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Definite Answer for Riemann Hypothesis Zeta 3/2 Function Provided by New Material $Yb_2Si_2O_7$ in Quantum Mechanics

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Abstract

This paper indicates the problem of the famous Riemann hypothesis (RH), which has been well-verified by a definite answering method using a Bose-Einstein Condensate (BEC) phase. We adopt mathematical induction, mappings, and laser photons governed by electromagnetically induced transparency (EIT) to examine the existence of the RH. In considering the well-developed as Riemann zeta function, we find that the existence of RH has a corrected and self-consistent solution. Specifically, there is the only one pole at $s = 1$ on the complex plane for Riemann's functions, which generalizes to all non-trivial zeros while $s > 1$. The essential solution is based on the BEC phases and on the nature of the laser photon(s). This work also incorporates Heisenberg commutators $[\hat{x}, \hat{p}] = 1/2$ in the field of quantum mechanics. We found that a satisfactory solution for the RH would be incomplete without the formalism of Heisenberg commutators, BEC phases, and EIT effects. Ultimately, we propose the application of qubits in connection with the RH.

Keywords

BEC Phases, EIT, Heisenberg Commutators, Laser Photons, Qubits, Riemann Hypothesis

1. Introduction

The study of the zeta function and its zeros by Riemann originated from his explicit formula for the number of primes $\pi(x)$ less than or equal to a given x, which he published in his 1859 paper, "On the Number of Primes Less Than a

Given Magnitude". This formula was described in terms of the related function [1]. Riemann's explicit formula relates the number of primes less than a given number to a sum over the zeros of the Riemann zeta function, indicating that the magnitude of the oscillations of primes around their expected positions is controlled by the real parts of the zeros of the zeta function. In particular, the error term in the prime number theorem is closely related to the position of these zeros; for example, if β is the upper bound of the real parts of the zeros. In harmonic analysis, third harmonic generation (THG) laser is achieved by taking a base wavelength of 1064 nm and by multiplying it by 1/3. Clearly one can explore related categories of these issues to address the RH problem. Von Koch proved that the RH implies the best possible bound for the error in the prime number theorem (1901) [2]. The prime number theorem suggests that, on average, the individual gap between a prime p and the successor is log p. However, some gaps between primes can be much larger than this average. Cramér proved that, assuming the RH, each gap is $O\left(\sqrt{p}\log p\right)$. This is a case where even the best bound that can be proved using the RH is far weaker than what seems true: Cramér's conjecture implies that every gap is $O\left((\log p)^2\right)$ much smaller than the bound implied by the RH, even though it is larger than the average gap. In addition, numerical evidence supports Cramér's conjecture [3]. Several applications use the generalized RH for Dirichlet's L-series or the zeta functions of number fields rather than the RH itself. The fundamental properties of the Riemann zeta function can without doubt be generalized to all Dirichlet's L-series. Therefore, it is plausible that a method proving the RH for the Riemann zeta function would also apply to the generalized RH for Dirichlet's L-functions. Hardy and Littlewood showed that the generalized RH implies a conjecture of Chebyshev $\lim_{x\to 1^-}\sum_{p>2}(-1)^{\frac{p+1}{2}}x^p=\infty$ (for $p > 2$), which suggests that primes 3 mod 4 are more common than primes 1 mod 4 in some sense (1921) [4]. They also showed that the generalized RH implies a weak form of the Goldbach conjecture for odd numbers: that every sufficiently large odd number is the sum of three primes (1923). Pólya proved that the Riemann hypothesis is equivalent to the hyperbolicity of Jensen polynomials for the Riemann zeta function at the point of symmetry (1927) [5]. The hyperbolicity has been proved for degrees $d \le 3$. The canonical commutation rule for the position x and the momentum p variables of a particle $[\hat{x},\hat{p}]=\hat{p}\hat{x}-\hat{x}\hat{p}=h/2\pi i$ (Heisenberg Uncertainty Principle). Chowla showed that the generalized RH implies that the first prime in the arithmetic progression a mod m is at most $K\cdot m^2\log(m)^2$ for some fixed constant K (1934) [6]. By utilizing the harmonic analysis, the zero points of the Riemann zeta function can be viewed as harmonics of the prime number distribution, as described by Lowell Schoenfeld (1962) [7]. The Paley-Wiener theorem provides the fundamental form of the Heisenberg Uncertainty Principle in harmonic analysis [8]-[11]. Weinberger showed that the generalized RH implies that Euler's list of ideal numbers is complete (1972) [12]. Odlyzko discussed how the generalized RH could be used to give sharper estimates for discriminants and class numbers of number fields (1990) [13]. Ono and Soundararajan showed that the generalized RH implies that Ramanujan's integral quadratic form $x^2 + y^2 + 10z^2$ represents all integers that it represents locally with exactly 18 exceptions (1997) [14]. Alexander Dunn and Maksym Radziwill proved Patterson's conjecture under the assumption of the GRH (2021) [15]. E. Bombieri has stated that "in the opinion of prevalent mathematicians the RH and the extension to general classes of L-functions, is probably today the most important open problem in pure mathematics" in his article titled "Problems of the Millennium: RH". From 2023 to 2024, there have recently been few physical methods introduced in pure mathematics to solve the problem, beyond such failure works. In contrast, our work in this paper assumes that an approach based on the spectrum structure of the Heisenberg commutators is satisfactory and successful. This approach applies concepts from pure mathematics to pure physics and then returns them to pure mathematics. In addition, using BEC phases and EIT effects [16]-[21] enhances the efficiency of this method: UVC ranges of EIT effects extend the lifetime of BEC phases. This is achieved because the hole $s := n = 0$ is uniquely laser-engraved by UVC wavelength ranges around 222 nm (*i.e.*, the excimer laser KrCl*), with EIT locking BEC modes (*i.e.*, optical phase lock) without heat production. As a result, s = 1 must be retained for that the only pole at s = 1 is well-behaved as a BEC vortex core. Here $n = 0$ is represented as ground state, which leads to the arrival of a BEC phase ($\lambda_{-Si-} \approx 1.1\,\mu\text{m}$, $T_c \to 0\text{ K}$) and simultaneously all prime numbers of atoms/molecules are evaporated from Magneto-Optical Trap (MOT) to ensemble of $n = 1/2 + i\alpha = (E_0/\hbar\omega) + i\alpha, n \in C$ (*i.e.*, which satisfies the Dirichlet's boundary conditions) thus the RH problem is completely solved by this paper for the first time.

2. Method

2.1. Formulism: Spectrum Structures (Mappings & Enclosure Adjoints)

If one asks a question: what are the relationships between Heisenberg commutators and the Riemann Hypothesis? What is the spectrum? Are their ground states the same? Start with $s = B + iH > 1$ and[1] the sequence $s^+ = B - iH^+ > 1$ ($B \in R$) [Heisenberg commutators in quantum mechanics (Q.M.)] and with widely-

[1]Associated with $s = B + iH > 1$ and Equation (2), this causes $\frac{3}{4} < \alpha < 1$ as the same as the disparate ranges of α by statement of theorem by Carlson (1920) [22]. However, if $i\alpha \equiv 0$ in complex planes (see the Riemann sphere) then Ingham's exponent (1940) [23] would be re-denoted as $3(1-\alpha)/(2-\alpha)\big|_{\alpha=0} = 3/2 = s$, such that it returns to $\zeta(3/2)$ and $\frac{1}{2} + i\alpha = \frac{1}{2}$ *i.e.*, all non-trivial zeros of $\zeta(s)$ have real part equal to 1/2. Namely the Riemann Hypothesis solution depends on BEC phasing term $\zeta(3/2)$. Note that $\alpha \equiv \Theta = 2\tan^{-1}\left(\left(D_f - D_i\right)/(2d)\right)$ laser $D_f = D_i$ results in $\alpha = 0$ for the BEC photon polarization. Note that thi$s$ is not indicated by spin quantum number.

known that $\zeta(1/2+iH)=0$ [2]. Where let

$$B \equiv 1/2 \tag{1}$$

If $s^2 = B\cdot B + 2iBH - H\cdot H^{+} > 1$ can be established so that

$$\begin{aligned} &1/4 + iH - H\cdot H^{+} > 1, \\ &iH - H\cdot H^{+} > 3/4 \end{aligned} \tag{2}$$

Obviously

$$2\left(iH - H\cdot H^{+}\right) > 3/2 \tag{3}$$

With operator $H = n + 1/2$ making quantum numbers of $n \in R$ in a particular phase to be mapping onto *s*-components in Riemann zeta functions in a physical system.

Therefore,

$$H = s + 1/2,\ s := n \tag{4}$$

And simultaneously the adjoint-operator in terms of $H^{+} = s^{+} - 1/2$ is kept for enclosure. Therefore,

$$2(s+1/2)\left(i - H^{+}\right) > 3/2 \tag{5}$$

This paper demonstrates that there exists a spectrum set being both for real and the complex numbers as structures below since the ***s-region*** is smooth and continuous. In case of simple harmonic oscillators (SHO), the superposition of wavefunctions with $n = 0,1,2,3,\cdots$ is overcome with a complete set expressed as sinc-function that the wave function of laser by the spectrum of observable Heisenberg photons appears. Therefore,

$$\left.\begin{aligned} &\zeta\left(2(s+1/2)(i+H)\right) > \zeta(3/2), \forall s>1, s\in R \\ &\zeta\left(2(s+1/2)\left(i-H^{+}\right)\right) > \zeta(3/2), \forall s^{+} \in C \end{aligned}\right\} \tag{6}$$

And $\left(\zeta(3/2)\right)^{n}$ to the power $n \in R$ (in later sections, one can see that $n = p = 3,7$ definitely in BEC phases). As far as one can see the problem of Riemann hypothesis must be relative to $\zeta(3/2)$ in expressions of representation by Heisenberg commutators[3]. Above is clear and powerful as it pertains directly to the Riemann Hypothesis (RH).

2.2. Riemann Zeta Function $\zeta(3/2)$ Used in BEC Phases with Fermionic Condensate in Cooper Pairs

To be continued with **Sect. 2.1** here, cited here as a widely known formula:

$$T_C = \left(\frac{n}{\zeta(3/2)}\right)^{2/3} \frac{2\pi\hbar^2}{mk_B} \propto \frac{1}{\left(\zeta(3/2)\right)^{2/3}} \tag{7}$$

[2]See Michael Berry and Jon Keating's work (1999), Ref. [24].

[3]This is not a coincidence but based on an unexplored reason, such as homology. Additionally, for the above sections, one can refer to *talk of Adjoint transformation of gauge fields*. See the topic about "transformation-of-gauge-fields" (2014) in Physics Forums, where the idea of $\hat{U} \equiv \hat{1} + i\omega^{a}T_{F}^{a}$ is similar with $s = 1/2 + it$ where one can define $t \equiv \alpha$.

If the densities of the particles in BEC phases are fixed, then the relationships between the critical temperature and the Riemann zeta function $\left(\zeta\left(3/2\right)\right)^{-2/3} = const$ shown as Equation (7). This approach offers a reputable way of significantly simplifying a problem through pure physics; for example, the prime number 3 can be considered in mathematics. This paper suggests that the critical temperature of BEC phases can be represented as Riemann zeta function of 3/2. Moreover, associated with spectrum in **Sect. 2.1**, $\zeta\left(3/2\right)$ is considered (**Figure 1**).

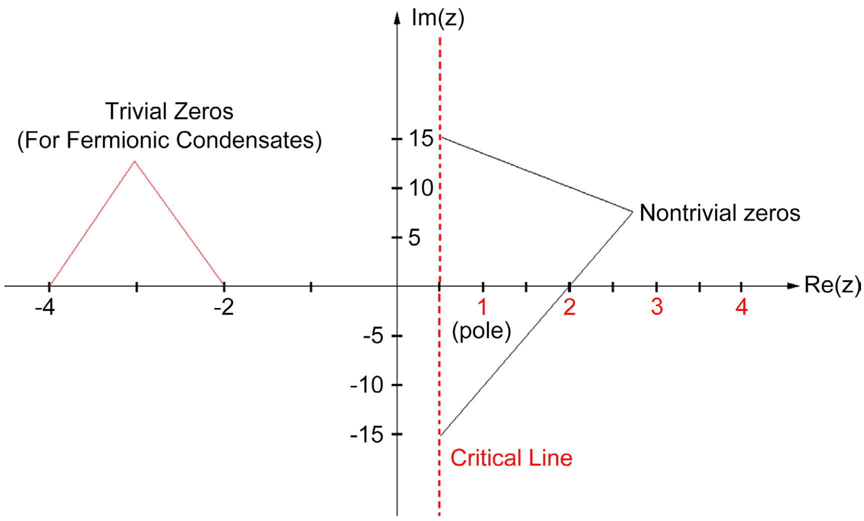

Figure 1. Given that the complex plane is associated with a BEC phase (*i.e.*, the regions on the right-hand side), one can quickly derive the solution to the RH. The BEC phase ($s :\neq n$) SHO quantum numbers, in terms of $s = \left(n+1/2\right)\big|_{n=0}$ for cooling ultra-cold atoms (with some of them escaped from magneto-optical traps (MOT)), are represented as prime numbers located on the critical line of $s = 1/2 + it$ where $|t| \to \infty$.

3. Results

3.1 Mathematical Induction: Substituting *p* = 3

Given Riemann zeta functions as

$$\zeta\left(s\right) = \prod_p \frac{1}{1-p^{-s}}, \quad s > 1,$$
$$\zeta\left(s\right) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\cdots}, \quad s > 1 \tag{8}$$

where $s := n$ is denoted as the quantum numbers of BEC phases in complex-planes. As far as one can see, since it is associated with

$$T_p = \left(1 + \frac{1}{10^p}\right) T_c \neq 0, \quad -\infty < p < \infty \tag{9}$$

Note that Equation (9) is an empirical expression based on data from the Cornell University Laboratory (1995), denoted as p = 7 for ~10^{-7} K (*i.e.*, the energy spectrum of photons of the BEC-laser).

Associated with the Bogoliubov transformation in Bardeen–Cooper–Schrieffer (BCS) theory,

$$E = \frac{hc}{\lambda} = 3.52 k_B T_c \sqrt{1 - \frac{T}{T_c}} \tag{10}$$

Therefore, in the fine-tuning process, where the parameter p must be self-regulated as $p = \|[3.52]\| = 3$ in progress[4]. As for the other terms of prime numbers, they need to be moved onto the left-hand side (LHS). Hence:

$$K \cdot \zeta(s) = \frac{1}{\left(1 - \frac{1}{3^s}\right)}, s > 1 \tag{11}$$

Where defined as

$$K \equiv \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{11^s}\right)\cdots, s > 1 \text{ for all prime numbers} \tag{12}$$

e.g., let $s = 3/2$ in a BEC phase, and next let $K\zeta(3/2)$ in terms of $\zeta'(3/2)$. Namely,

$$\begin{aligned}
&\zeta'(3/2) = \frac{1}{1 - 3^{-3/2}} = -0.2383135547 \equiv C \neq 0, \\
&\vdots \\
&\zeta'(s) = \frac{1}{1 - 3^{-s}} = C, \forall s > 1, \\
&\zeta'(s) - C = 0, \forall s > 1
\end{aligned} \tag{13}$$

On the critical line where $\mathrm{Re}(s) = 1/2$, it obviously turns to $\zeta(s) = 0, \forall s > 1$ for all *trivial zeros* (definitely with the critical line where $\mathrm{Re}(s) = 1/2$). This implies that a BEC phase on arbitrary atoms could possibly be denoted by Equation (13).

The famous *fine structure constant*, whose reciprocal approximates the prime $p = 137$, could have been included by this mathematical induction if $p = 137$ were picked. Hence, the method stated by the "Todd function" (2019) was in vain in the results[5] $\alpha = 1/137$.

As a running point, α is here assumed to be represented by a matrix and calculated via a statement of interactions. Note that the above p is not denoted as a quantum number, so one should be careful with it.

Equation (14) implies that the valence electrons of *Si atoms* or *Si-topological* superconductors (e.g., $Yb_2Si_2O_7$) could form a fermionic condensate (Cooper pairs), which is permitted. The total bulk of this material transitions into BEC phases. Additionally, due to the similar chemical periodic properties of Na/Si and

[4]The detailed solution is relative to the energy-gap of $Yb_2Si_2O_7$ in a BEC phase in solid-state physics or statistical mechanics. For instance, as shown:

$$\left|E_g(\mathrm{Si}, 0\ \mathrm{K})\right| = \frac{1}{1000} E_g(\mathrm{Si}, 300\ \mathrm{K}) = \frac{1}{10^3}(1.12\ \mathrm{eV}) = 0.00112\ \mathrm{eV} \approx 10^{-3}\ \mathrm{eV} \tag{14}$$

This complies with the requirements of BCS theory. The indicated $p = 3$ is a type of fine-tuning.

[5]For the relevant depiction, one can see Appendix B.

Cs/Yb, the connection of the Riemann Hypothesis (RH) to BEC phase materials (e.g., NaCs or $Yb_2Si_2O_7$) and the molecule numbers of a BEC phase as an observable distribution of prime numbers establishes a clear relation between the RH and material science, which is universal.

Referring to the paper cited by Figure 5 of Ref. [25], we have solved this problem in perspective: Figure 5 shows that the molecule quantities as prime numbers are clearly computed: 41, 53, 79, 97, 127, 163, 179, 181, 211, 241, 251, 257 on the list from RHS to LHS of the x-axis (Hold time as shown) in Figure 5 of Ref. [25]. From a quantum mechanical point of view, the coherent state of the unique BEC phase broadly reveals that the state of one molecule is inseparable from another (molecule-to-molecule, *i.e.*, molecular clouds) into two arbitrarily different states and must retain a unique state (*i.e.*, a coherent state). In quantum mechanics (e.g., Heisenberg commutators), the distribution of prime numbers naturally satisfies the characteristics of BEC phases. As a result, such matter strongly supports our viewpoints in this paper, sufficiently and convincingly. Likewise, our relevant derivations are undoubtedly confirmed.

Note that the physical significance of the Simple Harmonic Oscillator (SHO) average energy, where the chemical covalent bond is directed to the electron cloud, acts as an electron-gas shell around absolute zero. This forms simple harmonically-mixed condensed states while Yb and Si are cooling. Therefore, based on the Virial theorem, the Hamiltonian of the system is indicated as $\bar{E} = 2\bar{K} = 2\bar{V}$. Via approximation, one can only consider Si atoms which contribute the average kinetic energy to $Yb_2Si_2O_7$ in crystals, leading to Equation (14). Therefore, Yb atoms are regarded as a bound state having an average potential energy $\bar{V}$. The prime number $p \equiv 3$ is naturally options associated with Equation (9) for the Cooper pairs' energy $E_g \le 10^{-3}$ eV required.

3.2. Statements: Riemann Hypothesis (RH)

The Riemann series of absent $s > 1$ *i.e.*, $\zeta(s) = \infty$, $s = 0$ *i.e.*, $\zeta(0) = \infty$ [laser unique spot appears in BEC phases by using the famous EIT effects (quantum storages)[6] this implies singularity point $\zeta(0) = \infty$]. The BEC phase gives the atomic physical systems of the coherent states in complex spaces to be indicated as photons of laser [UVC ranges with metastable state $i \cdot n$ of s^+ (see the text on p-10) and $s^+ = s \to 2$ e.g., $n = s = 2^- > 1$ see **Figure 2**] to incident into a BEC phase to provide this phase as presentation stage, hence $\zeta(s) = 0$, $\forall s = -4, -2$ stands for *all trivial zeros*. However, if the Electromagnetically Induced Transparency (EIT) photons act on a BEC phase ($s = 1$ being in a BEC vortex core, see later sections), they could be retained for a longer lifetime due to this vortex core, thereby maintaining the BEC phase[7]. This causes:

$$\zeta(s) = 0, \forall s = -4, -2 \tag{15}$$

[6]Quantum entanglement promotes the exchange of wave functions between photon-electron pairs, causing the quantum numbers ($s = 1$) of the photons in a laser (*i.e.*, a BEC beam) to transfer onto the electron quantum numbers ($s = 1$).

[7]The same statements are as the same as above.

The BEC phase (where $s = 1$ exists in a BEC vortex core, see later sections) can be retained for a longer life-time, and then the BEC phase is remained.

As widely-known as, a beam of laser photons in UVC ranges can sculpt a hole (*i.e.*, with $t=\left[i/2,-i/2\right]$ to product one of results: $s=1/2+it$ where obtained s = 0 (see the famous **Riemann spheres**). Obviously, s = 0 indicated as absence of bosons (e.g., no photon gas) such naturally directs s = 0 itself to be the lowest energy-level within the material of $Yb_2Si_2O_7$ stationed in BEC phases. Based on this, the *Riemann hypothesis* is established, *i.e.*, the statement of the RH would be given by

$$\sum_{i=prime}\zeta_i\left(s\right)=\infty,\ s>1\ \text{ with }\ \operatorname{Re}\left(s\right)=\frac{1}{2},0\le s<2,\forall s=0 \tag{16}$$

Namely, for all prime numbers, there exists a unique pole $\zeta\left(s\right)=1$ as one of the non-trivial zero points of $\zeta\left(s\right)=\infty$, $s=0$ when considering both BEC phases and EIT effects applied in a wave range, e.g., with $\lambda=226.43\,\text{nm}$ UVC photons (see Appendix C for the calculations). We claim that this is a singular point in BEC phases, *i.e.*, the existence of the Riemann Hypothesis (RH) is supported by Equation (16). This is precisely the widely-expected definite solution to the Clay Mathematics Institute Millennium Prize Problem, which has been well-solved by this paper for the first time. We are justified in making the above statements and relevant conclusions. Note that above $\sum_{i=prime}\zeta_i\left(s\right)=\infty,\ s>1$, actually locates on 2D plane (see RHS of Figure 1).

The laser cooling (statistical mechanics): Consider that an atomic system of *reversal population* which has the reversion of Boltzmann distribution

$$N\propto \mathrm{e}^{-E/kT} \tag{17}$$

Let $E=nh\nu,n=1,2,3,\cdots$ while the laser photons incident into the energy-level of exited state in metastable states, such that

$$N\propto \mathrm{e}^{inh\nu/\left(-ikT\right)} \tag{18}$$

Moreover,

$$\mathrm{e}^{-nh\nu/kT}=\mathrm{e}^{inh\nu/\left(-ikT\right)}=\mathrm{e}^{in\hbar\omega/\left(-ikT\right)} \tag{19}$$

where

$$\kappa\equiv -ik \tag{20}$$

Hence

$$\mathrm{e}^{in\hbar\omega/\kappa T}=\cos\left(n\hbar\omega/\kappa T\right)+i\sin\left(i\cdot n\hbar\omega/kT\right) \tag{21}$$

Such gives $i\cdot n$ of s^{+} where atoms/or molecules occupied metastable states in case of $n>s^{+}$ since $\left.\begin{array}{l}\hat{s}:=s=n-\dfrac{1}{2}\equiv\hat{H}\\ \hat{s}^{+}:=s^{+}=n+\dfrac{1}{2}\equiv i\hat{H}\end{array}\right\}$ where $\hat{s}^{+}$ could be denoted as raising operator and $\hat{s}$ as lowering operator definitely $n:=\hat{n}$ in quantum mechanics. In an atomic system in reversal population (abbreviated as RP) with consideration of sufficient larger numbers of atoms $n\gg 1/2$ while the laser-RP is applied[8].

Therefore $s^{+} = s = n_1$ if $n \in R$. Defined $s := s^{+} = in_2$ if $n \in \mathrm{Im}$ [9] and $i\hat{H} := i\hat{n}$ strongly leaves alone that the real number 1/2 as the **critical line** such that causes all real parts of non-trivial zeros to be located on $\mathrm{Re}(s) = 1/2 = [\hat{x}, \hat{p}]$. See **Figure 2**.

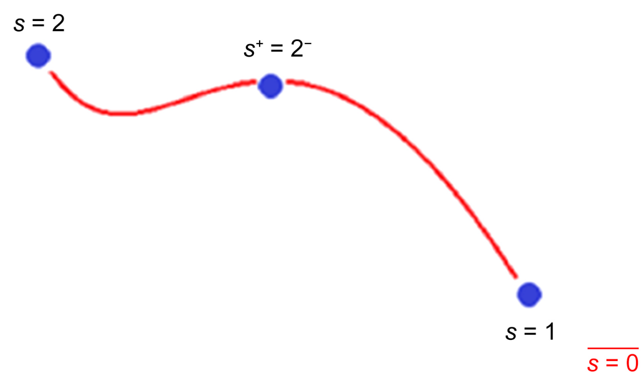

Figure 2. The Metastable State ($s^{+} = 2^{-}$) of atoms or electrons in the Riemann zeta function $\zeta(3/2)$ in a BEC phase. The population inversion by laser is indicated as greater numbers of electrons exist in $s = 2$ rather than in $s = 1$. However, the BEC phase demands it to exist in $s = 1$ much greater than in $s = 2$. At *absolute zero*, all particles located on the lowest energy-level (the ground state (g.s.)) that have zero-point energy and then $s := n = \zeta(1/2 + iH) = 0$ denoted as g.s. such implies the complete solution for non-trivial zeros existing on *the critical line* with $\mathrm{Re}(s) = \frac{1}{2}$, $0 \le s < 2$, $n := s$ (fits Dirichlet's boundary conditions).

Table 1. The statements of optical pumps in quantum mechanics shown. The pumps as functions of a *metastable state* hence terms of probability density of ***state 1*** is illustrated to be projected by $\langle s^{+} | \cdot | s \rangle$ such that a group of popular electrons located on ***state 1***. For the same reason, but ***state 2*** requires a work done by external circumstances (*i.e.*, the negative sign for ***state 2***).

——	Projection 1	Projection 2	Notes
Functions	$\langle s^{+} s \rangle = n_1^2$ ($T > 0\,\mathrm{K}$) (22)	$\langle s s^{+} \rangle = -n_2^2$ ($T > 0\,\mathrm{K}$) (23)	s indicated as a specific state of atoms.
Physical Significances	Spontaneous Emission (*i.e.*, Normal Distribution). And $\langle s^{+} s \rangle = n_1^2 = 0$ ($T = 0\,\mathrm{K}$) (24)	Pump-up. A work done by $E = nh\nu$ (Stimulated emission). (25)	The photon gas for **projection 1**: $f_V(v_x, v_y, v_z)\, dv_x dv_y dv_z > 0$ (26) The photon gas for **projection 2**: $f_V(v_x, v_y, v_z)[-dv_x][-dv_y][-dv_z] < 0$ (27)

[8]See Appendix E.

[9]Such that $\langle s^{+} s \rangle = n_1^2$, $\langle s s^{+} \rangle = -n_2^2$, $\sum_{s^{+} = n + 1/2} \lim_{\substack{n_1 \to n_2 \\ T \to 0\,\mathrm{K}}} \vec{j} \equiv \vec{j} = \frac{\hbar}{2mi}\left(\langle s^{+} s \rangle \mp \langle s s^{+} \rangle\right) = 0$ (the original negative sign is not a choice due to laser-RP (the conjugated image part)) produces the *conservation of probability density flux* which obeys the fundamental principle in quantum mechanics. The above can be written as $\vec{j} = \frac{1}{2\pi i} \frac{h}{2m}\left(\langle s^{+} s \rangle \mp \langle s s^{+} \rangle\right)$, which is a more suitable expression.

Important that by A. Einstein "The Quantum Theory of Radiation" (1917) the detailed derivations are as discussing as below:

Given that $\dot{\rho}(v)+\nabla\cdot\vec{j}=0$ where $(\dot{n}_1)_{B_{21}}=0=B_{21}n_2\cdot\rho(v)$, $n_1=Const$, such that $\dot{\rho}(v)\big|_{T=0\mathrm{K}}=0$. Note that $\rho(v)=\lim\limits_{T\to 0\mathrm{K}}\dfrac{2hv^3}{c^3\left(\mathrm{e}^{hv/kT}-1\right)}=0$ while laser is forcing a BEC phase arrival.

Note that the ideas are at the first time (Table 1).

Remark. ***"Einstein General Elevator"***: We generalize the *Einstein's elevator of gravity* (1916-1917) to the version of atoms in a laser system, as shown in Table 2.

Table 2. To highlight the differences between spontaneous emission and stimulated emission, the following distinctions are made.

Transition Types	$s^+\to s$	$s\to s^+$
Physical Mechanisms	$(s^+>s)$ Spontaneous Emission	$(s>s^+)$ Stimulated Emission
Requires $\rho(v)$	No	Yes

Remark. The above mechanism is named as *Einstein's general elevator*. Returning to the previous articles, let us arrange the equation stated by $(\dot{n}_1)_{B_{21}}=B_{21}n_2\rho(v)$ precisely. Yields

$$\rho(v)=\frac{1}{B_{21}}\frac{(\dot{n}_1)_{B_{21}}}{n_2} \tag{28}$$

If

$$n_2\equiv\frac{(\dot{n}_1)_{B_{21}}}{B_{21}}>0 \tag{29}$$

Then[10]

$$\rho(v)\big|_{T>0\mathrm{K}}=1\left[\mathrm{J/m^3}\right],\ \rho(v)\big|_{T=0\mathrm{K}}=0 \tag{30}$$

Or[11]

$$n_2\equiv\frac{(\dot{n}_1)_{A_{12}}}{-A_{12}}>0 \tag{31}$$

where $(\dot{n}_1)_{A_{12}}<0$ such produces

$$\rho(v)\big|_{T>0\mathrm{K}}=1\left[\mathrm{J/m^3}\right],\ \rho(v)\big|_{T=0\mathrm{K}}=0 \tag{32}$$

The reason of $B_{21}\to -A_{12}$ (a sphere mapping to an atom pointed *i.e.*, $[B]\to[A]$) is based on ideas of the G.R. by Einstein. Associated with Einstein's general elevator by this paper, note that ideas of probabilities in quantum mechanics are still established. According to the above derivation, Einstein's A and B coefficients are finally unified into the same concept. This uniformity helps in

[10]In normalization, SI units of energy density are usually discarded.

[11]The footnotes, written in reverse as $-A_{12}\equiv A_{21}$ are caused by the principle of detailed balancing.

understanding the BEC phase or the condensed state of the laser photons (g.s.) in quantum mechanics.

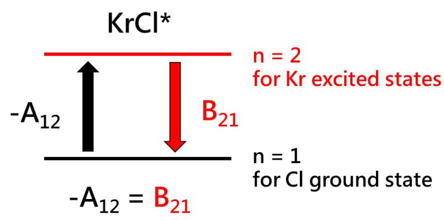

Figure 3. The illustration of $B_{21} \to -A_{12}$ (Taken **excimer molecule KrCl*** as example. See Appendix C) where regarded $-A_{12}$ as given chemical bonds to KrCl*. This is the representation for *Einstein's general elevator* deducted by this paper where Einstein's coefficients of A and B are united by this type of elevator. The excimer molecule laser using KrCl* which radiates wave lengths of 222 nm (smaller than 226.43 nm) of the impulse laser such supports the production of EIT effects. The systematic error is naturally controlled within < 2% and indicated in ranges off frequency response (using color-films to filter the laser: 226.43 nm→222 nm, see the remark below) of optical sensitive which is allowed. The derivation gives the mechanism of the impulse laser (used in EIT effects) for the reason of pumping source itself is an impulse. Note that $s := n = 2$ as excited states or $s^+ := n = 2^-$ as metastable states and $s := n = 1$ as ground states.

Remark. **Gaussian Beam:** By means of adjusting Rayleigh's length: $z_R = \pi\omega_0^2/\lambda$ is used to a positive lens to obtain the suitable beam waist ω_0 and then corresponds to the Rayleigh length and the wavelengths λ obtained. See the *confocal parameter*. On the other hand,

(A) The expressions by **Four-Level laser** in Figure 3:

$$\left.\begin{array}{l} 2 \to 2_m^- \to 2_l^- \to 1, \\ \Delta E = hv_m - hv_l = E_{2_m^-} - E_{2_l^-}, v_m > v_l \\ \Delta E' = 2hv = hv_2 - hv_1, v_2 > v_1 \end{array}\right\} \tag{33}$$

The frequency response:

$$\Delta E \equiv \Delta E' \quad \text{(Stimulated Emission)} \tag{34}$$

where the opposite direction $1 \to 2$ is indicated as chemical bonds of KrCl*.

(B) The expressions by **three-level laser as impulse** in Figure 3:

$$\left.\begin{array}{l} 2 \to \left(2_m^- \to 2_l^-\right)_{\Delta t \to 10^{-15}\,\text{sec}} \to 1, \\ \phi_i - Gain: \\ 2 > F.T.\left\{\left(2_m^- \to 2_l^-\right)_{\delta(t)}\right\} \ge 1, \Delta t \approx \delta(t) \to 10^{-15}\,\text{sec} \\ 2 \to 2^- \to 1 \end{array}\right\} \tag{35}$$

where the B-term complies with EIT effects. Since that there exists $\delta(t) \to 10^{-15}\,\text{sec} \ne 0$ so that the transition of $2_m^- \to 2_l^-$ is not simultaneously, *i.e.*, $2_m^- \equiv |i\rangle, 2_l^- \equiv |f\rangle, i \ne f$ such causes the transition probability per second to keep a constant. e.g., $|\delta(t)|^2 = \begin{cases} 1, g(E_i) = \hbar/(2\pi b), b \ne 0 \\ 1/2\pi, g(E_i) = \hbar/b, b \ne 0 \end{cases}$ for $|i\rangle \to |f\rangle$ where

$E_i \equiv \hbar\omega = E + dE$ ($g(E_i)$ is the density of the photon states at a given energy) and b is indicated as chemical bond lengths of KrCl* and has phase gain of $\phi_i - Gain$.

Given the slope as

$$h = E_n / v_{resp.} = E_{l-m} / v_{resp.} \quad \text{with} \quad \phi_i = 2\pi \tag{36}$$

Or

$$G_{dB} \equiv 20 \cdot \log_{10} \frac{\hbar}{b \cdot E_i / c} \approx -590 \text{ dB} < 0, \text{ for an operation of laser} \tag{37}$$

Which is too weak and human-beings cannot hear/sense this quantum noise where c is the vacuum light-speed (see the *Fermi Golden rule*). Thus appears a strong noise as

$$G_{dB} = 20 \cdot \log_{10} \left. \frac{\hbar}{b \cdot E_i / c} \right|_{b=l_p} \approx 190 \text{ dB} \sim 200 \text{ dB} \tag{38}$$

In ordered Equations (17) to (38), which do not induce any bias to our subjects but are as to discipline to use for persuading one that the RH solution is caused by the BEC-laser cooling, this is due to that the *zero-density* is naturally to be the lowest via formation of the BEC phase.

4. Conclusion

In conclusion, this paper presents the long-sought solution to the Riemann Hypothesis, providing a definitive solution for the first time. The facts reveal that the Riemann Hypothesis is valid and has no counterexample due to the presence of a single pole at $\text{Re}(s) = 1$ and all non-trivial zeros $\sum_{i=prime} \zeta_i(s) = \infty, s > 1$ located on the critical line with $\text{Re}(s) = \frac{1}{2}, 0 \le s < 2$, strictly $s \ne -2, -4$ This strict adherence is due to the properties of Dirichlet's boundary conditions as identified by the Riemann Hypothesis and the nature of Heisenberg commutators, $\text{Re}(s) = \frac{1}{2} = [\hat{x}, \hat{p}]$ being for the nature of Heisenberg commutators. Ultimately, we ceremoniously conclude that the solution to the Riemann Hypothesis problem relies significantly on contributions originating from the Bose-Einstein Condensate (BEC) phase. This is a universal property of prime numbers. Whenever a laser cools a portion of atoms or molecules, resulting in the observation of BEC phases, it must adhere to mathematical rules where the numbers of atoms or molecules are distributed as prime numbers.

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Note that the order and structure of the appendix follow the derivation logic of this article. The statements (including the remarks) in the appendix constitute the descriptions and support the content of the article.

Claims

We ceremonially claim that this paper was initially completed on March 12th, 2024. (Although the addition of Ref. [25] (June 3rd, 2024) was included later upon request for revisions.) This paper presented a stronger version than the paper by Larry Guth and James Maynard announced around June 5th, 2024. Our lead in this research is at least a period of around three months. Hereby, we claim that this paper holds the international priority rights.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendices

A. The Approximation

The paper titled *"Problems of the Millennium: the Riemann Hypothesis"* by E. Bombieri where the first page writes them down as

$$\xi(t)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{A.1}$$

Applying Equations (6) and (7), therefore Equation (A.1) becomes

$$\xi(t)=\frac{3\pi^{-3/4}\Gamma(3/4)\zeta(3/2)}{8}\to\infty,\ s=3/2 \tag{A.2}$$

This is due to $\Gamma(3/4)=\infty$, and other terms in Equation (A.2) are constants, which cause $\lim_{t\to\infty}\xi(t)\to\infty$ to be an entire function. This satisfactorily agrees with Lindelöf's hypothesis and the growth of the zeta function. Namely,

$$\zeta(1/2+it)=O\left(t^{\varepsilon}\right),\forall\varepsilon>0\ \text{ as }\ t\to\infty \tag{A.3}$$

These are obviously equivalent. Note that $\Gamma(3/4)=\Gamma(-1/4)=\infty$ is divergent but is well-behaved in complex analysis, with $1/0=\infty$ as pole(s) on Riemann sphere.

B. The Examination of Todd Function

It is well-known that Aliah (1929-2019) defined his Todd function, which has been previously announced. Here it is mentioned again for reference:

$$\lim_{y\to\infty}T\left(\frac{1}{2}+yi\right) \tag{B.1}$$

In the Riemann sphere, Equation (B.1) is indicated as $T\left(\frac{1}{2}+i\infty\right)$ (and agrees with the limitation). When returning to map itself in real space ($\pm i\mapsto\pm1$), one obtains:

$$\frac{1}{2}+y,\forall y\in R \tag{B.2}$$

For y is arbitrary in real space. Moreover, if options of $1/p=1/137$ so that

$$\frac{1}{2}+y=\frac{1}{137},|y|>0 \tag{B.3}$$

It can be concluded as

$$\lim_{y\to\infty}T\left(\frac{1}{2}+yi\right)=\frac{1}{137} \tag{B.4}$$

i.e., it is obviously coincidental and has no significant results in derivation because the limitation could be any arbitrary real number. In Sect. 3.1, the prime number 137 is included early in the mathematical induction [see Equations (8) to (13)]. Hence, Aliah's ideas make no sense. Assuming Aliah had read the paper published by J.K. Webb *et al.* (2001), he would not have made this mistake. Because α is an observed value in cosmology, the error or derivation is hard to avoid; one cannot place it directly into an equation to advance a result. The Todd function value of the limitation indicated as 1/137 is incorrect.

C. The Calculation: UVC Photons $\lambda \approx 226.43\ \text{nm}$ (KrCl* Radiates Photons $\lambda \approx 222\ \text{nm}$)

At low temperatures, considering the boiling point of **He-4** at T = 4.222 K and associating it with the electron minimum sizes of $1.954\ \text{fm} \approx 2.81(5)\ \text{fm}/1.44$ (where the coefficient 1.954 can be regarded as the **SCF constant**[12]) in BEC phases:

Hence

$$T = Const \equiv \frac{4.222}{1.954} \approx 2.1607\ \text{K} < 2.1768\ \text{K} \tag{C.1}$$

Therefore $p \equiv 3.825$ produced from Equation (9) while above values are substituted.

Take $T \approx 2.1607\ \text{K}$, $T_C = 2.1768\ \text{K}$ by Equation (C.1) to be substituted into Equation (10), via process of reversal solution, thus we obtain

$$\lambda \approx 226.43\ \text{nm} < 300\ \text{nm} \tag{C.2}$$

The wavelengths used in a laser cooling process for the condensation of Cooper pairs of electrons (*i.e.*, Electromagnetically Induced Transparency (EIT) effects) for $Yb_2Si_2O_7$ in a BEC phase correspond to wavelengths smaller than 300 nm (*i.e.*, the laser engravings produce $s = 0$)[13].

Remark. **Degenerate state matters for statements of number 1.954:** Given the *First Principle*: $\Delta x \Delta p \geq \hbar/2$ for He-4 such that $\Delta x_{\min}$ causes $\Delta p_{\max}$, where the momentum requires a more powerful force to maintain Δp. In a physical sense, this can be regarded as T = 0 K or $T = Const$ ($dT = 0$) (either way works). Hence, in this case, the lowest energy level that electrons must occupy is determined. As is known, this quantum state is discrete. Through the first author's calculations, we found that the SCF constant is 1.954 for He-4, and the corresponding temperature is around 2.1607 K. This value is permitted based on the *First Principle*. Regarding the superfluidity possessed by the BEC phases, the Half Quantum Vortex (HQV) is indeed involved [26]. The correlative issue about the HQV in BEC phases is discussed further in Appendix G.

D. The Tricky Way of Solving the RH Problem: Using the Laser Cooling

Complex-Plane Analysis: Similar to the nature of $f(z) = e^{\frac{1}{z}}, z = 0$ possessing an essential singularity, there is a need for at least one pole next to the number 1 on complex planes for Riemann functions (series) for the Riemann Hypothesis. This could be generalized for all non-trivial zeros while $s > 1$ (*i.e.*, the pole at s = 1) corresponds to a BEC vortex core [26]. The function exhibits extreme behavior

[12]The derivation is indubitably long and is detailed in the first author's private manuscript. For further context, see the study of helium white dwarfs in cosmology.

[13]The work of the sculpture $s = 0$ is achieved using the excimer laser KrCl*.

near the essential singularity, which can be searched for in a physical sense. The essential solution is actually based on laser cooling in a BEC phase of the complex plane because $s = 0$ (*i.e.*, an essential singularity) is indicated as a laser engraving hole in the progress of a BEC phase. The derivations in this paper are all based on this, and we primarily follow mathematical induction. By this method, the RH is quickly and correctly solved in this paper. Additionally, if one asks the question: "What could happen if these parameters change as a result of photon pumping, lattice distortion, or defects in the system? Will this influence or induce multiplicity of the pole?" This can be solved using the method of XPM under stable EIT, as proposed by Schmidt and Imamoğlu [27].

Therefore, in Quantum Field Theory, scale invariance can be explained in terms of particle physics. In scale-invariant theories, the strength of particle interactions does not depend on the energy of the particles (including lattice distortion or defects in the systems composed of particles) involved. Electromagnetically Induced Transparency (EIT) focuses on the exchange of wave functions, not on the energy absorbed by atoms. According to the mechanism of Cross-Phase Modulation (XPM), the number of particles caused by photon pumping does not influence the multiplicity of the pole at $s = 1$ at any time.

E. The Notation Distinguish: $s := n$ and $s :\neq n$

Note that $s := n$ ($n := s$) reveals that s can be denoted as n for energy levels of electrons in a BEC phase, but $s :\neq n$ (s is not denoted as n) in the case of Simple Harmonic Oscillators (SHOs). Therefore, $s :\neq n$ completely stands for escaped atoms in the laser cooling of a BEC phase, which gives that:

$$E_0 = \left(0 + \frac{1}{2}\right)\hbar\omega = \frac{1}{2}\hbar\omega, \mathrm{Re}(s) = \frac{1}{2}, \forall s = 0 \tag{E.1}$$

If the potential of atoms is not considered, it is permitted to treat the Hamiltonian of an atom as purely kinetic energy $3E_0 + E_{cl.} \approx \frac{3}{2}\hbar\omega$ in three-dimensional space. In the case of a BEC phase system, the classical terms (including heat) can be completely ignored due to the quantum extreme low temperature. See **Figure A1**.

Note that in a Magneto-Optical Trap (MOT), due to the simple harmonic potential constraint by BEC phases (*i.e.*, an expulsive parabolic potential within a certain time interval):

$$\frac{1}{2}\hbar\omega\bigg|_{n_0} + \frac{3\hbar\omega}{2} + \left|-\frac{3\hbar\omega}{2}\right|\bigg|_{m_j=-1} = \frac{7\hbar\omega}{2} \tag{E.2}$$

Based on this one can therefore have $E_n = \left(n + \frac{1}{2}\right)\hbar\omega \le \frac{7}{2}\hbar\omega$, $n = 1,2,3$. But out of this trap (LHS of **Figure A1**) $\forall p \neq 2,3$ (*i.e.*, $p = 5,7,\cdots,73,\cdots,107,\cdots$)[14].

[14]See the work of Pritchard's team at MIT, USA, which confined 107 atoms in a Magneto-Optical Trap (MOT), also known as the Ioffe-Pritchard Trap. See **Figure A1**, $\dot{p} = |-\dot{p}| = 107/\mathrm{sec}$ where $\dot{N} = 0$. Based on this, it is permitted to generalize to other prime numbers starting from $p = 5$.

Where N indicates the total number of atoms. Moreover, this aligns with statements made by Hardy and Littlewood's conjecture of Chebyshev.

$$\lim_{x \to 1^-} \sum_{p>2} (-1)^{\frac{p+1}{2}} x^p = \infty \quad (p > 2) \tag{E.3}$$

Note that Equation (E.2) complies with statement: $\operatorname{Re}(s), 0 \le s < 2$.

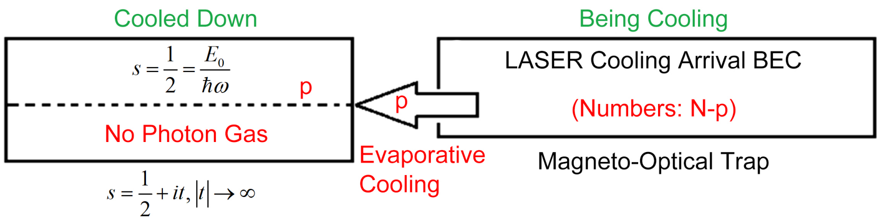

Figure A1. Two physical systems that successfully express the phenomenological process derived in this paper, where t can represent concepts of atomic positions in motion. In the context of scattering processes in many-body systems, the Feshbach resonance occurs when the energy of a bound state of an interatomic potential is equal to the kinetic energy of a colliding pair of atoms (see the next remark as shown). The figure strongly supports Figure 2, and this process has ensured that our work in this paper preserves correctness and completeness in compliance with Dirichlet's boundary conditions in pure mathematics. The ranges of p are grown as shown below.

Remark. $\psi = \left[A_c + A_s \frac{(\gamma \cosh\theta + \cos\varphi) + i(\alpha \sinh\theta + \beta \sin\varphi)}{\cosh\theta + \gamma\cos\varphi} \right] \times \exp\left(\frac{\lambda t}{2} + i\varphi_c \right)$ in **Equation (4)** of **Ref.** [21], where let

$$\psi \equiv \left[A_c + A_s \frac{(\gamma \cosh\theta_1 + \cos\varphi_1) + i(\alpha \sinh\theta_2 + \beta \sin\varphi_2)}{\cosh\theta_1 + \gamma\cos\varphi_1} \right] \times \exp\left(\frac{\lambda t}{2} + i\varphi_c \right) \tag{E.4}$$

Applying the Darboux transformation: $(\theta_1, \theta_2) \to (\varphi_1, \varphi_2)$ so that this type of transformation fits the one-dimensional Schrödinger equation in terms of photon beams of the BEC-laser. If the interaction angles of photon-atoms are given, then by the Darboux transformation: $\varphi_1 = \theta_1 + 2\partial_x^2(\ln(E))$, $\varphi_2 = \theta_2 - \frac{\partial_x E}{E}\theta_2$ with $E = E_n = const$ returns to **Equation (E.2)** which denotes one that

$$\begin{aligned} \varphi_1 &= \theta_1 = \pi/4, m_j = \pm 1 \\ \varphi_2 &= \theta_2 = 0, n_0 = 0 \end{aligned} \tag{E.5}$$

Associated with $\gamma = 1$, such leads to

$$\psi_0(x,t) = (A_c + A_s) \times \exp\left(\frac{\lambda t}{2} + i\varphi_c \right) \tag{E.6}$$

Comparing the above Equation (E.6) with Equation (3) of **Ref.** [28], we find that $A_s = 0$. Based on this, we can claim that photons of the BEC-laser (bright solitons) are embedded in the background (*i.e.*, the photons are embedded in a cavity system of the laser). The interactions between photons and atoms can be neglected, without any production of heat. In the introduction of this paper, we mentioned that $\lambda_{-Si-} \approx 1.1\,\mu\text{m}$, $T_c \to 0\,\text{K}$ is in good agreement with $a_{\perp} \approx 1.4\,\mu\text{m}$ [28]. Since the photons are embedded in the background, the splitting of new solitons is prevented, and simultaneously the number of atoms remains in dynamic stability (see p4 of **Ref.** [28]). We conclude that it is the prime number(s) at all times.

F. Qubits and Quantum Information

Commonly given that

$$\begin{aligned} &|\psi\rangle_a = \alpha_a|0\rangle_a + \beta_a|1\rangle_a, \alpha_i, \beta_i \in C, |\alpha_i|^2 + |\beta_i|^2 = 1, \\ &|\psi\rangle_b = \alpha_b|1\rangle_b + \beta_b|0\rangle_b \end{aligned} \tag{F.1}$$

where $i = \alpha$ is indicated as the pure state of **detection light** and $i = \beta$ is indicated as the pure state of **coupled light** (both types of light are involved in EIT effects). These states undergo linear superposition by Equation (F.1) before measurement. Therefore:

$$\sum_{i=a}^{b} \psi = |\psi\rangle_a + |\psi\rangle_b = \left(\alpha_a|0\rangle_a + \alpha_b|1\rangle_b\right) + \left(\beta_a|1\rangle_a + \beta_b|0\rangle_b\right) \tag{F.2}$$

The detection light and coupled light occur in EIT simultaneously (e.g., on the attosecond scale[15]). Based on this, the photons of the laser exhibit behavior consistent with quantum entanglement. Hence

$$|\psi\rangle_a = |\psi\rangle_b = |\psi\rangle_i, i = a, b \quad \text{with} \quad t = 10^{-18}\,\text{sec} \tag{F.3}$$

Yields

$$\begin{aligned} &\sum_{i=a}^{b} \psi_i = 2|\psi\rangle = \left(\alpha_a|0\rangle_a + \alpha_b|1\rangle_b\right) + \left(\beta_a|1\rangle_a + \beta_b|0\rangle_b\right), \\ &|\alpha_a|^2 + |\alpha_b|^2 + |\beta_a|^2 + |\beta_b|^2 = 2 \cdot \left(|i/2|^2 + |-i/2|^2\right) = 1 \end{aligned} \tag{F.4}$$

Such Equation (F.4) fits Equation (F.1) and obeys the variation principle in Q.M. In Equation (F.4), where $t = \pm i/2$ are obviously projected with $\varphi = \pi/4$ or $\varphi = 3\pi/4$ by $spin = \pm 1/2$, respectively[16]. Based on this, therefore $t = \pm i/2$ can be completely denoted as probability amplitudes in Q.M. (e.g. α_i, $\beta_i \in C$, $i = a, b$).

At present, the concept of qubits actually supports the work presented in this paper. If the solution for the Riemann Hypothesis problem proposed in this paper is validated in the future, it will significantly enhance the understanding and capabilities of quantum cryptography (excluding the technology involved). To summarize, one can write down:

$$RH \Leftrightarrow qubit \tag{F.5}$$

However, *Qubit Technology* is not included within the scope of this paper. We suggest that it be discussed in a separate theory. Specifically, Equations (F.1) to (F.4) fit the qutrit model $|0\rangle$, $|1\rangle$, $|2\rangle$, where photons ($s = 1$) have eigenvalues of 0, ±1.

[15]See literatures of the Nobel Prize in physics 2023.

[16]The spin(s) are indicated as the spin(s) of quantum particles in Riemann spheres.

G. The HQV of Formations of BEC Phases and The Applications

Equation (E.5) denotes one that $|1,\pm1\rangle$ as photons absorbed by surface atoms. On the other hand, the observation of a high density of $|1,1\rangle$ atoms while the photospin $S=-1$ is not completely depleted (where the atoms have a small bump and thus violate the formation of a BEC phase) occurs in a Half Quantum Vortex (HQV). In a Magneto-Optical Trap (MOT), where the magnetic field gradient $m_j=-1$ given by Equation (E.2) is normalized to $|m_j|=1$ by way of absolute values (a MOT: $\partial_z \hat{H}(x,-y,-z)\big|_{m_j=1} = \partial_z \hat{H}(-x,y,-z)\big|_{m_j=-1}$ are canceled), the frequencies of atom-atom collisions are reduced, thereby increasing the mean free path. By this method, one can promote the formation of BEC phases (*i.e.*, maintaining specific quantum states). This section makes a scientific statement that the applications of 1) the pole $s=1$ (in the RH problem) as core $(|1,-1\rangle)$ and 2) the nontrivial zeros located on the critical line $s=1/2+it$, $|t|\to\infty$ are both exactly due to this mechanism [26] [29].

H. The Multi-Solitons

Scientific investigation shows that our findings in Equation (21) can be further represented as expressions of multi-solitons (see Equation (3) and Equation (15) of **Ref.** [28]) of the generalized KDKK[17] equation for N-solitons with three lumps. These lumps correspond to atoms in the bright solitons at periodic time intervals: the ground state, the first excited state, and the metastable state of photons in the BEC-laser [28] [30]-[36].

[17]The abbreviation of the capitals by Konopelchenko-Dubrovsky-Kaup-Kupershmidt.