{13} \times \frac{13}{17} \times \frac{17}{19} \times \cdots \times \frac{p_k - 2}{p_k} \times p_k \times \sqrt{2m} \\ &\ge \frac{1}{4} \times \frac{3}{3} \times \frac{5}{5} \times \frac{5}{7} \times \frac{7}{11} \times \frac{11}{13} \times \frac{13}{17} \times \frac{17}{19} \times \cdots \times \frac{p_k - 2}{p_k} \times p_k \times \sqrt{2m} \\ &\ge \frac{\sqrt{2m}}{4} \end{aligned}$$

4) Analyze the Case B

Assume that any prime $p_i \mid m$ and $p_i \le \sqrt{2m}$, and for $\frac{p_k - 1}{p_k} > \frac{p_k - 2}{p_k}$, then

$$\begin{aligned} BIP(m) &\ge \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \frac{12}{13} \times \frac{16}{17} \times \frac{18}{19} \times \cdots \times \frac{p_k - 1}{p_k} \times m \\ &\ge \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \frac{12}{13} \times \frac{16}{17} \times \frac{18}{19} \times \cdots \times \frac{p_k - 1}{p_k} \times \frac{2m}{2} \\ &\ge \frac{1}{4} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \frac{12}{13} \times \frac{16}{17} \times \frac{18}{19} \times \cdots \times \frac{p_k - 1}{p_k} \times \sqrt{2m} \times \sqrt{2m} \\ &\ge \frac{\sqrt{2m}}{4} \times \frac{2}{3} \times \frac{4}{5} \times \cdots \times \frac{p_k - 1}{p_k} \times \sqrt{2m} \\ &\ge \frac{\sqrt{2m}}{4} \times \frac{1}{3} \times \frac{3}{5} \times \cdots \times \frac{p_k - 2}{p_k} \times p_k \\ &\ge \frac{\sqrt{2m}}{4} \end{aligned}$$

Hence, $\frac{\sqrt{2m}}{4} \ge 1$, this means that at least one Strong Goldbach partition to be found. Therefore, $m \ge 8$, the Strong Goldbach Conjecture must be true.

While $m < 8$, we list the solutions of the Strong Goldbach partition as below:

$m = 2$, then $4 = 2 + 2$.

$m = 3$, then $6 = 3 + 3$.

$m = 4$, then $8 = 3 + 5$.

$m = 5$, then $10 = 3 + 7$, or $10 = 5 + 5$.

$m=6$, then $12=5+7$.

$m=7$, then $14=3+11$, or $14=7+7$.

Therefore, every even integer $2m\geq 4$ can be expressed as the sum of two primes. The Strong Goldbach Conjecture is true.

3.5. Example

Given $2m=100$. Then the odd numbers feasible solutions listed below,

$$S(100)=\{(3,97),(5,95),(7,93),(9,91),(11,89),(13,87),(15,85),(17,83),(19,81),(21,79),(23,77),(25,75),(27,73),(29,71),(31,69),(33,67),(35,65),(37,63),(39,61),(41,59),(43,57),(45,55),(47,53),(49,51)\}.$$

Does exist at least one optimal solution to the Strong Goldbach problem?

Answer: The median $m=50$ and $\lceil\sqrt{100}\rceil=10$. So $p_i=\{3,5,7\}$.

1) First Stage, $S(100)$ modulo 3

Since $3\nmid 50$, we must efface one arithmetic progressions from $S(100)$. For details, see Table 1.

Table 1. Modulo 3.

Modulo 3	1	2	3	4	5	6	7	8
(0, *y*) (mod3)	(3, 97)	(9, 91)	(15, 85)	(21, 79)	(27, 73)	(33, 67)	(39, 61)	(45, 55)
(*x*, *y*) (mod3)	(5, 95)	(11, 89)	(17, 83)	(23, 77)	(29, 71)	(35, 65)	(41, 59	(47, 53)
(*x*, 0) (mod3)	(7, 93)	(13, 87)	(19, 81)	(25, 75)	(31, 69)	(37, 63)	(43, 57)	(49, 51)

2) The Second Stage, (x,y) list modulo 5

Since $5\mid 50$, choose the middle row list modulo 5 from Table 1. For details, see Table 2.

Table 2. Modulo 5.

Modulo 5	1	2	3	4	5	6
(0, 0) (mod5)	(5, 95)	(35, 65)	-	-	-	-
(*x*, *y*) (mod5)	(11, 89)	(17, 83)	(23, 77)	(29, 71)	(41, 59)	(47, 53)

3) The Third Stage, (x,y) list modulo 7

Since $7\nmid 50$, choose the middle row list modulo 7 from Table 2. For details, see Table 3.

Table 3. Modulo 7.

Modulo 7	1	2	3	4	5
(0, *y*) (mod7)	-	-	-	-	-
(*x*, *y*) (mod7)	(11, 89)	(17, 83)	(29, 71)	(41, 59)	(47, 53)
(*x*, 0) (mod7)	(23, 77)	-	-	-	-

4) The Fourth Stage

Since $11^2 = 121 > 100$, stop.

Now, we obtain four optimal solutions:

$S(100) = \{(11,89),(17,83),(29,71),(41,59),(47,53)\}$.

$BIP(47) = 5 > 1$, the Strong Goldbach Conjecture is correct.

Note: Ignore $p_i = \{3,5,7\}$ as primes.

4. Prove the Weak Goldbach's Conjecture

Now let us prove the Weak Goldbach Conjecture.

Theorem 3. *Every odd integer* (≥9) *can be represented as the sum of three odd primes.*

Proof of Theorem 3. The Weak Goldbach Conjecture can be derived from the Strong Goldbach Conjecture since every odd integer ≥ 9 subtracts 3, then it becomes a Strong Goldbach Conjecture Problem. Therefore, the Weak Goldbach Conjecture is true.

5. Conclusion

The Strong Goldbach problem is a finite number problem. We can sieve optimal solutions by recursively effacing those false objects while $m \geq 8$. While $m < 8$, we have listed all their optimal solutions. Hence, the Strong Goldbach Conjecture is true. The Weak Goldbach Conjecture is one of the special cases of the Strong Goldbach Conjecture. Therefore, the Goldbach Conjecture is true.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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