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Contour Approach for Analysis of Minimum Regions for the Economic Statistical Design of X-Bar Control Charts

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Abstract

Visualization of the patterns or behaviors of complex functions is an important approach to identifying their solution spaces. This approach can lead to improving the design of search algorithms for optimization purposes. This paper presents a 2D visualization method based on minimum cost scores (MCS) to identify the most likely regions to find optimal values for the economic statistical design of X-bar control charts. Uniform sampling was considered for the analysis of the cost function of Rahim and Banerjee (1993). Elliptical regions were found to model the minimum regions defined by MCS values estimated for this cost function through different values of sample size (n), length of the sampling interval (h) and coefficient of the control chart's limits (L). This method was assessed by finding the optimal set of n and h values within these elliptical regions. Optimal values reported in the literature with Genetic Algorithms (GA) were considered for this case. It was observed that the optimal values were located within the boundaries of the elliptical regions and were associated with sub-regions with the highest MCS values. This confirms the suitability of this approach to obtain the *a-priori* estimation of the solution space.

Keywords

Contour Plots, Economic-Statistical Design, 2D Visualization

1. Introduction

Visualization is an important approach to understanding the complex relationships between variables and objective functions to improve data analysis (Wang et al., 2010; Goel et al., 2001; Wang & Saunders, 1999). For combinatorial problems, visualization can improve the understanding of the static and dynamic characteristics of their solution spaces (Caballero-Morales, 2014; Caballero-Morales & Rahim, 2015). Consequently, visualization can be applied to understand the search mechanisms of solving algorithms and improve their performance (Halim et al., 2006; Halim & Yap, 2007; Pérez et al., 2013).

In this work, the economic statistical design (ESD) problem for X-bar control charts is considered. A control chart is defined by three main parameters: the sample size (n), the length of the sampling interval (h) and the coefficient of the chart's control limits (L). The ESD combinatorial problem consists in finding the optimal values for n, h and L, that globally minimize costs (modeled by a cost function) considering economic and statistical restrictions (Rahim & Banerjee, 1993).

Previously, full three-dimensional (3D) visualizations were performed to understand the dynamics of cost function models with general failure distribution (Caballero-Morales, 2014) and non-normality (Caballero-Morales & Rahim, 2015) for the ESD problem. In this work, a general contour approach is proposed to identify bi-dimensional (2D) minimum regions for the ESD problem. The estimation of minimum cost scores (MCS) is defined to obtain the 2D visualization of the minimum regions. It is expected that this approach can lead to improvements in the search mechanisms of solving algorithms to obtain the optimal chart parameters. In contrast to a 3D visualization (Caballero-Morales, 2014; Caballero-Morales & Rahim, 2015) the proposed MCS 2D visualization can be computed faster, improving its practical use.

The present paper is structured as follows: in Section 2 an overview of the cost function model for the ESD of X-bar control charts is presented. This overview includes a description of reference data sets for the assessment of the MCS 2D visualization approach which is described in Section 2. Then, in Section 3 the contour-based MCS 2D visualizations are presented and discussed for each reference data set. Finally, in Section 4 the conclusions and future work are presented.

2. Materials and Methods

In this work the cost function model of Rahim and Banerjee (Rahim & Banerjee, 1993) with Gamma failure distribution and uniform sampling was considered. The objective function of this cost model consists of minimizing $F(n, h, L) = E(C)/E(T)$, where $E(T)$ is the Expected Cycle Length, and $E(C)$ is the total Expected Cost per Cycle (Rahim & Banerjee, 1993; Chih et al., 2011). These costs are defined as follows:

$$E(T) = h + (\alpha Z_0 + h)\left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(1+\frac{\lambda h}{1-e^{-\lambda h}}\right) + \frac{h\beta}{1-\beta} + Z_1, \quad (1)$$

$$\begin{aligned} E(C) = {} & (\alpha + bn + \alpha Y + D_1 h)\left(\frac{e^{-\lambda h}}{1-e^{-\lambda h}}\right)\left(1+\frac{\lambda h}{1-e^{-\lambda h}}\right) \\ & + \frac{\alpha + bn}{1-\beta} + \frac{2D_0}{\lambda} + D_1\left(\frac{h}{1-\beta} - \frac{2}{\lambda}\right) + W, \end{aligned} \quad (2)$$

where $\alpha = 2\phi(-L)$ and $\beta = 1 - \left[\phi\left(\delta\sqrt{n} - L\right) + \phi\left(-\delta\sqrt{n} - L\right)\right]$. If α_U and p_L represent the maximum and minimum values for the type I and type II error probabilities respectively, the minimization of $F(n, h, L)$ is subject to the following restrictions: $\alpha \le \alpha_U$, $1 - \beta \ge p_L$, $n \ge 1$, $h \ge 0$ and $L \ge 0$. Constant elements of the cost function identify time and economic costs that are described as follows (Rahim & Banerjee, 1993; Chih et al., 2011):

- Z_0 is the expected search time associated with a false alarm (false positive detection of the "out-of-control" state) while Z_1 is the expected search and repairing time associated with a true alarm.
- a and b are the fixed cost per sample and the cost per unit sample respectively.
- D_0 is the expected production cost (per hour) of nonconforming items if the process is "in control" state. D_1 is the expected production cost of nonconforming items if the process is in an "out-of-control" state ($D_1 > D_0$).
- W is the expected cost of searching and repairing an assignable cause of failure (restoring the process to an "in control" state). Y is the expected cost of a false alarm (false positive detection of the "out-of-control" state).
- Finally, δ is the detected shift size of the process and λ is the scale parameter of the failure distribution which (in this case) is Gamma (λ, 2).

Reference Data Sets

As presented in (1) and (2) the cost model considers a total of 12 independent variables: Z_0, Z_1, D_0, D_1, W, Y, a, b, δ, α_U, p_L, and λ. To obtain a general overview of the behaviour of the cost function model, sets of value levels for these variables were considered. These value levels were selected according to the data presented in (Chih et al., 2011) where representative combinations of these levels were defined by an L27 orthogonal array. **Table 1** presents the value levels defined for each independent variable while **Table 2** presents the L27 orthogonal array (Chih et al., 2011).

Table 1. Value levels for the independent variables of the cost function model.

Variable	Level 1	Level 2	Level 3
Z_0	0.025	0.25	0.5
Z_1	0.1	1.0	10
D_0	25	50	100
D_1	475	950	1900
W	550	1100	2200
Y	250	500	1000
a	10	20	40
b	2.11	4.22	8.44
δ	0.25	0.50	1.00
α_U	0.01	0.05	0.1
p_L	0.85	0.9	0.95
λ	0.025	0.05	0.1

In (Chih et al., 2011), the assessment of PSO (Particle Swarm Optimization) was presented as a promising method for solving the ESD problem for X-bar control charts under (1) and (2). The performance of GA (Genetic Algorithms) was presented for comparison purposes. This was important for the present work as the results reported in (Chih et al., 2011) could be used for assessment of the regions described by the proposed 2D visualization method. Hence, **Table 2** includes the best n, h, and L values estimated by the GA method for each combination of the L27 orthogonal array (Chih et al., 2011). These reference data sets are the basis for the discussion presented in Section 3.

Table 2. Orthogonal array and best parameters for the control chart with genetic algorithms (data obtained from Chih et al., 2011).

Trial	Z_0	Z_1	D_0	D_1	W	Y	a	b	δ	α_U	$1-\beta$	λ	n	h	L	$E(C)/E(T)$
1	0.025	0.1	25	475	550	250	10	2.11	0.25	0.01	0.85	0.025	209	12.1305	2.5758	110.293
2	0.025	0.1	25	475	1100	500	20	4.22	0.50	0.05	0.90	0.050	43	6.5401	1.9600	124.138
3	0.025	0.1	25	475	2200	1000	40	8.44	1.00	0.10	0.95	0.100	17	5.3109	2.3862	213.903
4	0.025	1.0	50	950	550	250	10	4.22	0.50	0.05	0.95	0.100	52	3.4424	1.9600	213.171
5	0.025	1.0	50	950	1100	500	20	8.44	1.00	0.10	0.85	0.025	10	4.4158	1.9858	120.963
6	0.025	1.0	50	950	2200	1000	40	2.11	0.25	0.01	0.90	0.050	239	7.2545	2.5758	259.434
7	0.025	10.0	100	1900	550	250	10	8.44	1.00	0.10	0.90	0.050	10	2.0754	1.8181	177.653
8	0.025	10.0	100	1900	1100	500	20	2.11	0.25	0.01	0.95	0.100	286	3.8120	2.5758	334.421
9	0.025	10.0	100	1900	2200	1000	40	4.22	0.50	0.05	0.85	0.025	39	4.3231	1.9774	218.452
10	0.250	0.1	50	1900	550	500	40	2.11	0.50	0.10	0.85	0.050	41	2.3956	1.9233	190.626
11	0.250	0.1	50	1900	1100	1000	10	4.22	1.00	0.01	0.90	0.100	15	1.2664	2.5758	235.198
12	0.250	0.1	50	1900	2200	250	20	8.44	0.25	0.05	0.95	0.025	208	12.9624	1.9600	364.356
13	0.250	1.0	100	475	550	500	40	4.22	1.00	0.01	0.95	0.025	18	7.2209	2.5758	139.278
14	0.250	1.0	100	475	1100	1000	10	8.44	0.25	0.05	0.85	0.050	144	19.1449	1.9600	272.858
15	0.250	1.0	100	475	2200	250	20	2.11	0.50	0.10	0.90	0.100	35	3.7494	1.6449	254.619
16	0.250	10.0	25	950	550	500	40	8.44	0.25	0.05	0.90	0.100	169	8.5765	1.9600	294.623
17	0.250	10.0	25	950	1100	1000	10	2.11	0.50	0.10	0.95	0.025	65	5.2409	2.3863	90.688
18	0.250	10.0	25	950	2200	250	20	4.22	1.00	0.01	0.85	0.050	14	2.4755	2.5758	117.582
19	0.500	0.1	100	950	550	1000	20	2.11	1.00	0.05	0.85	0.100	18	1.7297	2.7208	201.686
20	0.500	0.1	100	950	1100	250	40	4.22	0.25	0.10	0.90	0.025	138	10.8119	1.6449	235.680
21	0.500	0.1	100	950	2200	500	10	8.44	0.50	0.01	0.95	0.050	72	8.4034	2.5758	310.532
22	0.500	1.0	25	1900	550	1000	20	4.22	0.25	0.10	0.95	0.050	174	6.1854	1.6449	311.456
23	0.500	1.0	25	1900	1100	250	40	8.44	0.50	0.01	0.85	0.100	53	3.1141	2.5758	390.370
24	0.500	1.0	25	1900	2200	500	10	2.11	1.00	0.05	0.90	0.025	15	1.8801	2.5090	101.673
25	0.500	10.0	50	475	550	1000	20	8.44	0.50	0.01	0.90	0.025	60	14.1236	2.5758	121.502
26	0.500	10.0	50	475	1100	250	40	2.11	1.00	0.05	0.95	0.050	15	3.8802	2.1575	94.864
27	0.500	10.0	50	475	2200	500	10	4.22	0.25	0.10	0.85	0.100	116	7.7401	1.6449	209.789

3. Results

3.1. The MCS 2D Contour Algorithm

The following steps were performed to process the MCS 2D visualization data for the contour plots of the minimum regions for $F(n, h, L) = E(C)/E(T)$:

1) Initialize the range vectors for the different values of n, h, and L. In this case, 20 different values within the range {1, 325} and {1, 20} were considered for n and h respectively. These values were stored in the vectors rang_n and rang_h. For L, 14 different values were considered. These were stored in the vector rang_L.

2) Estimate $E(C)/E(T)$ for each set of n, h, and L values defined by rang_n, rang_h and rang_L. This was performed as follows:

a) For each set of n, h, and L values, $E(C)/E(T)$ was estimated. This led to the creation of a square matrix of $E(C)/E(T)$ values for a particular L value.

b) Each $E(C)/E(T)$ matrix is stored within a three-dimensional array called costs. While the x and y axes are associated to the values of rang_h and rang_n respectively, the z-axis is associated to the values of rang_L.

3) Normalize the values within the array costs to pixel values. This was performed in two steps:

a) Normalize the values of costs for each L to {0, 1}:

$$\text{Vnorm}_L = \frac{\text{costs}_L - \min(\text{costs}_L)}{\max(\text{costs}_L) - \min(\text{costs}_L)} \tag{3}$$

b) Scale the normalized values of costs for each L (Vnorm_L) to {1, 255} (pixel values):

$$Pix_L = (255-1)(\text{Vnorm}_L)+1. \tag{4}$$

4) Keep only the pixels that represent minimum cost values. If a pixel in Pix_L has a magnitude higher than 1.0 then it gets a constant value of 0; however, if it has a magnitude less or equal to 1.0 then it gets a constant value of 1.0.

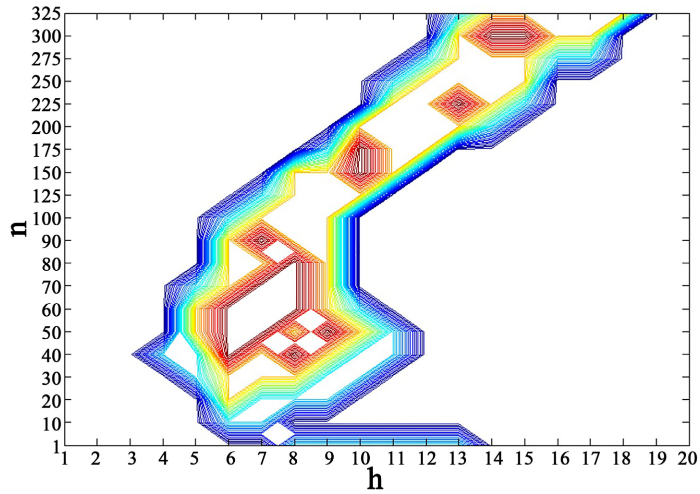
5) Estimation of minimum cost regions through all L values (estimation of MCS values).

This was performed by adding all normalized (and scaled) Pix_L matrices through all L values (z-axis). This led to the creation of a single matrix called minimum_regions where the region is most likely to contain overall minimum cost values have higher values of minimum cost scores (MCS) which are given by the cumulative sum of Pix_L's.

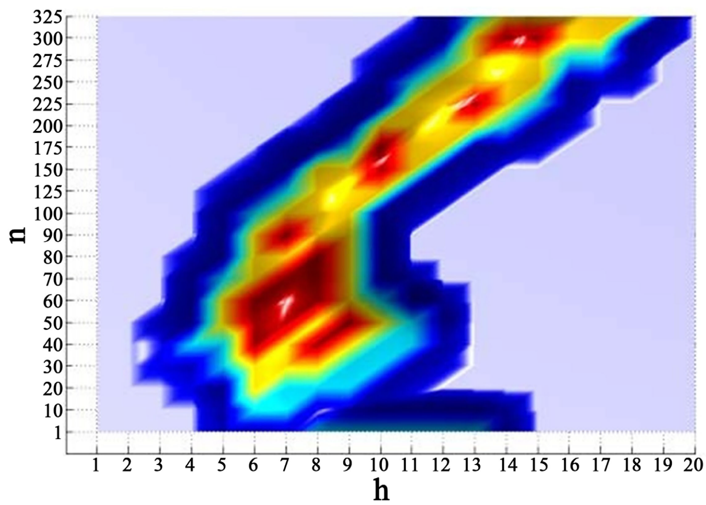
The minimum_regions matrix presents the following advantages: (a) it only contains integer MCS values within the range {0, 14}, (b) its numerical data (MCS values) can be used directly by the solving algorithm to adjust the limit restrictions for the chart's parameters (and thus, reduce the search space), (c) it can be visualized in the 2D domain with any contour or surface visualization method. **Figure 1(a)** presents an example of the minimum_regions matrix in its numerical form (MCS values) while **Figure 1(b)** and **Figure 1(c)** present two different contour visualizations of the same matrix.

0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	2	2	2	0	0
0	0	0	0	0	0	0	0	0	0	0	0	2	3	3	2	2	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	2	2	2	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	2	2	2	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	2	2	3	2	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	2	2	2	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	3	2	2	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	2	2	3	2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	2	2	2	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	2	2	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	2	3	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	3	3	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	3	2	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	3	3	3	2	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	3	3	2	3	2	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	2	2	2	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0
0	0	0	1	1	3	2	3	2	1	1	0	0	0	0	0	0	0	0	0

(a)



(b)



(c)

Figure 1. Examples of numerical data (MCS) and 2D contour visualizations from a minimum_regions matrix. (a) Numerical Data (MCS); (b) Contour Visualization 1; (c) Contour Visualization 2 (Surface).

As presented in Figure 1(b) and Figure 1(c), minimum regions that were consistent through L have the highest numerical data (highest MCS values). These regions are represented with dark red tones. Blue tones represent minimum regions that were observed just for very few (if any) values of L. In the following section, the visualization of the minimum regions for all the cases (trails) of Table 2 is presented and discussed.

3.2. MCS 2D Contour Plots

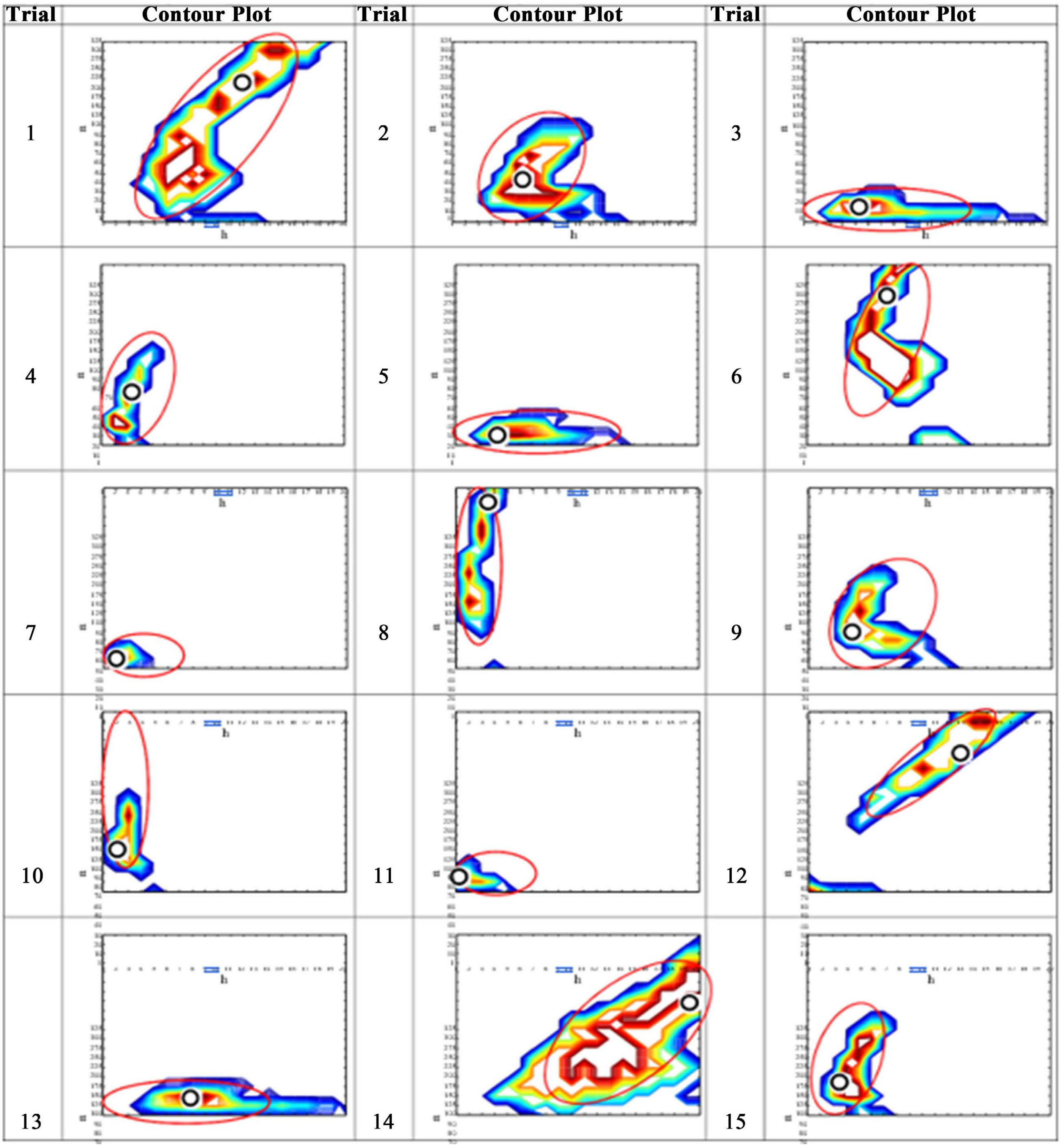


Figure 2. MCS 2D contour plots for the trails of the orthogonal array (Part I).

Figure 2 and Figure 3 present the MCS 2D visualization contour plots for the minimum_regions matrix of each trail from the L27 orthogonal array (see Table 2). The x-y coordinate points defined by the best n-h values reported in Table 2 (Chih et al., 2011) are plotted within the contour plots. These points are identified with the symbol "O".

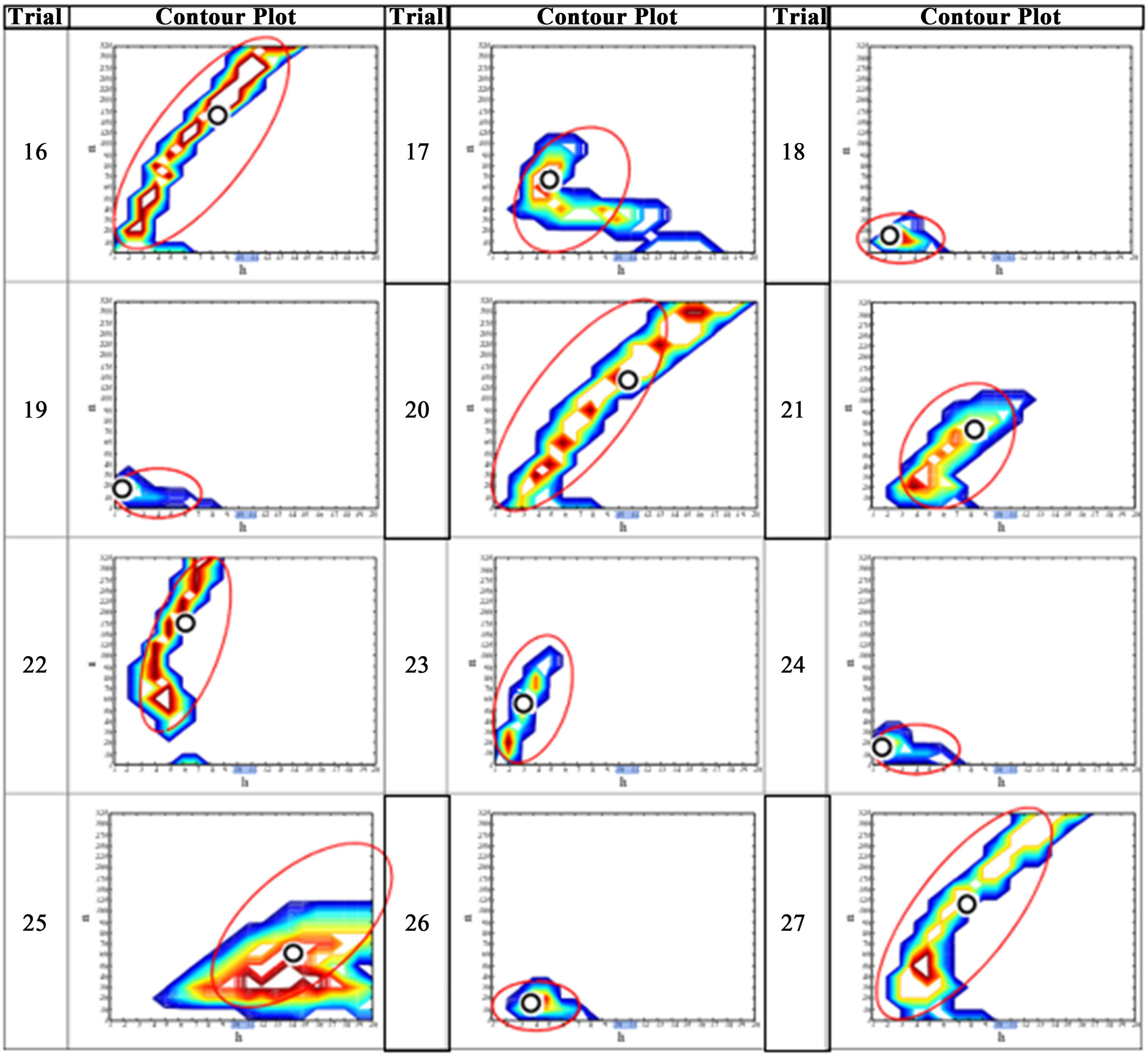


Figure 3. MCS 2D contour plots for the trails of the orthogonal array (Part II).

The contours of the minimum regions presented in Figure 2 and Figure 3 were observed to be elliptical contours. Figure 4 presents the ten elliptical contours that were identified to cover the minimum regions of the 27 trails. These elliptical contours were labeled from A-to-I. Note that, although C* is the same size as C, C* is located closer to the y-axis (n). Table 3 presents the elliptical contours that model the minimum regions of each trail.

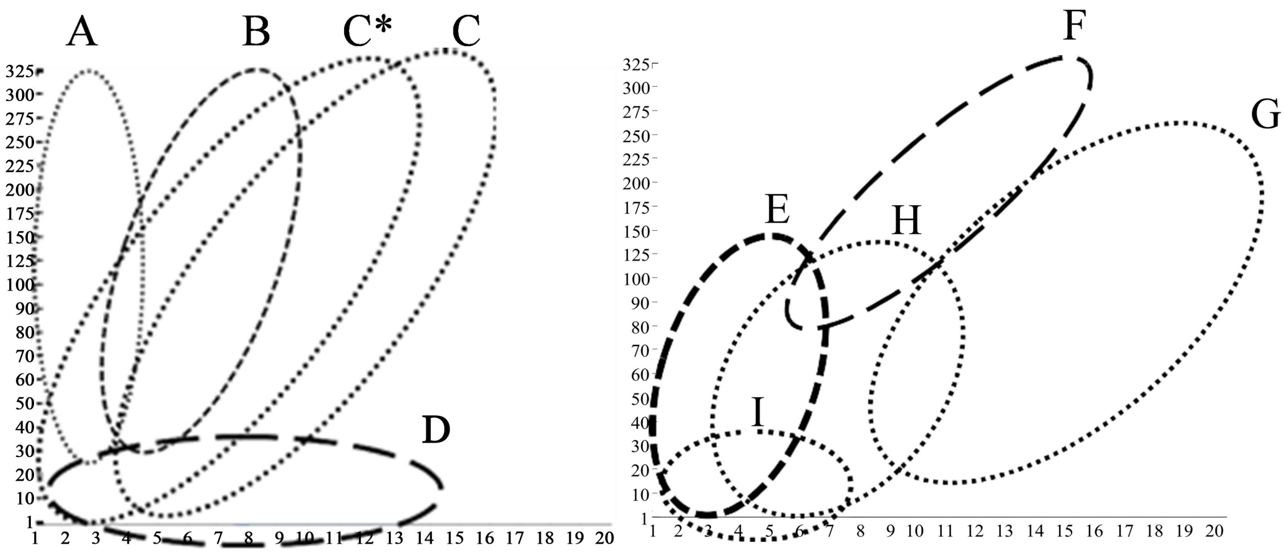


Figure 4. Elliptical contours that cover the MCS 2D minimum regions of the cost function for each trail of the L27 array.

Table 3. Elliptical contours for each trail.

Trail	Elliptical Contour	Trail	Elliptical Contour	Trail	Elliptical Contour
1	C	2	H	3	D
4	E	5	D	6	B
7	I	8	A	9	H
10	A	11	I	12	F
13	D	14	G	15	E
16	C*	17	H	18	I
19	I	20	C*	21	H
22	B	23	E	24	I
25	G	26	I	27	C*

If the search mechanisms of a solving algorithm are only focused on the minimum regions with the highest MCS values, then the size of these regions (and of the search space) can be reduced further. As presented in Figure 2 and Figure 3, for most of the trials, the best n-h pairs (as presented in Table 2) were located within the minimum regions with the highest MCS values. This confirms the advantages and suitability of the proposed approach to improve the search mechanisms of solving algorithms.

4. Discussion

This paper presented a fast estimator of the solution space for a cost function model for the economic statistical design (ESD) of X-bar control charts. This estimator introduced the concept of 2D visualization of minimum cost scores (MCS) to illustrate the patterns of the cost function model for the ESD problem.

It was observed that the minimum regions of the cost function for different reference data sets could be modelled with elliptical boundaries. Also, it was observed that sub-regions with the highest MCS were more likely to contain the optimal values for the chart's parameters. Hence, this approach can be integrated within a heuristic method to perform an initial sampling of the solution space, making the search process more specific.

Nevertheless, more work must be performed to measure the suitability of this approach. The following points are considered for future work:

- Explore alternatives to (or improvements on) the metric of minimum cost scores (MCS). Because this metric can be seen as a coding metric, other metrics can be considered for representation of minimum values for a cost function model. This could make the search processes of heuristic and exact-solving algorithms faster.
- Apply the MCS 2D visualization approach to track the search performance of solving algorithms.

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Conflicts of Interest

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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