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Neutral Pion Electromagnetic Form Factor as a Bound System of 3 + 1 Dimensional QCD

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Abstract

We investigate a neutral pion electromagnetic form factor in momentum space and obtain Gaussian-like function for it. The characteristic form of our neutral pion electromagnetic form factor is consistent with the results published by Jefferson Lab Hall A Collaboration.

Keywords

Neutral Pion Electromagnetic Form Factor, Gaussian-Like Function

1. Introduction

After M. Dlamini *et al.* (Jefferson Lab Hall A Collaboration) have published new results of neutral pion electromagnetic form factor [1], J. Arrington *et al.* (number of co-authors is 31) have been starting to consider that the lightest pseudo-scalar mesons appear to be the key to the further understanding of the emergent mass and structure mechanisms [2]. In ref. [1], M. Dlamini *et al.* insist that the t-dependence of the cross section, usually parametrized by Regge-like profile function, is no longer valid at typical values of $-t > 1\,\text{GeV}^2$ and also they use a functional form of $C\left(Q^2\right)^A \exp\left(-Bt'\right)$ to fit their data where Q^2 is proton momenta and $t' = t_{\min} - t\left(t = \left(q - q'\right)^2\right)$. Here q is virtual photon momentum and q' is π^0 momentum. This means that neutral pion electromagnetic form factor is not Regge-like but Gaussian-like. M. Diehl and P. Kroll has shown this in the GPD (generalized parton distributions) analysis of nucleon form factors [3]. E. Arriola *et al.* have shown the behavior of π^0 wave function in configuration space in Figure 2 which looks like Gaussian-like by using the quenched Lattice QCD calculation [4]. We also obtained a π^0 wave function as a bound system of 3 + 1 dimensional QCD with massive quarks in configuration space of which characteristic form is Gaussian-like [5]. Therefore, we have to investigate an π^0 electromagnetic form factor in momentum space from our π^0 wave function.

2. Formulation

We briefly describe our formalism and the equation of motion we obtained previously [5]. Suura [6] [7] defined the Bethe-Salpeter-like amplitude as

$$\chi_{\xi\eta}(1,2) = \langle 0 | q_{\xi\eta}(1,2) | P \rangle \tag{1}$$

where $|0\rangle$ and $|P\rangle$ denote the vacuum and physical states, respectively, and the gauge invariant bi-local operator $q(1,2)$ is defined in the non-Abelian gauge field as

$$q_{\xi\eta}(1,2) \equiv T_r^c q_\eta^\dagger(2) P \exp\left(ig\int_1^2 \mathrm{d}\vec{x}\,\vec{A}^a(\vec{x})\left(\frac{\lambda_a}{2}\right)\right) q_\xi(1) \tag{2}$$

Here ξ and η denote the Dirac indices, P denotes the path ordering, and the $\frac{\lambda_a}{2}$ components are generators of the adjoint representation of SU(N) color gauge group. The Trace is calculated for color spin a. For massive quarks and anti-quarks case, Dirac equation is expressed as

$$i\frac{\partial q}{\partial t} = -i\alpha^k D_{Ak} q - \beta m q \tag{3}$$

The Dirac equation of the complex conjugate $q^\dagger$ becomes as the following.

$$i\frac{\partial q^\dagger}{\partial t} = -i\alpha^k D_{Ak} q^\dagger + \beta m q^\dagger \tag{4}$$

where $D_{Ak} \equiv \partial_k + igA_k^a\left(\frac{\lambda_a}{2}\right)$.

Note that we choose the plus sign for covariant derivative following Erratum [8].

We employ the metric system and γ matrices as follows, according to Weinberg [9].

$$\eta^{00} = -1,\ \ \eta^{11} = \eta^{22} = \eta^{33} = 1$$

$$\gamma^0 = (-i)\begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix},\ \ \gamma^k = (-i)\begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}$$

where σ_0 is a unit matrix of a 2 × 2 matrix and σ_k is the 2 × 2 Pauli-matrix specified by k = 1, 2, 3.

$$\alpha^k = \gamma^0\gamma^k \ \text{ and } \ \beta = i\gamma^0$$

Then for the non-chiral limit case, Equation (3) and Equation (4) lead to the starting equation of motion as follows.

$$\begin{aligned} i\frac{\partial}{\partial t} q(1,2) = {} & -i\vec{\alpha}\cdot\vec{\nabla}(2) q(1,2) - q(1,2) i\vec{\alpha}\cdot\vec{\nabla}(1) + \beta m_2 q(1,2) - q(1,2)\beta m_1 \\ & + g\int_1^2 \mathrm{d}\vec{x}\, q_E(1,2;x) + g\vec{\alpha}\cdot\int_1^2 \mathrm{d}\vec{x}\times q_B(1,2;x) \end{aligned} \tag{5}$$

where $q_{\bar{O}}(1,2;x) \equiv q^{\dagger}(2)U(1,x)\bar{O}^{a}U(x,2)q(1)$.

O is any operator and

$$U(1,2) \equiv P\exp\left(ig\int_{1}^{2} \mathrm{d}\vec{x}\,\vec{A}^{a}(\vec{x})\left(\frac{\lambda_a}{2}\right)\right)$$

Equation (5) except mass terms is derived in [10], although the derivation way is slightly revised by using the consideration of Erratum [8]. Then, we obtained the following equations as shown in [5].

$$P_0\chi_0(r) = -\frac{2}{r}\chi_1(r) + \frac{g^2|L_1|}{2}r\chi_1(r) + \frac{g^2|L_1|\delta(0)}{2P_0}\chi_0(r) \tag{6}$$

$$P_0\chi_1(r) = \frac{g^2|L_1|}{2}r\chi_0(r) + \frac{g^2|L_1|\delta(0)}{2P_0}\chi_1(r) - (m_1+m_2)\chi_3(r) \tag{7}$$

$$P_0\chi_2(r) = -2\frac{\partial}{\partial r}\chi_3(r) - \frac{2}{r}\chi_3(r) - \frac{g^2|L_1|\delta(0)}{2}r\chi_2(r) \tag{8}$$

$$P_0\chi_3(r) = 2\frac{\partial}{\partial r}\chi_2(r) - \frac{g^2L_1}{2}r\chi_2(r) + \frac{g^2|L_1|\delta(0)}{2P_0}\chi_3(r) - (m_1+m_2)\chi_1(r) \tag{9}$$

Because we use the consideration of Erratum [8], L_1 is replaced by $|L_1|$ in Equation (6)-(9).

We are interested in pion cases. It is obvious that the following pion wave function with an eigenvalue of $P_0^2 = \frac{g^2L_1\delta(0)}{2} = m_\pi^2$ exactly satisfy Equations (6)-(9).

$$\chi_0^{(\pi)} = const\frac{2(m_1+m_2)}{g^2|L_1|}\frac{1}{r^2}\exp\left(-\frac{g^2|L_1|}{8}r^2\right) \tag{10}$$

$$\chi_1^{(\pi)} = 0 \tag{11}$$

$$\chi_2^{(\pi)} = 0 \tag{12}$$

$$\chi_3^{(\pi)} = const\frac{1}{r}\exp\left(-\frac{g^2|L_1|}{8}r^2\right) \tag{13}$$

Therefore, we obtain $\chi_0^{(\pi)}(r) = const\frac{2(m_1+m_2)}{g^2|L_1|}\frac{1}{r^2}\exp\left(-\frac{g^2|L_1|}{8}r^2\right)$ as a neutral pion wave function.

3. Derivation

In Section 2, we show the following form as a neutral pion wave function in configuration space as

$$\chi_0^{(\pi)}(r) = const\frac{2(m_1+m_2)}{g^2|L_1|}\frac{1}{r^2}\exp\left(-\frac{g^2|L_1|}{8}r^2\right)$$

An electromagnetic form factor is defined in momentum space so that we can use three dimensional Fourier transform as follows.

$$F_\pi(q) = \iiint \mathrm{d}^3\vec{r}\exp(i\vec{q}\cdot\vec{r})\left|\chi_0^{(\pi)}(r)\right|^2$$
$$= 2\pi\int_0^\infty r^2\mathrm{d}r\int_0^\pi \sin\theta\mathrm{d}\theta \mathrm{e}^{iqr\cos\theta}A^2\frac{1}{r^4}\exp\left(-\frac{g^2|L_1|}{4}r^2\right) \tag{14}$$

where $A = const\dfrac{2(m_1+m_2)}{g^2|L_1|}$.

From now on we denote $2\pi A^2$ and $\dfrac{g^2|L_1|}{4}$ as A_1 and β, respectively. First, we take θ integration, then Equation (14) becomes as

$$F_\pi(q) = A_1\frac{1}{iq}\int_0^\infty \mathrm{d}r\frac{\mathrm{e}^{-\beta r^2}\left(\mathrm{e}^{iqr}-\mathrm{e}^{-iqr}\right)}{r^3} \tag{15}$$

Because an integral for r becomes infinite at $r = 0$, we have to use regularization at origin.

This means that we set r_0 instead of 0 and after evaluation we take r_0 going to 0.

$$F_\pi(q) = A_1\frac{1}{iq}\int_{r_0}^\infty \mathrm{d}r\frac{1}{r^3}\left(\mathrm{e}^{-\beta r^2+iqr}-\mathrm{e}^{-\beta r^2-iqr}\right)$$
$$= A_1\frac{1}{iq}\int_{r_0}^\infty \mathrm{d}r\frac{1}{r^3}\left(\mathrm{e}^{-\beta\left(r-\frac{iq}{2\beta}\right)^2-\frac{q^2}{4\beta}}-\mathrm{e}^{-\beta\left(r+\frac{iq}{2\beta}\right)^2-\frac{q^2}{4\beta}}\right) \tag{16}$$

Here, changing variable $r-\dfrac{iq}{2\beta}=\overline{r_1}$ for the first term and changing variable $r+\dfrac{iq}{2\beta}=\overline{r_2}$ for the second term and omitting $A_1\dfrac{1}{iq}$ term, Equation (16) becomes

$$\text{Equation (16)} = \mathrm{e}^{-\frac{q^2}{4\beta}}\left(\int_{r_0-\frac{iq}{2\beta}}^\infty \mathrm{d}\overline{r_1}\frac{\mathrm{e}^{-\beta\overline{r_1}^2}}{\left(\overline{r_1}+\frac{iq}{2\beta}\right)^3}-\int_{r_0+\frac{iq}{2\beta}}^\infty \mathrm{d}\overline{r_2}\frac{\mathrm{e}^{-\beta\overline{r_2}^2}}{\left(\overline{r_2}-\frac{iq}{2\beta}\right)^3}\right) \tag{17}$$

First term of Equation (17)

$$= I^{(1)} = \mathrm{e}^{-\frac{q^2}{4\beta}}\left(\int_0^\infty \mathrm{d}r\frac{\mathrm{e}^{-\beta r^2}}{\left(r+\frac{iq}{2\beta}\right)^3}-\int_0^{r_0-\frac{iq}{2\beta}}\mathrm{d}r\frac{\mathrm{e}^{-\beta r^2}}{\left(r+\frac{iq}{2\beta}\right)^3}\right) \tag{18}$$

Second term of Equation (17)

$$= I^{(2)} = \mathrm{e}^{-\frac{q^2}{4\beta}}\left(\int_0^\infty \mathrm{d}r\frac{\mathrm{e}^{-\beta r^2}}{\left(r-\frac{iq}{2\beta}\right)^3}-\int_0^{r_0+\frac{iq}{2\beta}}\mathrm{d}r\frac{\mathrm{e}^{-\beta r^2}}{\left(r-\frac{iq}{2\beta}\right)^3}\right) \tag{19}$$

Thus, denoting the first terms of Equation (18) and Equation (19) as $I_1^{(1)}$ and $I_1^{(2)}$, respectively, we can combine these two terms as

$$I_1^{(1)} - I_1^{(2)} = \mathrm{e}^{-\frac{q^2}{4\beta}} \int_0^{\infty} \mathrm{d}r \frac{-6r^2 \frac{iq}{2\beta} + i\left(\frac{q}{2\beta}\right)^3}{\left(r^2 + \left(\frac{q}{2\beta}\right)^2\right)^3} \tag{20}$$

To deal with the second terms of Equation (18) and Equation (19), we use the method of change of variables again. We denote the second term of Equation (18) as $I_2^{(1)}$ and the second term of Equation (19) as $I_2^{(2)}$. For $I_2^{(1)}$, changing variable as $r + \frac{iq}{2\beta} = \overline{r}$, $I_2^{(1)}$ becomes

$$I_2^{(1)} = -\mathrm{e}^{-\frac{q^2}{4\beta}} \int_{\frac{iq}{2\beta}}^{r_0} \mathrm{d}\overline{r} \frac{\mathrm{e}^{-\beta\left(\overline{r} - \frac{iq}{2\beta}\right)^2}}{\overline{r}^3} = -\int_{\frac{iq}{2\beta}}^{r_0} \mathrm{d}\overline{r} \frac{\mathrm{e}^{-\beta\overline{r}^2 + iq\overline{r}}}{\overline{r}^3} \tag{21}$$

Similarly, for $I_2^{(2)}$, changing variable as $r - \frac{iq}{2\beta} = \overline{\overline{r}}$, $I_2^{(2)}$ becomes

$$I_2^{(2)} = -\mathrm{e}^{-\frac{q^2}{4\beta}} \int_{-\frac{iq}{2\beta}}^{r_0} \mathrm{d}\overline{\overline{r}} \frac{\mathrm{e}^{-\beta\left(\overline{\overline{r}} + \frac{iq}{2\beta}\right)^2}}{\overline{\overline{r}}^3} = -\int_{-\frac{iq}{2\beta}}^{r_0} \mathrm{d}\overline{\overline{r}} \frac{\mathrm{e}^{-\beta\overline{\overline{r}}^2 - iq\overline{\overline{r}}}}{\overline{\overline{r}}^3} \tag{22}$$

For the last integral of Equation (22), changing variable as $-\overline{\overline{r}} = r$, $I_2^{(2)}$ becomes

$$I_2^{(2)} = -\int_{\frac{iq}{2\beta}}^{-r_0} (-\mathrm{d}r) \frac{\mathrm{e}^{-\beta + iqr}}{(-r)^3} = \int_{-r_0}^{\frac{iq}{2\beta}} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2 + iqr}}{r^3} \tag{23}$$

Because minus sign in the integrand is cancelled out, we can obtain the form of the last term. Then denoting $\overline{r}$ in Equation (21) as r, $I_2^{(1)} - I_2^{(2)}$ becomes

$$\begin{aligned} I_2^{(1)} - I_2^{(2)} &= -\int_{\frac{iq}{2\beta}}^{r_0} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2 + iqr}}{r^3} - \int_{-r_0}^{\frac{iq}{2\beta}} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2 + iqr}}{r^3} \\ &= -\int_{-r_0}^{r_0} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2 + iqr}}{r^3} \\ &= -\int_0^{r_0} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2 + iqr}}{r^3} - \int_{-r_0}^{0} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2 + iqr}}{r^3} \end{aligned} \tag{24}$$

Because of the fact that $\frac{iq}{2\beta} = 0 + \frac{iq}{2\beta}$, that is, real part of $\frac{iq}{2\beta}$ is 0, we can obtain the form of the last term of Equation (24) within the framework of complex analysis. Especially, we refer to the integration contour in Ref. [11].

Again recalling the change of variable as $r = -r'$ for the second term of the last form of Equation (24), this term becomes

$$\int_{r_0}^{0} (-\mathrm{d}r') \frac{\mathrm{e}^{-\beta r'^2 - iqr'}}{(-r')^3} = \int_0^{r_0} \mathrm{d}r' \frac{\mathrm{e}^{-\beta r'^2 - iqr'}}{r'^3} \tag{25}$$

Denoting r' as r in Equation (25), we can combine this term and the first term of Equation (24).

Then, recalling the omitting terms of $A_1 \frac{1}{iq}$, we obtain the following form for $I_2^{(1)} - I_2^{(2)}$ as

$$I_2^{(1)} - I_2^{(2)} = -\frac{2A_1}{q} \int_0^{r_0} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2}}{r^3} \sin(qr) \tag{26}$$

Thus, our neutral pion electromagnetic form factor is described as

$$F_\pi(q) = A_1 \mathrm{e}^{-\frac{q^2}{4\beta}} \int_0^\infty \mathrm{d}r \frac{-3r^2 \frac{1}{\beta} + \frac{q^2}{(2\beta)^3}}{\left(r^2 + \left(\frac{q}{2\beta}\right)^2\right)^3} \mathrm{e}^{-\beta r^2} - \frac{2A_1}{q} \int_0^{r_0} \mathrm{d}r \frac{\mathrm{e}^{-\beta r^2}}{r^3} \sin(qr) \tag{27}$$

We obtain the form of Equation (27) as a neutral pion electromagnetic form factor, however, we have to check the behavior of this form when q approaches zero because the first term of Equation (27) looks negative at $q \to 0$ when r_0 goes to 0.

To check this, taking $q = \varepsilon \ll 1$ and changing variable as $r = \frac{\varepsilon}{2\beta} \tan \bar{\theta}$, the first term of Equation (27) becomes

$$F_\pi^{(1)}(q = \varepsilon) = \lim_{\varepsilon \to 0} \int_{\bar{\theta}_1}^{\frac{\pi}{2}} \mathrm{d}\bar{\theta} \left(-3\frac{A_1}{\beta}\right) \left(\frac{2\beta}{\varepsilon}\right)^3 \left(\sin\bar{\theta} \cos\bar{\theta}\right)^2 \tag{28}$$

where $\bar{\theta}_1$ is a function of ε and it becomes zero when ε approaches zero. Actual form of $\bar{\theta}_1$ is determined later.

$$\begin{aligned} F_\pi^{(1)} &\to \left(-\frac{3A_1}{\beta}\right)\left(\frac{2\beta}{\varepsilon}\right)^3 \left[\frac{\bar{\theta} - \frac{\sin 4\bar{\theta}}{4}}{8}\right]_{\bar{\theta}_1}^{\frac{\pi}{2}} = \left(-\frac{3A_1}{\beta}\right)\left(\frac{2\beta}{\varepsilon}\right)^3 \left(\frac{\pi}{16} - \frac{2}{3}\bar{\theta}_1^3\right) \\ &= -\frac{3A_1}{\beta}\left(\frac{2\beta}{\varepsilon}\right)^3 \frac{\pi}{16} + \frac{2A_1}{\beta}\left(\frac{2\beta}{\varepsilon}\right)^3 \bar{\theta}_1^3 \end{aligned} \tag{29}$$

To obtain the last term, we use the fact that $\bar{\theta}_1 \ll 1$ and use Taylor expansion of sin function.

From Equation (29) we notice that $\bar{\theta}_1$ should be a linear function of ε because the second term of Equation (29) should be independent of ε. If the first term of Equation (29) is cancelled out by the second term of Equation (27), $F_\pi(q)$ would be positive at all ranges of q^2.

For the second term of Equation (27), namely $F_\pi^{(2)}$, we evaluate this term when q approaches 0 as follows. To do this, it is sufficient to consider only case of r be near 0 (q also near 0).

$$F_\pi^{(2)} \to \lim_{\varepsilon \to 0} \left[-2A_1 \int_{0+i\varepsilon^3}^{r_0} \mathrm{d}r \frac{1}{r^2}\right] = 2A_1 \left(\frac{1}{r_0} - \frac{1}{i\varepsilon^3}\right) \tag{30}$$

Thus, in order to cancel out the term of $-\frac{3A_1}{\beta}\left(\frac{2\beta}{\varepsilon}\right)^3 \frac{\pi}{16}$, it is sufficient to set this result equal to $\frac{3A_1}{\beta}\left(\frac{2\beta}{\varepsilon}\right)^3\frac{\pi}{16}$. Then, we obtain r_0 as

$$r_0=\frac{\varepsilon^3\left(\frac{\pi\beta^2}{2}+i\right)}{\left(\frac{\pi\beta^2}{2}\right)^2+1} \tag{31}$$

Note that real part of r_0 is positive and that r_0 goes to 0 at $\varepsilon\to0$.

Thus our form factor described as Equation (27) actually can be considered as a neutral pion electromagnetic form factor.

4. Conclusions

In Section 3, we obtain a neutral pion electromagnetic form factor $F_\pi(q)$ as follows.

$$F_\pi(q)=A_1\mathrm{e}^{-\frac{q^2}{4\beta}}\int_0^\infty\mathrm{d}r\frac{-3r^2\frac{1}{\beta}+\frac{q^2}{(2\beta)^3}}{\left(r^2+\left(\frac{q}{2\beta}\right)^2\right)^3}\mathrm{e}^{-\beta r^2}-\frac{2A_1}{q}\int_0^{r_0}\mathrm{d}r\frac{\mathrm{e}^{-\beta r^2}}{r^3}\sin(qr)$$

Because the second term of $F_\pi(q)$ becomes zero when we take r_0 to 0 as definition of regularization, its behavior at large q^2 is

$$F_\pi(q)\to\frac{\mathrm{e}^{-\frac{q^2}{4\beta}}}{q^4} \tag{32}$$

We can use this argument if the integrand of $F_\pi^{(2)}$ is regular. To examine this, we use Tayler expansion of Gaussian and sin functions of the integrand of $F_\pi^{(2)}$ as follows.

$$\begin{aligned}\text{In tegrand of } F_\pi^{(2)}&=\frac{1}{q}\left[\frac{1}{r^3}\mathrm{e}^{-\beta r^2}\sin(qr)\right]\\&=\frac{1}{r^2}-\left(\beta+q^2\right)+O\left(r^2\right)+\cdots\end{aligned} \tag{33}$$

Choosing r_0 as $\frac{\varepsilon^3\left(\frac{\pi\beta^2}{2}+i\right)}{\left(\frac{\pi\beta^2}{2}\right)^2+1}$ shown in Equation (31), first term is cancelled out by first term of $F_\pi^{(1)}$ because singularity occurs at $r=0$, we can use the same argument around from Equation (28) to Equation (31). Then, the remaining terms of integrand of $F_\pi^{(2)}$ are regular. In addition, $\frac{1}{q}\sin(qr)$ approaches 0 when q is large, then we can use the above argument.

According to Ref. [1], the fitting function of neutral pion momentum to the differential cross section is a simple Gaussian. Comparing our result to their results, our form factor has an extra power function of $q=\sqrt{q^2}$.

5. Discussion

We obtain characteristically Gaussian-like form factor in momentum space and this is comparable to Jefferson Lab. results [1]. However, this is different from Brodsky's results [12] and normal Lattice QCD results, namely these are Regge-like behavior. We do not know precise reason but we could point out that Holographic treatment and normal Lattice QCD reflect non-perturbative calculation of Feynman Diagram. Our path order calculation and Arriora's linked quench Lattice QCD [4] do not refer Feynman Diagram. In addition, Kroll's method [3], namely changing kinematics, is also different from Feynman Diagram calculation. This suggests that QCD for strong interaction may not be just matrices extension version of QED but we need some more else.

We would like to point out another thing that the results of Jefferson Lab [1] suggest. There has been no normal Lattice QCD calculation for the neutral pion form factor of which characteristic form is Gaussian (at least to my knowledge). Because the quenched Lattice QCD [4] is a linked Lattice calculation, it is different from normal Lattice QCD calculation. In addition, in the GPD analysis, there have been no papers using Dyson-Schwinger equation that shows Gaussian-like form factor for a neutral pion (at least to my knowledge). By using kinematic approximation, such as $\xi \approx \frac{x_B}{2-x_B}$, Kroll *et al.* [3] succeed to realize that the t-dependence of cross section, usually parametrized by Regge-like profile functions, is no longer valid. All these four cases are formulated in a covariant way. Also, Suura's hadronic operator (our case) is not manifestly formulated in a covariant way, however, our results are comparable to that of Arriola's quenched Lattice QCD calculation (see Figure 2 in Ref. [4]). This suggests that covariant way, namely, relative motion between quark and anti-quark is relativistic, is not the principal requirement to obtain a valid t-dependence. An important point is that we must figure out what is the actual physics requirement to obtain a valid t-dependence.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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