Scientific Research Publishing

Advances in Pure Mathematics, 2024, 14, 487-494
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

On Perron's Formula and the Prime Numbers

Michael M. Anthony

Department of Engineering, Enertron Inc., Hohenwald, Tennessee, USA
Email: uinvent@aol.com

How to cite this paper: Anthony, M.M. (2024) On Perron's Formula and the Prime Numbers. *Advances in Pure Mathematics*, **14**, 487-494.
https://doi.org/10.4236/apm.2024.146027

Received: May 23, 2024
Accepted: June 18, 2024
Published: June 21, 2024

Abstract

The Riemann hypothesis is intimately connected to the counting functions for the primes. In particular, Perron's explicit formula relates the prime counting function to fixed points of iterations of the explicit formula with particular relations involving the trivial and non-trivial roots of the Riemann Zeta function and the Primes. The aim of the paper is to demonstrate this relation at the fixed points of iterations of explicit formula, defined by functions of the form $\lim_{T\in\mathbb{N}\to\infty} f^T(z_w) = z_w$,where, z_w is a real number.

Keywords

Perron, Fixed Points, Iterations, Number Theory, Riemann Hypothesis, Iterations, Invariance, Primes

1. Fixed Points of Iterations

Tetration and iterations of common functions such as the Riemann Zeta function, and the log function produce results that follow similar characteristics. For tetrations, this property of fixed points comes from the existence of infinite procedure that focus on certain the fixed points where the iterations produce the same result. A few examples will be given here, and then the Perron's function will be examined for fixed points that infer the prime number theorem. Tetration is important due to:

a) there is a fixed point of the tetration that determines the final end point of the process, and,

b) to get from the general complex plain to the real values of a function, a tetration must go through the zeroes of the function.

Tetration also provides a link between the roots of a function and its fixed values.

Given an Abel function that satisfies $A_f(f(z)) = A_f(z)+1$ there exist another function by adding any constants $A'_f(f(z)) = A_f(z)+c$. Then, the super-logarithm is defined by $\operatorname{slog}_b(1)=0$, and the Abel function could be determined completely. If the existence and uniqueness of the analytic extension of tetration is provided by the condition of its asymptotic approach to the fixed points $z_w = 0.3181315052 - 1.337235701i$, $\bar{z} = 0.3181315052 - 1.337235701i$ of $z_w = \log(z_w)$, in the upper and lower part of the complex plain, then the inverse functions are also unique. Such a function becomes real at some point of the tetrations. In this paper, we will use ζ', ζ'', to represent the first and second derivatives of the Zeta function $\zeta(s)$.

Define the T^{th} iteration of an argument *z*, of the *ζ*-function, as the super-zeta function $\zeta^T(z)$, and the T^{th} iteration of an argument *z*, of the log-function, $\log(z)$, as the super-log function:

$$\zeta^T(z) = \underbrace{\zeta\left(\zeta\left(\zeta\left(\cdots\zeta(z)\right)\right)\right)}_{T\text{ times}},\ \operatorname{slog}_T(z) = \underbrace{\log\left(\log\left(\log\left(\cdots\log\left(\log z\right)\right)\right)\right)}_{T\text{ times}} \tag{1}$$

Then, there exists a fixed point z_T for $\zeta^T(z)$ given by the continuity of iterations of any argument $z = \rho + it$, such that:

$$\lim_{T\to\infty}\zeta^T(z) = \zeta\left(\lim_{T\to\infty}\operatorname{slog}_n(z)\right) = \zeta(z_w) = -0.29590500557213955\cdots \tag{2}$$

These constants, $z_w = 0.3181315052 \pm 1.337235701i$, and $z_w - 0.29590500557213955\cdots$ were discovered by the author in [1]. It is important to note that the following is true as a result of the LambertW function.

$$z_w = \mathrm{e}^{z_w} \tag{3}$$

Thus, given an argument $z = \rho + it$ of the *ζ*-function, $\zeta(z)$, *the only set of complex numbers that satisfy the fixed-point solutions,* z_w, *are the roots of the ζ-function and the constant* z_w. The roots take any tetration to the zeroes and thus the reals of the functions iterated for both the Riemann Zeta function, $\zeta(z)$ and the $\operatorname{slog}_T(z)$ function.

It follows that there must exist fixed of Perron's explicit formula,

$$\psi(x) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - x^{-2}\right) - \ln 2\pi \tag{4}$$

And the importance highlights that fact that if the Perron relation is satisfied, then, this fixed point must highlight the roots of the Riemann Zeta function as complex arguments that take the Zeta-function from the complex arguments to real arguments to converge to the constant $z_w = -0.29590500557213955$.

I.N. Galidakis [2] discussed the $\operatorname{slog}_T(z)$ function extensively.

A general series extension of the function $\operatorname{slog}_k(s)$ is given by:

$$\operatorname{slog}_k(s) = \sum_{k=0}^{\infty}\frac{s^k}{k!}v_k \tag{5}$$

where, for $0 \le s \le 1$, and,

$$v_k = \frac{1}{(k-1)!}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[s^{k-1}\operatorname{slog}_k\left(\zeta(s)\right)\right]_{s=0} \tag{6}$$

and where, v_k are normalized constants by the factorial. The same kind of relation in (6) was found by Xi Li, [3] for the vanishing of the Riemann Zeta functions.

Let $\{\lambda_k\}$ be a sequence of numbers λ_k, given by

$$\lambda_k = \frac{1}{(k-1)!}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[s^{k-1}\log\left(\xi_k(s)\right)\right]_{s=1} \tag{7}$$

for all positive integers *k*, where,

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{8}$$

with $\zeta(s)$ being the Riemann ζ-function. Xi Li [3] showed that the Riemann hypothesis is equivalent to the positivity of the sequence of the real numbers λ_k defined by the relation,

$$\lambda_k = \sum_{\rho}\left[1-\left(1-\frac{1}{\rho}\right)^k\right] \geq 0 \tag{9}$$

where the sum is taken over all *nontrivial zeros* of the function, $\xi(s)$ with ρ and $1-\rho$ being paired together.

The aim of this paper is to show that the fixed points of an infinite number of iterations of any number theoretic function defined by an infinite number field are defined by functions such that $\lim_{T\to\infty} f^T(z) = z_\infty$, since $f^\infty(z) = f(z_\infty) = z_\infty$.

2. Iterations of Perron's Formula

Define the Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p, & \text{if } x = p^a,\ p \text{ is a prime} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Obviously, the function $\Lambda(n)$ is a weighing function on the primes. Thus, the function is related to the Least Common Multiple of numbers *n* and below,

$$\sum_{m\leq n}\Lambda(m) = \log\left(\mathrm{LCM}(1,2,3,\cdots,n)\right) \tag{11}$$

The relation to the Riemann zeta function is given by: J. B. Conrey [4], and by Titchmarsh [5].

$$\sum_{n=1}^{\infty}\frac{\Lambda(m)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \tag{12}$$

Obviously when $s=0$, (12) gives the sum:

$$\sum_{n=1}^{\infty}\Lambda(m) = -\frac{\zeta'(0)}{\zeta(0)} = -\log 2\pi \tag{13}$$

Define the function:

$$\psi(x) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right)\mathrm{d}s \tag{14}$$

The function (14) is well-known as Perron's formula, [5], and is related to the roots of the Riemann Zeta function via residue theorem as the explicit formula:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - x^{-2}\right) - \ln 2\pi \tag{15}$$

Define the iteration

$$\psi^{k}(x) = \underbrace{\psi\left(\psi\left(\psi\cdots\left(\psi(x)\right)\right)\right)}_{k\text{-times}}$$

with $\psi^{0}(x) = x$. Here k is not a power but a level for the iterations. Then,

$$\psi^{1}(x) = \psi\left(\psi^{0}(x)\right) = \psi^{0}(x) - \sum_{\rho} \frac{\left(\psi^{0}(x)\right)^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - \psi^{0}(x)^{-2}\right) - \ln 2\pi \tag{16}$$

Then,

$$\psi^{2}(x) = \psi\left(\psi^{1}(x)\right) = \psi^{1}(x) - \sum_{\rho} \frac{\left(\psi^{1}(x)\right)^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - \psi^{1}(x)^{-2}\right) - \ln 2\pi \tag{17}$$

i.e.,

$$\begin{aligned}\psi^{2}(x) = &\left[\psi^{0}(x) - \sum_{\rho} \frac{\left(\psi^{0}(x)\right)^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - \psi^{0}(x)^{-2}\right) - \ln 2\pi\right] \\ &- \sum_{\rho} \frac{\left(\psi^{1}(x)\right)^{\rho}}{\rho} - \ln 2\pi - \frac{1}{2}\ln\left(1 - \psi^{1}(x)^{-2}\right)\end{aligned} \tag{18}$$

$$\begin{aligned}\psi^{2}(x) = &\psi^{0}(x) - \sum_{\rho} \frac{\left(\psi^{0}(x)\right)^{\rho} + \left(\psi^{1}(x)\right)^{\rho}}{\rho} \\ &- \frac{1}{2}\ln\left(1 - \psi^{0}(x)^{-2}\right)\left(1 - \psi^{1}(x)^{-2}\right) - 2\ln 2\pi\end{aligned} \tag{19}$$

$$\psi^{3}(x) = \psi\left(\psi^{2}(x)\right) = \psi^{2}(x) - \sum_{\rho} \frac{\left(\psi^{2}(x)\right)^{\rho}}{\rho} - \frac{1}{2}\ln\left(1 - \left(\psi^{2}\right)^{-2}\right) - \ln 2\pi \tag{20}$$

$$\begin{aligned}\psi^{3}(x) = &\psi^{0}(x) - \sum_{\rho} \frac{\left(\psi^{0}(x)\right)^{\rho} + \left(\psi^{1}(x)\right)^{\rho} + \left(\psi^{2}(x)\right)^{\rho}}{\rho} \\ &- \frac{1}{2}\ln\left[\left(1 - \psi^{0}(x)^{-2}\right)\left(1 - \psi^{1}(x)^{-2}\right)\left(1 - \left(\psi^{2}\right)^{-2}\right)\right] - 3\ln 2\pi\end{aligned} \tag{21}$$

We conclude by induction that the k^{th} term is given by for $n \le k$:

$$\psi^{k}(x) = \psi^{0}(x) - \sum_{\rho}\sum_{n=0}^{k} \frac{\left(\psi^{n}(x)\right)^{\rho}}{\rho} - \frac{1}{2}\ln\prod_{n=0}^{k}\left(1 - \left(\psi^{n}(x)\right)^{-2}\right) - k\ln 2\pi \tag{22}$$

And,

$$\psi^{k}(x) = x - \sum_{\rho}\sum_{n=0}^{k} \frac{\left(\psi^{n}(x)\right)^{\rho}}{\rho} - \frac{1}{2}\ln\prod_{n=0}^{k}\left(1 - \left(\psi^{n}(x)\right)^{-2}\right) - k\ln 2\pi \tag{23}$$

Note that x is a stationary term in all the inducts.

Iterative functions are discussed by Baker, I.N. and Rippon, P.J. [6].

The function $\psi^k(x)$, follows the same rules as the original (15).

The question arises, what is the meaning of $\psi^k(x)$?

The function $\psi(x)$ is the log of the least common multiple of the ***integers*** from 1 through *x*. For example,

$$\begin{aligned}\psi(10) &= \sum_{n=0}^{10}\Lambda(n)\\ &= 0+\log 2+\log 3+\log 2+\log 5+0+\log 7+\log 2+\log 3+0 \\ &= \log(2520)\end{aligned} \tag{24}$$

It is clear that $\psi(x)=\sum_p \log p$, is defined over primes, *p*, and the iterations become,

$$\psi(\psi(x)) = \psi(\log(\mathrm{LCM}(x))) = \log(\mathrm{LCM}(\log(\mathrm{LCM}(x)))) = \log(q)$$

where, *q* is the expression of some numbers that serves as the "prime factors" of the super-logs,

$$\Lambda(x) = \begin{cases}\log q, & \text{if } x=q^a, q \text{ is a "primitive number"}\\ 0 & \text{otherwise}\end{cases} \tag{25}$$

$$\psi(\psi(x)) = \psi(\log(\mathrm{LCM}(x))) = \log(\mathrm{LCM}(\log(\mathrm{LCM}(x)))) = \log(q)$$

STATEMENT 1:

An infinite number of iterations can only be carried out when $k\to\infty$, $x\to\infty$, otherwise, the iteration is finite.

LEMMA1:

$$\psi^{k+2}(x) - 2\psi^{k+1}(x) + \psi^k(x) = 0 \tag{26}$$

Proof:

$$\psi^k(x) = x - \sum_\rho \sum_{n=0}^{k} \frac{(\psi^n(x))^\rho}{\rho} - \frac{1}{2}\ln\prod_{n=0}^{k}\left(1-(\psi^n(x))^{-2}\right) - k\ln 2\pi \tag{27}$$

$$\psi^{k+1}(x) = x - \sum_\rho \sum_{n=0}^{k+1} \frac{(\psi^n(x))^\rho}{\rho} - \frac{1}{2}\ln\prod_{n=0}^{k+1}\left(1-(\psi^n(x))^{-2}\right) - (k+1)\ln 2\pi \tag{28}$$

$$\begin{aligned}&\psi^{k+1}(x) - \psi^k(x)\\ &= x - \sum_\rho \sum_{n=0}^{k+1} \frac{(\psi^n(x))^\rho}{\rho} - \frac{1}{2}\ln\prod_{n=0}^{k+1}\left(1-(\psi^n(x))^{-2}\right) - (k+1)\ln 2\pi\\ &\quad - x + \sum_\rho \sum_{n=0}^{k} \frac{(\psi^n(x))^\rho}{\rho} + \frac{1}{2}\ln\prod_{n=0}^{k}\left(1-(\psi^n(x))^{-2}\right) + k\ln 2\pi\end{aligned}$$

$$\psi^{k+1}(x) - \psi^k(x) = -\sum_\rho \frac{(\psi^{k+1}(x))^\rho}{\rho} - \frac{1}{2}\ln\left(1-(\psi^{k+1}(x))^{-2}\right) - \ln 2\pi \tag{29}$$

$$\psi(\psi^{k+1}(x)) - \psi^{k+1} = -\sum_\rho \frac{(\psi^{k+1}(x))^\rho}{\rho} - \frac{1}{2}\ln\left(1-(\psi^{k+1}(x))^{-2}\right) - \ln 2\pi \tag{30}$$

$$\psi^{k+1}(x)-\psi^{k}(x)=\psi^{k+2}(x)-\psi^{k+1}(x) \tag{31}$$

$$\psi^{k+2}(x)=2\psi^{k+1}(x)-\psi^{k}(x) \tag{32}$$

We will now use LEMMA 1 and STATEMENT 1 to prove the following Theorem.

THEOREM: $\lim_{k\to\infty}\psi^{k}(x)=x$ *if and only if* $\lim_{x\to\infty}\psi(x)=x$.

Proof:

It is clear from (32) for example, that

$$\psi(x)=x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{1}{2}\ln\left(1-x^{-2}\right)-\ln 2\pi \tag{33}$$

And for $k=0$, in

$$\psi^{k+2}(x)=2\psi^{k+1}(x)-\psi^{k}(x) \tag{34}$$

$$\psi^{2}(x)=2\psi(x)-x=0 \tag{35}$$

$$\psi^{2}(x)=x-2\sum_{\rho}\frac{x^{\rho}}{\rho}-\ln\left(1-x^{-2}\right)-2\ln 2\pi \tag{36}$$

Similarly, for $k=1$,

$$\psi^{3}(x)=2\psi^{2}(x)-\psi(x) \tag{37}$$

$$\psi^{3}(x)=x-3\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{3}{2}\ln\left(1-x^{-2}\right)-3\ln 2\pi \tag{38}$$

and for $k=2$,

$$\psi^{4}(x)=2\psi^{3}(x)-\psi^{2}(x) \tag{39}$$

$$\begin{aligned}\psi^{4}(x)=&2\left(x-3\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{3}{2}\ln\left(1-x^{-2}\right)-3\ln 2\pi\right)\\&-\left(x-2\sum_{\rho}\frac{x^{\rho}}{\rho}-\ln\left(1-x^{-2}\right)-2\ln 2\pi\right)\end{aligned} \tag{40}$$

$$\psi^{4}(x)=x-4\sum_{\rho}\frac{x^{\rho}}{\rho}-2\ln\left(1-x^{-2}\right)-4\ln 2\pi \tag{41}$$

and for $k=4$,

$$\psi^{5}(x)=2\psi^{4}(x)-\psi^{3}(x) \tag{42}$$

$$\psi^{5}(x)=x-5\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{5}{2}\ln\left(1-x^{-2}\right)-5\ln 2\pi \tag{43}$$

In general, by induction,

$$\psi^{k}(x)=x-k\left(\sum_{\rho}\frac{x^{\rho}}{\rho}+\frac{1}{2}\ln\left(1-x^{-2}\right)+\ln 2\pi\right) \tag{44}$$

We arrive at the beautiful relation:

$$x-\psi^{k}(x)=k(x-\psi(x)) \tag{45}$$

The iterations can proceed to infinite values of *k*. In most iterations, the fixed points are dependent on the starting value of the argument for the function and terminate at some fixed value that becomes independent of *k*. However, Perron's formula indicates that the functional relations for the iterated flow of arguments depends and is proportional to the number of iterations, *k*. This implies a continuity of the iterants to infinite values, hence the limits of the theorem. This completes the proof, since, when $x=\psi(x)$, then, $x=\psi^{k}(x)$, and x is a fixed value, that is only satisfied by the condition:

$$\lim_{k\to\infty}\psi^{k}(x)=x \text{ if and only if } \lim_{x\to\infty}\psi(x)=x \tag{46}$$

3. Conclusions

The existence of fixed points of iterations provides a unique perspective on number theoretic functions. An iteration can continue indefinitely if the validity of the function is adhered to within the range of the iterants. It is well known that the prime number theorem indicates the main theorem given above. The main conclusion of the theorem requires the validity of the exponentiation of the Perron's function for an infinite sequence. Not all functions require an infinite number of iterations to converge. For example, the functional iterations of the Sums of Divisors $\sigma(N_0)$ following the sequence,

$$\begin{gathered}(N_0)\equiv N_0,\\ (\sigma(N_0)-1)=N,\\ (\sigma(\sigma(N_0)-1)-1)=N_2\\ (\sigma(\sigma(\sigma(N_0)-1)-1)-1)=N_3\\ \cdots(\sigma(\sigma(\cdots\sigma(N_0)-1)-1)\cdots-1)=N_k=N_{k+1}=N_{k+2}=\cdots=N_\infty\end{gathered} \tag{47}$$

terminates at different intervals of iterations. Amazingly, the sequence lengths *k*, can be conclusively determined to be less than some finite number for any integer. An example of this iteration is given in the table below. The question remains as to whether there exists a function that describes *k* for all N_0, since, the iterations are dependent on only a *specific set of primes and such a function will have its zeroes defined by the primes.*

N_0	Iterations	k	Fixed points N_∞
2	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]	0	3
3	[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]	0	4
4	[4, 6, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11]	3	12
5	[5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5]	0	6
6	[6, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11]	2	12

Continued

7	[7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7]	0	8
8	[8, 14, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23]	3	24
9	[9, 12, 27, 39, 55, 71, 71, 71, 71, 71, 71, 71, 71]	6	72
10	[10, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17]	2	18
11	[11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11]	0	12
12	[12, 27, 39, 55, 71, 71, 71, 71, 71, 71, 71, 71, 71]	5	72
13	[13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13]	0	14
14	[14, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23]	2	24
15	[15, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23]	2	24
16	[16, 30, 71, 71, 71, 71, 71, 71, 71, 71, 71, 71, 71]	3	72
17	[17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17]	0	18
18	[18, 38, 59, 59, 59, 59, 59, 59, 59, 59, 59, 59, 59]	3	60
19	[19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19, 19]	0	20
20	[20, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41, 41]	2	42
3000	[3000, 9359, 10943, 11327, 11615, 14687, 15479, 16175, 20087, 20519, 22103, 23831, 23831]	12	23832

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Anthony, M.M. (2016) The Towering Zeta Function. *Advances in Pure Mathematics*, **6**, 351-392. https://doi.org/10.4236/apm.2016.65026

[2] Galidakis, I.N. (2004) On an Application of Lambert'swfunction to Infinite Exponentials. *Complex Variables*, *Theory and Application*: *An International Journal*, **49**, 759-780. https://doi.org/10.1080/02781070412331298796

[3] Li, X. (1997) The Positivity of a Sequence of Numbers and the Riemann Hypothesis. *Journal of Number Theory*, **65**, 325-333. https://doi.org/10.1006/jnth.1997.2137

[4] Conrey, J.B. (1989) More Than Two Fifths of the Zeros of the Riemann Zeta Function Are on the Critical Line, *Journal fur die Reine und Angewandte Mathematik*, **399**, 1-16. https://doi.org/10.1515/crll.1989.399.1

[5] E. C. Titchmarsh F.R.S. (1986) The Theory of the Riemann Zeta Function. Clarendon Press.

[6] Baker, I.N. and Rippon, P.J. (1985) A Note on Complex Iteration. *The American Mathematical Monthly*, **92**, 501-504. https://doi.org/10.1080/00029890.1985.11971666