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Mathematical Modeling of the Co-Infection Dynamics of HIV and Tuberculosis Incorporating Inconsistency in HIV Treatment

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Abstract

A non-linear HIV-TB co-infection has been formulated and analyzed. The positivity and invariant region has been established. The disease free equilibrium and its stability has been determined. The local stability was determined and found to be stable under given conditions. The basic reproduction number was obtained and according to findings, co-infection diminishes when this number is less than unity, and persists when the number is greater than unity. The global stability of the endemic equilibrium was calculated. The impact of HIV on TB was established as well as the impact of TB on HIV. Numerical solution was also done and the findings indicate that when the rate of HIV treatment increases the latent TB increases while the co-infected population decreases. When the rate of HIV treatment decreases the latent TB population decreases and the co-infected population increases. Encouraging communities to prioritize the consistent treatment of HIV infected individuals must be emphasized in order to reduce the scourge of HIV-TB co-infection.

Keywords

Co-Infection Modeling, HIV-TB Co-Infection, Mathematical Modeling, Reproduction Number, Inconsistent Treatment

1. Introduction

Tuberculosis is caused by Mycobacterium tuberculosis. It occurs in two strains, the active and the latent strain. The active strain must be treated to avoid death of individuals. It has, as its reservoir, one-third of human population [1]. It mostly affects the lungs. The latent type becomes active when infected people acquire HIV. On the other hand, Human immunodeficiency Virus (HIV) is a

deadly virus which when not treated results to AIDS, which is incurable. It damages body organs like, the kidney, heart, and the brain [2]. If treatment lacks, the HIV patient has 9 - 11 years to live. It is impossible to eradicate the virus during the life time of a host [3]. Usually HIV appears to be associated with other diseases [4]. It can be transmitted through blood transfusion, mother to child during birth or by breastfeeding, different sexual relations. (WHO, 2008). The HIV virus falls to low levels when treatment is adhered to, but when treatment is withdrawn the viral load increases to the levels that existed before the administration.

Co-infection of HIV and TB refers to the existence of the two deadly pathogens, Human Immuno deficiency virus and Mycobacterium tuberculosis, in an individual. Their interaction in epidemiological characteristics is similar [5]. According to reports from World Health Organization, a person with both HIV and TB infections is 30 times more likely to develop TB illness than a person with TB only. According to findings, coordinated care, adherence support and drug interactions, play a great role in controlling and treating co-infection. Healthcare providers as well as patients encounter a complex set of challenges as they deal with co-infection. Therefore this study finds great need in investigating the impact of inconsistent HIV treatment on the spread dynamics of the co-infection of HIV and TB. In particular, inconsistent uptake of HIV medication is common in low income countries especially the sub saharan Africa region [6].

World Health Organization (WHO) estimates that around 100 million people are infected by TB worldwide. Every year 8.3 m new cases appear, worldwide, and 1.8 m deaths occur of which 93 percent of deaths appear in developing countries. [7]. Tuberculosis is among the top ten causes of death globally. WHO has declared the end of TB strategy by end of 2035. Targeting 90 percent reduction of incidence rate [8]. There is a growing tuberculosis epidemic whose data is scarce or insufficient to contain it in high HIV prevalence areas, especially sub-Saharan Africa ([6]). In 2012 twenty percent of 2.8 million people, infected with TB worldwide, were HIV positive, out of whom 42 percent were from sub-Saharan Africa (WHO 2013). There was a positive skewness in the distribution of young people. More than half of this population was aged between 15 - 49 years. All these were infected with TB and developed active TB more frequently [9] [10]. As such, this paper will develop a co-infection mathematical model that will analyze the spread dynamics of HIV-TB co-infection to provide possible control measures that can contain the spread of the two diseases.

Among people living with HIV, TB is the leading cause of death, since their immune systems are already compromised and are unable to effectively fight off the TB bacterium. It has become a major global challenge to public health sector. In 2012 there were an estimated 8.6 million new cases of tuberculosis (TB), of which 13 percent were HIV positive, globally ([11]). In 2015, only three fifths of the 10.5 m people infected by TB were reported to the public health authorities [12]. The co-occurrence of HIV and TB is known as TB/HIV co-infection. In

middle and low-income countries there is a significant public health issue of HIV/TB co-infection, where HIV and TB are prevalent. In countries with limited resources TB and HIV/AIDS make the main burden of infectious diseases [13].

HIV and TB are several times referred to as a "deadly pair" [5]. Co-infection is largely unknown but it is clear that every time an epidemic of HIV occurs, a severe TB epidemic accompanies it. There is a high possibility of HIV patients to develop TB infection and also activate latent to active TB ([14]). Adherence to medication and ongoing monitoring and follow-up care is very critical for the successful management of HIV and TB co-infection.

Co-infection of HIV and TB is associated with low lean body mass, in adults [15]. After destroying the immune system, HIV renders it incapable of performing its protective functions to the human body. It destroys this system by decreasing the number of T-cells that are responsible for fighting infections [16]. It reduces their number as it progresses in stages and hence the reason for calling it a retrovirus. Tuberculosis brings about death of HIV infectives. It is actually the leading factor of this mortality [17] [18].

It is recommended that the initiation of ART for any HIV infected individual who develops TB should be done immediately after testing positive [19]. The rate of recurrent TB is increased by HIV infection due to reinfection and not relapse [20]. The two pathogens (M. tuberculosis and HIV) potentiate one another and accelerate the deterioration of the immune functions in the body [13].

HIV and TB co-infection remain a major threat to public health and a challenge to health systems in middle income and low income countries [21]. Current strategies are not sufficient to contain the glowing tuberculosis (TB) epidemic in areas of high HIV prevalence, such as sub Saharan Africa. Testing and treatment at community level is recommended ([22]).

In this study the main objective is to develop and analyze a HIV-TB confection mathematical model to investigate the impact of inconsistent treatment of HIV on co-infection dynamics. The results were analyzed and conclusions made to inform appropriate recommendations to alleviate the problem.

2. Model Formulation

In this model we have nine compartments: Susceptible(S), Infected by HIV (I_H), those infected with Active TB (I_{Tb}) those with latent TB (L_{Tb}), those treated for TB (T_{Tb}), those who have recovered from TB R_{Tb} those who are co-infected by the two diseases (I_c) and those with clinical symptoms of AIDS (A_H).

Humans join the Susceptible compartment(S) by natural descent or immigration, at the rate of Λ. When exposed they join the compartment L_{Tb} at the rate of β_2, through mass action. From (S) they can join those infected by HIV, in the compartment (I_H), at the rate of α_1, through mass action.

From (I_H) they join the compartment (I_c), as they get co-infected by the two diseases, at the rate of τ_c. They can also join the compartment of (A_H) at the rate of γ_1 by mass action, and hence get infected by AIDS.

From the same compartment T_H they can get infected by Aids and join the compartment A_H, at the rate of ϕ_1 through mass action. When people are co-infected they can easily move to the compartment (A_H), at the rate of γ_c. Thus they get AIDS. From this compartment, (A_H) people die at the rate of σ_2 due to the disease.

People get co-infected when they are infected by TB and move from the compartment I_{TB}, to the compartment I_C, at the rate of α_c. Those with latent TB move from the compartment L_{Tb} to the co-infection compartment (I_C at the rate of ω_2. Those with TB but are treated could also become co-infected, if adherence to medication is not observed, or treatment wanes, and move to the compartment (I_C, at the rate of ω_c, through mass action. They could also get infected with TB when medication wanes, and from (T_{Tb}) to I_{TB} at the rate of α_2.

Those in this compartment, I_{TB}, die due to disease at the rate of σ_2. Those who have recovered from TB move to the susceptible (S) compartment at the rate of η_2, through mass action. Those treated for HIV can develop co-infection and move from T_H to the compartment (I_C), at the rate of κ_1, by mass action. Those co-infected can die of the co-infection at the rate of σ_c.

Those infected with TB recover at the rate of κ_2 and move from the compartment T_{Tb} to R_{Tb}. From every compartment the rate of natural death is μ. These compartments and parameters are summarized in the illustrative model drawn in Figure 1 below.

A summary of the parameters used to describe Figure 1 and their interpretation is given in Table 1 below.

2.1. Model Assumptions

The study has utilized the following model assumption

- There is no vertical transmission

Table 1. Co-infection transmission dynamics parameters.

Parameter	Interpretation
Λ	Recruitment rate
τ_c	Rate of progression from HIV to coinfection
α_c	Progression rate from TB infection to coinfection
β_c	Progression rate from Latent TB to coinfection
ω_c	Rate of progression from TB treatment to coinfection
ϕ_c	Rate of progression from Treatment of HIV to Coinfection
γ_c	Rate of progression from coinfection to AIDS
σ_c	Death due to coinfection
ω_2	Rate of progression from Latent TB to coinfection

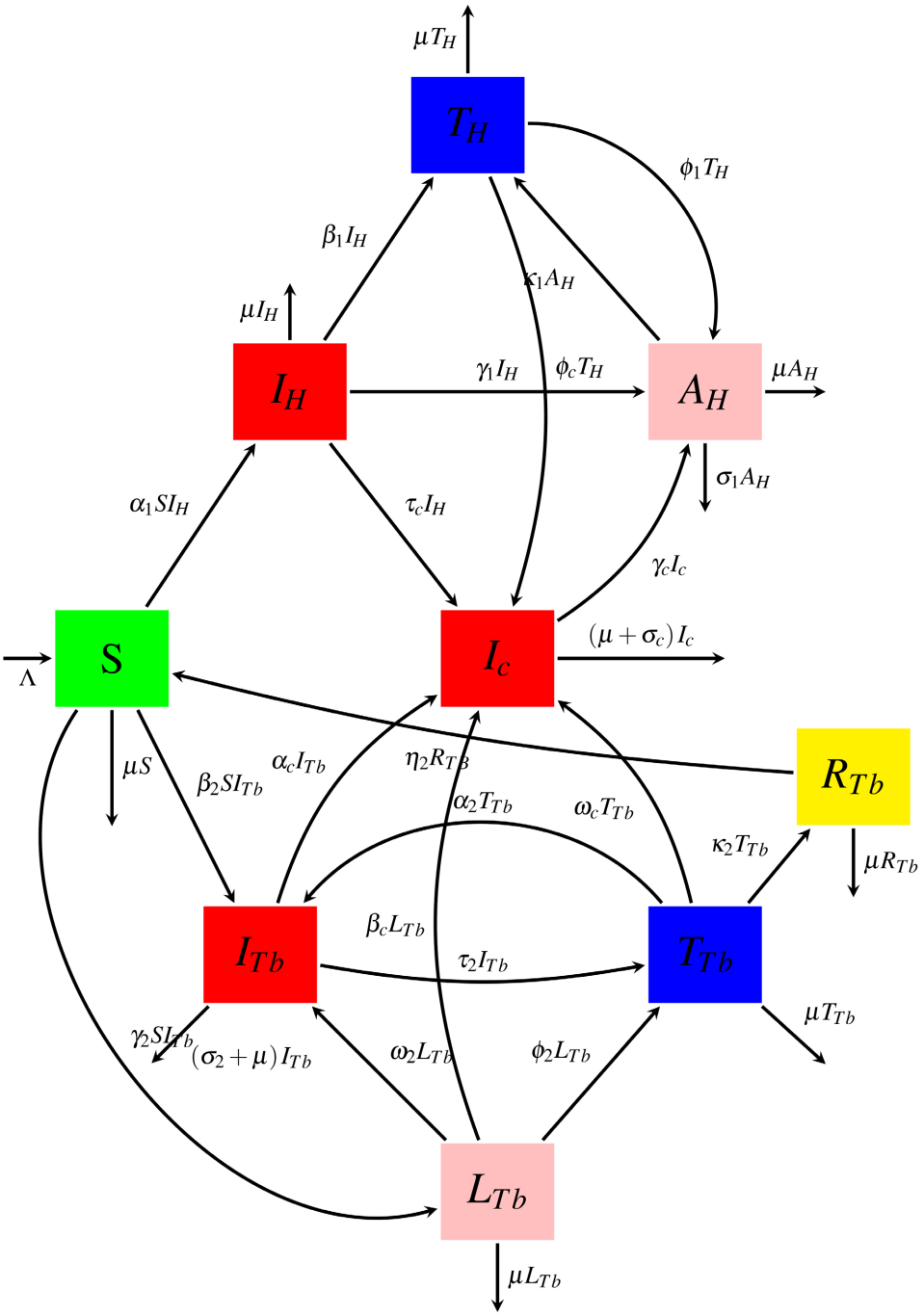

Figure 1. Model flow chart for HIV and TB co-infection spread dynamics.

- There is no immigration or emmigration
- There is free association among individuals.
- Susceptible humans are recruited at a constant rate

2.2. Model Equations for Coinfection

The model equations arising from the schematic model flow in **Figure 1** are given as.

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \alpha_1 SI_H - \beta_2 SI_{TB} - \gamma_2 SI_{TB} - \mu S + \eta_2 R_{TB} \\ \dfrac{\mathrm{d}I_H}{\mathrm{d}t} = \alpha_1 SI_H - (\mu + \tau_c + \gamma_1 + \beta_1) I_H \\ \dfrac{\mathrm{d}T_H}{\mathrm{d}t} = \beta_1 I_H + \kappa_1 A_H - (\mu + \phi_1 + \phi_c) T_H \\ \dfrac{\mathrm{d}A_H}{\mathrm{d}t} = \gamma_1 I_H + \phi_1 T_H + \gamma_c I_c - (\mu + \sigma_1 + \kappa_1) A_H \\ \dfrac{\mathrm{d}I_C}{\mathrm{d}t} = \tau_c I_H + \phi_c T_H + \omega_c T_{TB} + \beta_c L_{TB} + \alpha_c I_{TB} - (\mu + \sigma_c) I_c \\ \dfrac{\mathrm{d}I_{TB}}{\mathrm{d}t} = \beta_2 SI_{TB} + \omega_2 L_{TB} + \alpha_2 T_{TB} - (\alpha_c + \mu + \tau_2 + \sigma_2) I_{TB} \\ \dfrac{\mathrm{d}T_{TB}}{\mathrm{d}t} = \tau_2 I_{TB} + \phi_2 L_{TB} - (\omega_c + \kappa_2 + \alpha_2 + \mu) T_{TB} \\ \dfrac{\mathrm{d}L_{TB}}{\mathrm{d}t} = \gamma_2 SI_{TB} - (\omega_2 + \beta_c + \phi_2 + \mu) L_{TB} \\ \dfrac{\mathrm{d}R_{TB}}{\mathrm{d}t} = \kappa_2 T_{TB} - (\mu + \eta_2) R_{TB} \end{cases} \tag{2.1}$$

3. Basic Properties of the Model

3.1. Positivity and Invariant Region for Co-Infection

The total population is represented by $N(T)$ and is given by

$$N(t) = S(t) + I_H(t) + T_H(t) + A_H(t) + I_c(t) + I_{TB} + T_{TB} + L_{TB} + R_{TB} \tag{3.1}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I_H}{\mathrm{d}t} + \frac{\mathrm{d}T_H}{\mathrm{d}t} + \frac{\mathrm{d}A_H}{\mathrm{d}t} + \frac{\mathrm{d}I_c}{\mathrm{d}t} + \frac{\mathrm{d}I_{TB}}{\mathrm{d}t} + \frac{\mathrm{d}T_{TB}}{\mathrm{d}t} + \frac{\mathrm{d}L_{TB}}{\mathrm{d}t} + \frac{\mathrm{d}R_{TB}}{\mathrm{d}t} \tag{3.2}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \Lambda - \alpha_1 SI_H - \beta_2 SI_{TB} - \gamma_2 SI_{TB} - \mu S + \eta_2 R_{TB} + \tag{3.3}$$

$$\alpha_1 SI_H - (\mu + \tau_c + \gamma_1 + \beta_1) I_H + \gamma_1 I_h + \phi_1 \tau_H + \gamma_c I_c - (\mu + \sigma_1 + \kappa_1) A_H + \tag{3.4}$$

$$\tau_c I_H + \phi_c \tau_H + \omega_c T_{TR} + \beta_c L_{TB} + \alpha_c I_{TB} - (\mu + \sigma_c) I_c \tag{3.5}$$

$$\beta_2 SI_{TB} + \omega_2 L_{TB} + \alpha_2 T_{TB} - (\alpha_c + \mu + \tau_c + \sigma_2) I_{TB} \tag{3.6}$$

$$\tau_2 I_{TB} + \phi_2 L_{TB} - (\omega_c + \kappa_2 + \alpha_2 + \mu) T_{TB} + \gamma_2 SI_{TB} - (\omega_2 + \beta_c \phi_2 + \mu) T_{TB} + \tag{3.7}$$

$$\gamma_2 SI_{TB} - (\omega_2 + \beta_c + \phi_2 + \mu) L_{TB} + \tag{3.8}$$

$$\kappa_2 T_{TB} - (\mu + \eta_2) R_{TB} \tag{3.9}$$

Simplifying we obtain the following,

$$\frac{\mathrm{d}N}{\mathrm{d}t}(t) = \Lambda - \mu S - \mu I_H - \mu T_H - (\mu + \sigma_1) A_H - (\mu + \sigma_c) I_c - (\mu + \sigma_2) I_{TB} \tag{3.10}$$

In the absence of HIV, TB and Co-infection, We have,

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} = \Lambda - \mu(S(t)) + I_H(T) + T_H(t) + A(t) + I_c(t) \\ + I_{TB}(T) + T_{TB}(T) + L_{TB}(t) + R_{TB}(t) \end{aligned} \tag{3.11}$$

Hence,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \Lambda - \mu N(t) \tag{3.12}$$

Integrating both sides, we have,

$$\int \frac{\mathrm{d}N(t)}{\mathrm{d}t} = \int \left(\Lambda - \mu N(t)\right) \tag{3.13}$$

Let μ be=

$$\Lambda - \mu N(t) \tag{3.14}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}t} = -\mu \tag{3.15}$$

Hence,

$$\mathrm{d}N = \int \frac{\mathrm{d}U}{-\mu} \tag{3.16}$$

This is the same as,

$$= \frac{-1}{\mu} \int \frac{\mathrm{d}u}{\mu} = \frac{-1}{\mu} \ln u \tag{3.17}$$

but,

$$u = \Lambda - \mu N(t) \tag{3.18}$$

Therefore,

$$\frac{-1}{\mu} \ln\left(\Lambda - \mu N(t)\right) = t \tag{3.19}$$

So,

$$\ln\left(\Lambda - \mu N(t)\right) = -\mu t + c \tag{3.20}$$

We shall exponiate both sides to obtain,

$$\mathrm{e}^{\ln\left(\Lambda - \mu N(t)\right)} = A\mathrm{e}^{-\mu t + c} \tag{3.21}$$

So we have,

$$\Lambda - \mu N(t) = \mathrm{e}^{-\mu t + c} \tag{3.22}$$

Hence,

$$A\mathrm{e}^{-\mu t} \tag{3.23}$$

where

$$A = \mathrm{e}^{c} \tag{3.24}$$

We already have,

$$\Lambda - \mu N(t) = A\mathrm{e}^{-\mu t} \tag{3.25}$$

$$\frac{\mu N(t)}{\mu} = \frac{\Lambda - A\mathrm{e}^{-\mu t}}{\mu} \tag{3.26}$$

$$N(t) = \frac{\Lambda}{\mu} - \frac{A}{\mu}\mathrm{e}^{-\mu t} \tag{3.27}$$

But the initial condition is,

$$t = 0$$

Therefore,

$$N(0)=\frac{\Lambda}{\mu}-\frac{A}{\mu} \tag{3.28}$$

So,

$$A=\Lambda-\mu N(0) \tag{3.29}$$

Since,

$$N(t)=\frac{\Lambda}{\mu}-\frac{A}{\mu}\mathrm{e}^{-\mu t} \tag{3.30}$$

and,

$$A=\Lambda-\mu N(0) \tag{3.31}$$

Then,

$$N(t)=\frac{\Lambda}{\mu}-\frac{A-\mu N(0)}{\mu}\mathrm{e}^{-\mu t} \tag{3.32}$$

As,

$$Ast\to\infty, \tag{3.33}$$

$$\frac{A-\mu N(0)}{\mu}\to 0 \tag{3.34}$$

Hence,

$$N(t)\le\frac{\Lambda}{\mu} \tag{3.35}$$

The region generated is positively invariant.

3.2. Disease Free Equilibrium for Co-Infection

From the model Equations,

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t}&=\Lambda-\alpha_1 SI_H-\beta_2 SI_{TB}-\gamma_2 SI_{TB}-\mu S+\eta_2 R_{TB}\\
\frac{\mathrm{d}I_H}{\mathrm{d}t}&=\alpha_1 SI_H-\left(\mu+\tau_c+\gamma_1+\beta_1\right)I_H\\
\frac{\mathrm{d}T_H}{\mathrm{d}t}&=\beta_1 I_H+\kappa_1 A_H-\left(\mu+\phi_1+\phi_c\right)T_H\\
\frac{\mathrm{d}A_H}{\mathrm{d}t}&=\gamma_1 I_H+\phi_1 T_H+\gamma_c I_c-\left(\mu+\sigma_1+\kappa_1\right)A_H\\
\frac{\mathrm{d}I_C}{\mathrm{d}t}&=\tau_c I_H+\phi_c T_H+\omega_c T_{TB}+\beta_c L_{TB}+\alpha_c I_{TB}-\left(\mu+\sigma_c\right)I_c\\
\frac{\mathrm{d}I_{TB}}{\mathrm{d}t}&=\beta_2 SI_{TB}+\omega_2 L_{TB}+\alpha_2 T_{TB}-\left(\alpha_c+\mu+\tau_2+\sigma_2\right)I_{TB}\\
\frac{\mathrm{d}T_{TB}}{\mathrm{d}t}&=\tau_2 I_{TB}+\phi_2 L_{TB}-\left(\omega_c+\kappa_2+\alpha_2+\mu\right)T_{TB}\\
\frac{\mathrm{d}L_{TB}}{\mathrm{d}t}&=\gamma_2 SI_{TB}-\left(\omega_2+\beta_c+\phi_2+\mu\right)L_{TB}\\
\frac{\mathrm{d}R_{TB}}{\mathrm{d}t}&=\kappa_2 T_{TB}-\left(\mu+\eta_2\right)R_{TB}
\end{aligned} \tag{3.36}$$

At DFE,

$$I_H = 0, T_H = 0, A_H = 0, I_c = 0, I_{TB} = 0, L_{TB} = 0, R_{TB} = 0 \tag{3.37}$$

Within any given time, $\frac{\mathrm{d}}{\mathrm{d}t} = 0$ [23]. This implies that,

$$0 = \Lambda_1 - \alpha_1 S I_H - \beta_2 S I_{TB} - \gamma_2 S I_{TB} - \mu S + \eta_2 R_{TB} \tag{3.38}$$

$$0 = \alpha_1 S I_H - \left(\mu + \tau_c + \gamma_1 + \beta_1\right) I_H \tag{3.39}$$

$$0 = \beta_1 I_H + \kappa_1 A_H - \left(\mu + \phi_1 + \phi_c\right) T_H \tag{3.40}$$

$$0 = \gamma_1 I_H + \phi_1 T_H + \gamma_c I_c - \left(\mu + \sigma_1 + \kappa_1\right) A_H \tag{3.41}$$

$$0 = \tau_c I_H + \phi_c T_H + \omega_c T_{TB} + \beta_c L_{TB} + \alpha_c I_{TB} - \left(\mu + \sigma_c\right) I_c - \gamma_c I_c \tag{3.42}$$

$$0 = \beta_2 S I_{TB} + \omega_2 L_{TB} + \alpha_2 T_{TB} - \left(\alpha_c + \mu + \tau_2 + \sigma_2\right) I_{TB} \tag{3.43}$$

$$0 = \tau_2 I_{TB} + \phi L_{TB} - \left(\omega_c + \kappa_2 + \alpha_2 + \mu\right) T_{TB} \tag{3.44}$$

$$0 = \gamma_2 S I_{TB} - \left(\omega_2 + \beta_c + \phi_2 + \mu\right) L_{TB} \tag{3.45}$$

$$0 = \kappa_2 T_{TB} - \left(\mu + \eta_2\right) R_{TB} \tag{3.46}$$

We can write,

$$\frac{\mathrm{d}}{\mathrm{d}t} = 0 \tag{3.47}$$

and,

$$0 = \Lambda - \mu S \tag{3.48}$$

Therefore,

$$S^* = \frac{\Lambda}{\mu} \tag{3.49}$$

This implies that,

$$\left(S^*, I_H^*, T_H^*, A_H^*, I_c^*, I_{TB}^*, L_{TB}^*, R_{TB}^*\right) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0\right) \tag{3.50}$$

Stability of DFE of Co-Infection

The jacobian matrix from the model equations is:

$$J = \begin{bmatrix}
\frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I_H} & \frac{\partial f_1}{\partial T_H} & \frac{\partial f_1}{\partial A_H} & \frac{\partial f_1}{\partial I_C} & \frac{\partial f_1}{\partial I_{TB}} & \frac{\partial f 1}{\partial T_{TB}} & \frac{f_1}{\partial L_{TB}} & \frac{\partial f_1}{\partial R_{TB}} \\
\frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I_H} & \frac{\partial f_1}{\partial T_H} & \frac{\partial f_2}{\partial A_H} & \frac{\partial f_2}{\partial I_C} & \frac{\partial f_2}{\partial I_{TB}} & \frac{\partial f_2}{\partial T_{TB}} & \frac{\partial f_2}{\partial L_{TB}} & \frac{\partial f}{\partial R_{TB}} \\
\frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial I_H} & \frac{\partial f_3}{\partial T_H} & \frac{\partial f_3}{\partial A_H} & \frac{\partial f_3}{\partial I_C} & \frac{\partial f_3}{\partial I_{TB}} & \frac{\partial f_3}{\partial T_{TB}} & \frac{\partial f_3}{\partial L_{TB}} & \frac{\partial f_3}{\partial R_{TB}} \\
\frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial I_H} & \frac{\partial f_4}{\partial T_H} & \frac{\partial f_4}{\partial A_H} & \frac{\partial f_4}{\partial I_C} & \frac{\partial f_4}{\partial I_{TB}} & \frac{\partial f_4}{\partial T_{TB}} & \frac{\partial f_4}{\partial L_{TB}} & \frac{\partial f_4}{\partial R_{TB}} \\
\frac{\partial f_5}{\partial S} & \frac{\partial f_5}{\partial I_H} & \frac{\partial f_4}{\partial T_H} & \frac{\partial f_5}{\partial A_H} & \frac{\partial f_5}{\partial I_C} & \frac{\partial f_5}{\partial I_{TB}} & \frac{\partial f_5}{\partial T_{TB}} & \frac{\partial f_5}{\partial L_{TB}} & \frac{\partial f_5}{\partial R_{TB}} \\
\frac{\partial f_6}{\partial S} & \frac{\partial f_6}{\partial I_H} & \frac{\partial f_6}{\partial T_H} & \frac{\partial f_6}{\partial A_H} & \frac{\partial f_6}{\partial I_C} & \frac{\partial f_6}{\partial I_{TB}} & \frac{\partial f_6}{\partial T_{TB}} & \frac{\partial f_6}{\partial L_{TB}} & \frac{\partial f_6}{\partial R} \\
\frac{\partial f_7}{\partial S} & \frac{\partial f_7}{\partial I_H} & \frac{\partial f_7}{\partial T_H} & \frac{\partial f_7}{\partial A_H} & \frac{\partial f_7}{\partial I_C} & \frac{\partial f_7}{\partial I_{TB}} & \frac{\partial f_7}{\partial T_{TB}} & \frac{\partial f_7}{\partial L_{TB}} & \frac{\partial f_7}{\partial R_{TB}} \\
\frac{\partial f_8}{\partial S} & \frac{\partial f_8}{\partial I_H} & \frac{\partial f_8}{\partial T_H} & \frac{\partial f_8}{\partial A_H} & \frac{\partial 8}{\partial I_c} & \frac{\partial f_8}{\partial I_{TB}} & \frac{\partial f_8}{\partial T_{TB}} & \frac{\partial f_8}{\partial L_{TB}} & \frac{\partial f_8}{\partial R_{TB}} \\
\frac{\partial f_9}{\partial S} & \frac{\partial f_9}{\partial I_H} & \frac{\partial f_9}{\partial T_H} & \frac{\partial f_9}{\partial A_H} & \frac{\partial f_9}{\partial A_H} & \frac{\partial f_9}{\partial A_H} & \frac{\partial f_9}{\partial T_{TB}} & \frac{\partial f_9}{\partial L_{TB}} & \frac{\partial f_9}{\partial R_{TB}}
\end{bmatrix} \tag{3.51}$$

$$\begin{bmatrix} -\mu & -\frac{\alpha_1\Lambda}{\mu} & 0 & 0 & 0 & -\frac{\Lambda\beta_2}{\mu}-\frac{\Lambda\gamma_2}{\mu} & 0 & 0 & \eta_2 \\ 0 & \frac{\alpha_1\Lambda}{\mu}-\mu-\gamma_1-\beta_1-\tau_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & -\mu-\phi_1-\phi_c & \kappa_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & \phi_1 & -\kappa_1-\sigma_1-\mu & \gamma_c & 0 & 0 & 0 & 0 \\ 0 & \tau_c & \phi_c & 0 & -\mu-\sigma_c & \alpha_c & \omega_c & \beta_c & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Lambda\beta_2}{\mu}-\mu-\alpha_c-\sigma_2-\tau_2 & \alpha_2 & \omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_2 & -\kappa_2-\mu-\omega_c-\alpha_2 & \phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Lambda\gamma_2}{\mu} & 0 & -\beta_2-\mu-\phi_2-\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_2 & 0 & -\mu-\eta_2 \end{bmatrix} \tag{3.52}$$

At DFE, the Jacobian matrix will be,

$$\begin{bmatrix} -\mu & -\frac{\alpha_1\Lambda}{\mu} & 0 & 0 & 0 & -\frac{\Lambda\beta_2}{\mu}-\frac{\Lambda\gamma_2}{\mu} & 0 & 0 & \eta_2 \\ 0 & \frac{\alpha_1\Lambda}{\mu}-\mu-\gamma_1-\beta_1-\tau_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & -\mu-\phi_1-\phi_c & \kappa_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & \phi_1 & -\kappa_1-\sigma_1-\mu & \gamma_c & 0 & 0 & 0 & 0 \\ 0 & \tau_c & \phi_c & 0 & -\mu-\sigma_c & \alpha_c & \omega_c & \beta_c & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Lambda\beta_2}{\mu}-\mu-\alpha_c-\sigma_2-\tau_2 & \alpha_2 & \omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_2 & -\kappa_2-\mu-\omega_c-\alpha_2 & \phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\Lambda\gamma_2}{\mu} & 0 & -\beta_2-\mu-\phi_2-\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_2 & 0 & -\mu-\eta_2 \end{bmatrix} \tag{3.53}$$

In this matrix, the eigenvalues of the first and last columns are negative. We shall now work at the seven by seven matrix formed to determine whether the eigenvalues will be negative.

$$\begin{bmatrix} \frac{\alpha_1\Lambda}{\mu}-\mu-\gamma_1-\beta_1-\tau_c & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & -\mu-\phi_1-\phi_c & \kappa_1 & 0 & 0 & 0 & 0 \\ \gamma_1 & \phi_1 & -\kappa_1-\sigma_1-\mu & \gamma_c & 0 & 0 & 0 \\ \tau_c & \phi_c & 0 & -\mu-\sigma_c & \alpha_c & \omega_c & \beta_c \\ 0 & 0 & 0 & 0 & \frac{\Lambda\beta_2}{\mu}-\mu-\alpha_c-\sigma_2-\tau_2 & \alpha_2 & \omega_2 \\ 0 & 0 & 0 & 0 & \tau_2 & -\kappa_2-\mu-\omega_c-\alpha_2 & \phi_2 \\ 0 & 0 & 0 & 0 & \frac{\Lambda\gamma_2}{\mu} & 0 & -\beta_2-\mu-\phi_2-\omega_2 \end{bmatrix} \tag{3.54}$$

can be written as

$$\begin{bmatrix} \frac{\alpha_1\Lambda}{\mu}-k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & -k_2 & \kappa_1 & 0 & 0 & 0 & 0 \\ \gamma_1 & \phi_1 & -k_3 & \gamma_c & 0 & 0 & 0 \\ \tau_c & \phi_c & 0 & -k_4 & \alpha_c & \omega_c & \beta_c \\ 0 & 0 & 0 & 0 & \frac{\Lambda\beta_2}{\mu}-k_5 & \alpha_2 & \omega_2 \\ 0 & 0 & 0 & 0 & \tau_2 & -k_6 & \phi_2 \\ 0 & 0 & 0 & 0 & \frac{\Lambda\gamma_2}{\mu} & 0 & -k_7 \end{bmatrix} \tag{3.55}$$

whose trace is

$$\frac{\alpha_1\Lambda}{\mu}-k_1-k_2-k_3-k_4+\frac{\Lambda\beta_2}{\mu}-k_5-k_6-k_7 \tag{3.56}$$

and determinant

$$\frac{\left(\alpha_1\Lambda-\mu k_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2-\mu k_5k_6k_7\right)\left(\gamma_c\kappa_1\phi_c-k_2k_3k_4+k_4\kappa_1\phi_1\right)}{\mu^2} \tag{3.57}$$

For local stability of DFE to be attained the trace must be must be greater than zero while the determinant is less than zero. Since the trace is

$$\frac{\alpha_1\Lambda}{\mu}-k_1-k_2-k_3-k_4+\frac{\Lambda\beta_2}{\mu}-k_5-k_6-k_7>0 \tag{3.58}$$

We shall take all the positive values to the left hand side of the inequality and the negative ones to the right of the inequality. We have,

$$\frac{\alpha_1\Lambda}{\mu}+\frac{\Lambda\beta_2}{\mu}>k_1+k_2+k_3+k_4+k_5+k_6+k_7 \tag{3.59}$$

This Equation (3.59) is the first condition for local stability of the DFE of co-infection of the TB and HIV spread transmission dynamics. Similarly the second condition of local stability of the DFE of co-infection of TB and HIV can be determined by taking the positive values of the determinant in Equation (3.57) on the left hand side of the inequality and the negative values on the right hand side of the inequality. We have,

$$\frac{\left(\alpha_1\Lambda-\mu k_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2-\mu k_5k_6k_7\right)\left(\gamma_c\kappa_1\phi_c-k_2k_3k_4+k_4\kappa_1\phi_1\right)}{\mu^2}>0 \tag{3.60}$$

$$\left(\alpha_1\Lambda-\mu k_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2-\mu k_5k_6k_7\right)\left(\gamma_c\kappa_1\phi_c-k_2k_3k_4+k_4\kappa_1\phi_1\right)>0 \tag{3.61}$$

$$\begin{aligned} &\left(\alpha_1\Lambda\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2\right)-\alpha_1\Lambda\left(\mu k_5k_6k_7\right)\right. \\ &-\mu k_1\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2\right) \\ &\left.+\mu k_1\left(\mu k_5k_6k_7\right)\right)\left(\gamma_c\kappa_1\phi_c-k_2k_3k_4+k_4\kappa_1\phi_1\right)>0 \end{aligned} \tag{3.62}$$

$$\begin{aligned} &\alpha_1\Lambda\left(\gamma_c\kappa_1\phi_c+k_4\kappa_1\phi_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2\right) \\ &-\alpha_1\Lambda\left(\gamma_c\kappa_1\phi_c\right)\left(\mu k_5k_6k_7\right)-\mu k_1\left(\gamma_c\kappa_1\phi_c+k_4\kappa_1\phi_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7\right. \\ &\left.+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2\right)+\mu k_1\left(\gamma_c\kappa_1\phi_c+k_4\kappa_1\phi_1\right)\left(\mu k_5k_6k_7\right) \\ &-\left(k_2k_3k_4\right)\left(\alpha_1\Lambda\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2\right) \\ &+\left(k_2k_3k_4\right)\left(\alpha_1\Lambda\right)\left(\mu k_5k_6k_7\right)+\mu k_1\left(k_2k_3k_4\right)\left(\Lambda\alpha_2\gamma_2\phi_2+\Lambda\beta_2k_6k_7\right. \\ &\left.+\Lambda\gamma_2k_6\omega_2+\mu\alpha_2k_7\tau_2\right)-\mu k_1\left(k_2k_3k_4\right)\left(\mu k_5k_6k_7\right)>0 \end{aligned} \tag{3.63}$$

$$\begin{aligned}
&\alpha_1\Lambda\left(\gamma_c\kappa_1\phi_c + k_4\kappa_1\phi_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2 + \Lambda\beta_2k_6k_7 + \Lambda\gamma_2k_6\omega_2 + \mu\alpha_2k_7\tau_2\right)\\
&+ \mu k_1\left(\gamma_c\kappa_1\phi_c + k_4\kappa_1\phi_1\right)\left(\mu k_5k_6k_7\right) + \left(k_2k_3k_4\right)\left(\alpha_1\Lambda\right)\left(\mu k_5k_6k_7\right)\\
&+ \mu k_1\left(k_2k_3k_4\right)\left(\Lambda\alpha_2\gamma_2\phi_2 + \Lambda\beta_2k_6k_7 + \Lambda\gamma_2k_6\omega_2 + \mu\alpha_2k_7\tau_2\right)\\
&> \alpha_1\Lambda\left(\gamma_c\kappa_1\phi_c\right)\left(\mu k_5k_6k_7\right) + \mu k_1\left(\gamma_c\kappa_1\phi_c + k_4\kappa_1\phi_1\right)\left(\Lambda\alpha_2\gamma_2\phi_2\right.\\
&\left.+ \Lambda\beta_2k_6k_7 + \Lambda\gamma_2k_6\omega_2 + \mu\alpha_2k_7\tau_2\right) + \mu k_1\left(k_2k_3k_4\right)\left(\mu k_5k_6k_7\right)
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
&\left(\gamma_c\kappa_1\phi_c + k_4\kappa_1\phi_1\right)\left[\alpha_1\Lambda\left(\Lambda\alpha_2\gamma_2\phi_2 + \Lambda\beta_2k_6k_7 + \Lambda\gamma_2k_6\omega_2 + \mu\alpha_2k_7\tau_2\right)\right.\\
&\left.+ \mu k_1\left(\mu k_5k_6k_7\right)\right] + \left(k_2k_3k_4\right)\left(\alpha_1\Lambda\right)\left(\mu k_5k_6k_7\right)\\
&+ \mu k_1\left(\Lambda\alpha_2\gamma_2\phi_2 + \Lambda\beta_2k_6k_7 + \Lambda\gamma_2k_6\omega_2 + \mu\alpha_2k_7\tau_2\right)\\
&> \left(\gamma_c\kappa_1\phi_c + k_4\kappa_1\phi_1\right)\left[\alpha_1\Lambda\left(\mu k_5k_6k_7\right) + \mu k_1\left(\Lambda\alpha_2\gamma_2\phi_2 + \Lambda\beta_2k_6k_7\right.\right.\\
&\left.\left.+ \Lambda\gamma_2k_6\omega_2 + \mu\alpha_2k_7\tau_2\right)\right] + \mu k_1\left(k_2k_3k_4\right)\left(\mu k_5k_6k_7\right)
\end{aligned} \tag{3.65}$$

Equation (3.65) is the second condition for local stability of DFE of co-infection, where $k_1 = \mu + \gamma_1 + \beta_1 + \tau_c$, $k_2 = \mu + \phi_1 + \phi_c$, $k_3 = \kappa_1 + \sigma_1 + \mu$, $k_4 = \mu + \sigma_c$, $k_5 = \mu + \alpha_c + \sigma_2 + \tau_2$, $k_6 = \kappa_2 + \mu + \omega_c + \alpha_2$, $k_7 = \beta_2 + \mu + \phi_2 + \omega_2$. If the two conditions are met then the DFE is locally asymptotically stable.

3.3. Basic Reproductive Number for Co-Infection

From the model equations given below

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \alpha_1SI_H - \beta_2SI_{TB} - \gamma_2SI_{TB} - \mu S + \eta_2R_{TB}\\
\frac{\mathrm{d}I_H}{\mathrm{d}t} &= \alpha_1SI_H - \left(\mu + \tau_c + \gamma_1 + \beta_1\right)I_H\\
\frac{\mathrm{d}T_H}{\mathrm{d}t} &= \beta_1I_H + \kappa_1A_H - \left(\mu + \phi_1 + \phi_c\right)T_H\\
\frac{\mathrm{d}A_H}{\mathrm{d}t} &= \gamma_1I_H + \phi_1T_H + \gamma_cI_c - \left(\mu + \sigma_1 + \kappa_1\right)A_H\\
\frac{\mathrm{d}I_C}{\mathrm{d}t} &= \tau_cI_H + \phi_cT_H + \omega_cT_{TB} + \beta_cL_{TB} + \alpha_cI_{TB} - \left(\mu + \sigma_c\right)I_c\\
\frac{\mathrm{d}I_{TB}}{\mathrm{d}t} &= \beta_2SI_{TB} + \omega_2L_{TB} + \alpha_2T_{TB} - \left(\alpha_c + \mu + \tau_2 + \sigma_2\right)I_{TB}\\
\frac{\mathrm{d}T_{TB}}{\mathrm{d}t} &= \tau_2I_{TB} + \phi_2L_{TB} - \left(\omega_c + \kappa_2 + \alpha_2 + \mu\right)T_{TB}\\
\frac{\mathrm{d}L_{TB}}{\mathrm{d}t} &= \gamma_2SI_{TB} - \left(\omega_2 + \beta_c + \phi_2 + \mu\right)L_{TB}\\
\frac{\mathrm{d}R_{TB}}{\mathrm{d}t} &= \kappa_2T_{TB} - \left(\mu + \eta_2\right)R_{TB}
\end{aligned} \tag{3.66}$$

From the system of differential Equations (3.66), we shall take the infectious compartments as X and the non-infectious as Y and write them down as follows;

$$X = \begin{bmatrix} I_H \\ A_H \\ I_C \\ I_{TB} \\ T_{TB} \end{bmatrix} \tag{3.67}$$

$$Y = \begin{bmatrix} S \\ T_H \\ L_{TB} \\ R_{TB} \end{bmatrix} \tag{3.68}$$

Taking the parameters that bring in infection to X as f

$$f = \begin{bmatrix} \alpha_1 S I_H \\ \gamma_1 I_H + \phi_1 T_H \\ \phi_c T_H + \beta_c L_{TB} \\ \beta_2 S I_{TB} + \omega_2 L_{TB} \\ \phi_2 L_{TB} \end{bmatrix} \tag{3.69}$$

and those taking out infection from x as v, given below we have,

$$v = \begin{bmatrix} \left(\mu + \tau_c + \gamma_1 + \beta_1\right) I_H \\ -\gamma_1 I_H - \gamma_c I_c + \left(\mu + \sigma_1 + \kappa_1\right) A_H \\ -\tau_c I_H - \omega_c T_{TB} - \alpha_c I_{TB} + \left(\mu + \sigma_c\right) I_c \\ -\alpha_2 T_{TB} + \left(\alpha_c + \mu + \tau_2 + \sigma_2\right) I_{TB} \\ -\tau_2 I_{TB} + \left(\omega_c + \kappa_2 + \alpha_2 + \mu\right) T_{TB} \end{bmatrix} \tag{3.70}$$

Taking the equation of (3.69) as f_1, f_2, f_3, f_4 and f_5 respectively and obtaining their partial differentials, we have,

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial I_H} & \frac{\partial f_1}{\partial A_h} & \frac{\partial f_1}{\partial I_c} & \frac{\partial f_1}{\partial I_{TB}} & \frac{\partial f_1}{\partial T_{TB}} \\ \frac{\partial f_2}{\partial I_H} & \frac{\partial f_2}{\partial A_H} & \frac{\partial f_2}{\partial I_c} & \frac{\partial f_2}{\partial I_{TB}} & \frac{\partial f_2}{\partial T_{TB}} \\ \frac{\partial f_3}{\partial I_H} & \frac{\partial f_3}{\partial A_H} & \frac{\partial f_3}{\partial I_c} & \frac{\partial f_3}{\partial I_{TB}} & \frac{\partial f_3}{\partial T_{TB}} \\ \frac{\partial f_4}{\partial I_H} & \frac{\partial f_4}{\partial A_H} & \frac{\partial f_4}{\partial I_c} & \frac{\partial f_4}{\partial I_{TB}} & \frac{\partial f_4}{\partial T_{TB}} \\ \frac{\partial f_5}{\partial I_H} & \frac{\partial f_5}{\partial A_H} & \frac{\partial f_4}{\partial I_C} & \frac{\partial f_5}{\partial I_{TB}} & \frac{\partial f_5}{\partial T_{TB}} \end{bmatrix} \tag{3.71}$$

Which is equal to

$$\begin{bmatrix} \alpha_1 S & 0 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 S & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.72}$$

At DFE Equation (3.72) becomes,

$$F = \begin{bmatrix} \frac{\alpha_1 \Lambda}{\mu} & 0 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.73}$$

Similarly, solving for V from Equation (3.70), we have,

$$V=\begin{bmatrix} \frac{\partial v_1}{\partial I_H} & \frac{\partial v_1}{\partial A_H} & \frac{\partial v_1}{\partial I_c} & \frac{\partial v_1}{\partial I_{TB}} & \frac{\partial v_1}{\partial T_{TB}} \\ \frac{\partial v_2}{\partial I_H} & \frac{\partial v_2}{\partial A_H} & \frac{\partial v_2}{\partial I_c} & \frac{\partial v_2}{\partial I_{TB}} & \frac{\partial v_2}{\partial T_{TB}} \\ \frac{\partial v_3}{\partial I_H} & \frac{\partial v_3}{\partial A_H} & \frac{\partial v_3}{\partial I_c} & \frac{\partial v_3}{\partial I_{TB}} & \frac{\partial v_3}{\partial T_{TB}} \\ \frac{\partial v_4}{\partial I_H} & \frac{\partial v_4}{\partial A_H} & \frac{\partial v_4}{\partial I_c} & \frac{\partial v_4}{\partial I_{TB}} & \frac{\partial v_4}{\partial T_{TB}} \\ \frac{\partial v_5}{\partial I_H} & \frac{\partial v_5}{\partial A_h} & \frac{\partial v_5}{\partial I_c} & \frac{\partial v_5}{\partial I_{TB}} & \frac{\partial v_5}{\partial T_{TB}} \end{bmatrix} \tag{3.74}$$

Which is equal to,

$$V=\begin{bmatrix} \beta_1+\mu+\gamma_1+\tau_c & 0 & 0 & 0 & 0 \\ -\gamma_1 & \kappa_1+\sigma_1+\mu & -\gamma_c & 0 & 0 \\ -\tau_c & 0 & \mu+\sigma_c & -\alpha_c & -\omega_c \\ 0 & 0 & 0 & \tau_2+\sigma_2+\alpha_c+\mu & -\alpha_2 \\ 0 & 0 & 0 & -\tau_2 & \kappa_2+\mu+\omega_c+\alpha_2 \end{bmatrix} \tag{3.75}$$

We shall now obtain the inverse of V here below,

$$V^{-1}=\begin{bmatrix} A_1^{-1} & 0 & 0 & 0 & 0 \\ \frac{(\mu+\sigma_c)\gamma_1+\gamma_c\tau_c}{A_1(\mu+\sigma_c)A_2} & A_2^{-1} & \frac{\gamma_c}{(\mu+\sigma_c)A_2} & \frac{(\mu\alpha_c+A_5\alpha_c+\omega_c\tau_2)\gamma_c}{(\mu+\sigma_c)B_1A_2} & \frac{\gamma_c(\omega_c\mu+\omega_c+\alpha_2\alpha_c)}{(\mu+\sigma_c)\left(\mu^2+A_3\mu+A_4\omega_c+(\alpha_2+\kappa_2)\alpha_c+(\alpha_2+\kappa_2)\sigma_2+\kappa_2\tau_2\right)A_2} \\ \frac{\tau_c}{(A_1)(\mu+\sigma_c)} & 0 & (\mu+\sigma_c)^{-1} & \frac{\mu\alpha_c+A_5\alpha_c+\omega_c\tau_2}{(\mu+\sigma_c)B_1} & \frac{\omega_c\mu+A_4\omega_c+\alpha_2\alpha_c}{(\mu+\sigma_c)\left(\mu^2+A_3\mu+A_4\omega_c+(\alpha_2+\kappa_2)\alpha_c+(\alpha_2+\kappa_2)\sigma_2+\kappa_2\tau_2\right)} \\ 0 & 0 & 0 & \frac{\kappa_2+\mu+\omega_c+\alpha_2}{\mu^2+B_2} & \frac{\alpha_2}{\mu^2+A_3\mu+(\alpha_c+\sigma_2)\alpha_2+(\kappa_2+\omega_c)A_4} \\ 0 & 0 & 0 & \frac{\tau_2}{\mu^2+A_3\mu+A_5\alpha_c+A_6\kappa_2+A_6\omega_c+\alpha_2\sigma_2} & \frac{\tau_2+\sigma_2+\alpha_c+\mu}{\mu^2+A_3\mu+A_5\alpha_c+A_5\sigma_2+\tau_2(\kappa_2+\omega_c)} \end{bmatrix} \tag{3.76}$$

where

$$A_1=\beta_1+\mu+\gamma_1+\tau_c$$

$$A_2=\kappa_1+\sigma_1+\mu$$

$$A_3=\alpha_2+\alpha_c+\kappa_2+\omega_c+\sigma_2+\tau_2$$

$$A_4=\tau_2+\sigma_2+\alpha_c$$

$$A_5=\kappa_2+\omega_c+\alpha_2$$

$$A_6=\tau_2+\sigma_2$$

$$B_1=\mu^2+A_3\mu+A_5\alpha_c+A_6\kappa_2+A_6\omega_c+\alpha_2\sigma_2$$

$$B_2=A_3\mu+A_4\kappa_2+A_4\omega_c+(\alpha_c+\sigma_2)\alpha_2$$

$$B_3=A_3\mu+A_5\alpha_c+A_5\sigma_2+\tau_2(\kappa_2+\omega_c)$$

We shall now multiply F by the inverse of V to obtain

$$FV^{-1}=\begin{bmatrix} \frac{\alpha_1\Lambda}{\mu(\beta_1+\mu+\gamma_1+\tau_c)} & 0 & 0 & 0 & 0 \\ \frac{\gamma_1}{\beta_1+\mu+\gamma_1+\tau_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta_2\Lambda(\kappa_2+\mu+\omega_c+\alpha_2)}{(\mu^2+(\alpha_2+\alpha_c+\kappa_2+\omega_c+\sigma_2+\tau_2)\mu+(\tau_2+\sigma_2+\alpha_c)\kappa_2+(\tau_2+\sigma_2+\alpha_c)\omega_c+(\alpha_c+\sigma_2)\alpha_2)\mu} & \frac{\beta_2\Lambda\alpha_2}{(\mu^2+(\alpha_2+\alpha_c+\kappa_2+\omega_c+\sigma_2+\tau_2)\mu+(\alpha_c+\sigma_2)\alpha_2+(\kappa_2+\omega_c)(\tau_2+\sigma_2+\alpha_c))\mu} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.77}$$

The eigenvalues will be

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\beta_2\Lambda(\kappa_2+\mu+\omega_c+\alpha_2)}{(\mu^2+(\alpha_2+\alpha_c+\kappa_2+\omega_c+\sigma_2+\tau_2)\mu+(\tau_2+\sigma_2+\alpha_c)\kappa_2+(\tau_2+\sigma_2+\alpha_c)\omega_c+(\alpha_c+\sigma_2)\alpha_2)\mu} \\ \frac{\alpha_1\Lambda}{\mu(\beta_1+\mu+\gamma_1+\tau_c)} \end{bmatrix} \tag{3.78}$$

The most dominant eigenvalue will be,

$$R_0=\frac{\beta_2\Lambda(\kappa_2+\mu+\omega_c+\alpha_2)}{(\mu^2+(\alpha_2+\alpha_c+\kappa_2+\omega_c+\sigma_2+\tau_2)\mu+(\tau_2+\sigma_2+\alpha_c)\kappa_2+(\tau_2+\sigma_2+\alpha_c)\omega_c+(\alpha_c+\sigma_2)\alpha_2)\mu} \tag{3.79}$$

Equation (3.79) is the basic reproduction number of the co-infection spread dynamics. When R_0 is greater than 1 the co-infection persist in the population while when R_0 is less than one the co-infection dies down.

4. Global Stability of the Endemic Equilibrium

In order to determine the global stability we shall use Lypanov's method.

$$\begin{aligned} &L(S^{**},I^{**},T_H^{**},A_H^{**},I_C^{**},I_{TB}^{**},T_{TB}^{**},L_{TB}^{**},R_{TB}^{**}) \\ &=\left(S-S^{**}-S^{**}\ln\frac{S^{**}}{S}\right)+\left(I_H-I_H^{**}-I_H^{**}\ln\frac{I_H^{**}}{I_H}\right)+\left(T_H-T_H^{**}-T_H^{**}\ln\frac{T_H^{**}}{T_H}\right) \\ &+\left(A_H-A_H^{**}-A_H^{**}\ln\frac{A^{**}}{A_H}\right)+\left(I_C-I_C^{**}-I_C^{**}\ln\frac{I_C^{**}}{I_C}\right)+\left(I_{TB}-I_{TB}^{**}-I_{TB}^{**}\ln\frac{I_{TB}^{**}}{I_{TB}}\right) \\ &+\left(T_{TB}-T_{TB}^{**}-T_{TB}^{**}\ln\frac{T_{TB}^{**}}{T_{TB}}\right)+\left(L_{TB}-L_{TB}^{**}-L_{TB}^{**}\ln\frac{L_{TB}^{**}}{L_{TB}}\right)+\left(R_{TB}-R_{TB}^{**}-R_{TB}^{**}\ln\frac{R_{TB}^{**}}{R_{TB}}\right) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t}&=\left(\frac{S-S^{**}}{S}\right)\frac{\mathrm{d}S}{\mathrm{d}t}+\left(\frac{I_H-I_H^{**}}{I_H}\right)\frac{\mathrm{d}I_H}{\mathrm{d}t}+\left(\frac{T_H-T_H^{**}}{T_H^{**}}\right)\frac{\mathrm{d}T_H}{\mathrm{d}t} \\ &+\left(\frac{A_H-A_H^{**}}{A_H}\right)\frac{\mathrm{d}A_H}{\mathrm{d}t}+\left(\frac{I_C-I_C^{**}}{I_c}\right)\frac{\mathrm{d}I_c}{\mathrm{d}t}+\left(\frac{I_{TB}-I_{TB}^{**}}{I_{TB}}\right)\frac{\mathrm{d}I_{TB}}{\mathrm{d}t} \\ &+\left(\frac{T_{TB}-T_{TB}^{**}}{T_{TB}}\right)\frac{\mathrm{d}T_{TB}}{\mathrm{d}t}+\left(\frac{L_{TB}-L_{TB}^{**}}{L_{TB}}\right)\frac{\mathrm{d}L_{TB}}{\mathrm{d}t}+\left(\frac{R_{TB}-R_{TB}^{**}}{R_{TB}}\right)\frac{\mathrm{d}R_{TB}}{\mathrm{d}t} \end{aligned}$$

Differentiating we have,

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\left(I-\frac{S^{**}}{S}\right)\left(\Lambda-\alpha_1 SI-\beta_2 SI_{TB}+\gamma_2 SI_{TB}-\mu S+\eta_2 R_{TB}\right)$$
$$+\left(1-\frac{I_H^{**}}{I_H}\right)\left(\alpha_1 SI_H-\left(\mu+\tau_1+\gamma_1+\beta_1\right)I_H\right)$$
$$+\left(1-\frac{T_H^{**}}{T_H}\right)\left(\beta_1 I_H+\kappa_1 A_H-\left(\mu+\phi_1+\phi_c\right)T_H\right)$$
$$+\left(1-\frac{A_H^{**}}{A_H}\right)\left(\gamma_1 I_H+\phi_1 T_H+\gamma_c I_C-\left(\mu+\sigma_1+\kappa_1\right)A_H\right)$$
$$+\left(1-\frac{I_C^{**}}{I_C}\right)\left(\gamma_1 I_H+\phi_c T_H+\omega_c T_{TB}+\beta_c L_{TB}+\alpha_c I_{TB}-\left(\mu+\sigma_c\right)I_c\right)$$
$$+\left(1-\frac{I_{TB}^{**}}{I_{TB}}\right)\left(\beta_2 SI_{TB}+\omega_2 L_{TB}+\alpha_2 T_{TB}-\left(\alpha_c+\mu+\tau_2+\sigma_2\right)I_{TB}\right)$$
$$+\left(1-\frac{T_{TB}^{**}}{T_{TB}}\right)\left(\tau_2 L_{TB}+\phi_2 L_{TB}+\left(\omega_c+\kappa_2+\alpha_2+\mu\right)T_{TB}\right)$$
$$+\left(1-\frac{L_{TB}^{**}}{L_{TB}}\right)\left(\gamma_2 SI_{TB}-\left(\omega_2+\beta_c+\phi_2+\mu\right)L_{TB}\right)$$
$$+\left(1-\frac{R_{TB}^{**}}{R_{TB}}\right)\left(\kappa_2 T_{TB}-\left(\mu+\eta_2\right)R_{TB}\right)$$

Opening the brackets and simplifying we obtain the following values,

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\Lambda-\alpha_1 SI_H-\beta_2 SI_{TB}-\gamma_2 SI_{TB}-\mu S+\eta_2 R_{TB}-\Lambda\frac{S^{**}}{S}+\alpha_1 S^{**}I_H+\beta_2 S^{**}I_{TB}$$
$$+\gamma_2 S^{**}I_{TB}+\mu S^{**}-\eta_2 R_{TB}\frac{S^{**}}{S}+\alpha_1 SI_H-\left(\mu+\tau_c+\gamma_1+\beta_1\right)I_H-\alpha_1 SI_H^{**}$$
$$+\left(\mu+\tau_c+\gamma_1+\beta_1\right)I_H^{**}+\beta_1 I_H+\kappa_1 A_H-\left(\mu+\phi_1+\phi_c\right)T_H-\beta_1 I_H\frac{I_H^{**}}{I_H}$$
$$-\kappa_1 A_H\frac{T_H^{**}}{T_H}+\left(\mu+\phi_1+\phi_c\right)T_H^{**}+\gamma_1 I_H+\phi_1 T_H+\gamma_c I_c-\left(\mu+\sigma_1+\kappa_1\right)A_H$$
$$-\gamma_1 I_H\frac{A_H^{**}}{A_H}-\phi_1 T_H\frac{A_H^{**}}{A_H}-\gamma_c I_c\frac{A_H^{**}}{A_H}+\left(\mu+\sigma_1+\kappa_1\right)A_H^{**}+\tau_c I_H+\phi_c T_H$$
$$+\omega_c T_{TB}+\beta_c L_{TB}+\alpha_c I_{TB}-\left(\mu+\phi_c\right)I_c-\left(\tau_c T_H-\phi_c T_H-\omega_c T_{TB}-\beta_c L_{TB}\right.$$
$$-\alpha_c I_{TB}+\left(\mu+\sigma c\right)I_c^{**}+\beta_2 SI_{TB}+\omega_2 L_{TB}+\alpha_2 T_{TB}+\left(\alpha_c+\mu+\tau_2+\sigma_2\right)I_{TB}$$
$$-\beta_2 SI_{TB}^{**}-\omega_2\frac{I_{TB}^{**}}{I_{TB}}-\alpha_2 T_{TB}\frac{I_{TB}^{**}}{I_{TB}}+\left(\alpha_c+\mu+\tau_2+\sigma_2\right)I_{TB}^{**}+\left(\tau_2 I_{TB}+\phi_2 L_{TB}\right.$$
$$\left.-\left(\omega_2+\kappa_2+\alpha_2+\mu\right)T_{TB}\right)-\tau_2 I_{TB}\frac{L_{TB}^{**}}{L_{TB}}+\left(\omega_2+\kappa_2+\alpha_2+\mu\right)T_{TB}^{**}+\gamma_2 SI_{TB}$$
$$-\left(\omega_2+\beta_c+\phi_c+\mu\right)L_{TB}-\gamma_2 SI_{TB}\frac{L_{TB}^{**}}{L_{TB}}+\kappa_2 T_{TB}-\left(\mu+\eta_2\right)R_{TB}-\kappa_2 T_{TB}\frac{R_{TB}^{**}}{R_{TB}}$$
$$+\left(\mu+\eta_2\right)R_{TB}^{**}$$

Expanding by using parameter values we have,

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \left(\Lambda+\eta_2 R_{TB}+\alpha_1 S^{**} I_H+\beta_2 S^{**} I_{TB}+\gamma_2 S^{**} I_{TB}+\alpha_1 S I_H+\left(\mu+\tau_c+\gamma_1+\beta_1\right) I_H^{**}\right.\\ &+\beta_1 I_H+\kappa_1 A_H+\left(\mu+\phi_1+\phi_c\right) T_H^{**}+\gamma_1 I_H+\phi_1 T_H+\gamma_c I_c+\left(\mu+\sigma_1+\kappa_1\right) A_H^{**}\\ &+\tau_c I_H+\phi_c T_H+\omega_c T_{TB}+\beta_c L_{TB}+\left(\mu+\phi_c\right) I_c^{**}+\beta_2 S I_{TB}+\omega_2 L_{TB}+\alpha_2 T_{TB}\\ &+\left(\alpha_c+\tau_2+\sigma_2\right) I_{TB}+\left(\alpha_c+\mu+\tau_2+\sigma_2\right) I_{TB}^{**}+\tau_2 I_{TB}^{**}+\tau_2 I_{TB}\phi_2 L_{TB}+\left(\omega_2+\kappa_2\right.\\ &\left.+\alpha_2+\mu\right) T_{TB}^{**}+\gamma S I_{TB}+\left(\omega_2+\beta_c+\phi_c+\mu\right) L_{TB}^{**}\kappa_2 T_{TB}+\left(\mu+\eta_2\right) R_{TB}^{**}+\mu S^{**}\Big)\\ &-\left(\alpha_1 S I_H+\beta_2 S I_{TB}+\gamma_2 S I_{TB}+\mu S+\Lambda\frac{S^{**}}{S}+\eta_2 R_{TB}\frac{S^{**}}{S}+\left(\mu+\tau_c+\gamma_1+\beta_1\right) I_H\right.\\ &+\alpha_1 S I_H^{**}+\left(\mu+\phi_1+\phi_c\right) T_H+\beta_1 I_H\frac{T_H^{**}}{T_H}+\kappa_1 A_H\frac{T_H^{**}}{T_H}+\left(\mu+\sigma_1+\kappa_1\right) A_H\\ &+\gamma_1 I_H\frac{A_H^{**}}{A_h}+\phi_1 T_H\frac{A_H^{**}}{A_H}+\gamma_c I_c\frac{A_H^{**}}{A_h}+\left(\mu+\sigma_c\right) I_c+\tau_c T_H+\omega_c T_{TB}+\beta_c L_{TB}+\alpha_c I_{TB}\\ &+\beta_2 S I_{TB}^{**}+\omega_2 L_{TB}\frac{I_{TB}^{**}}{I_{TB}}+\alpha_2 T_{TB}\frac{I_{TB}^{**}}{I_{TB}}+\left(\omega_2+\kappa_2+\alpha_2+\mu\right) T_{TB}+\tau_2 I_{TB}\frac{T_{TB}^{**}}{T_{TB}}\\ &\left.+\phi_2 L_{TB}\frac{T_{TB}^{**}}{T_{TB}}+\left(\omega_2+\beta_c+\phi_c+\mu\right) L_{TB}+\gamma_2 S I_{TB}\frac{L_{TB}^{**}}{L_{TB}}+\left(\mu+\eta_2\right) R_{TB}+\kappa_2 T_{TB}\frac{R_{TB}^{**}}{R_{TB}}\right)\end{aligned}$$

where

$$\begin{aligned}T &= \Lambda+\eta_2 R_{TB}+\alpha_1 S^{**} I_H+\beta_2 S^{**} I_{TB}+\gamma_2 S^{**} I_{TB}+\alpha_1 S I_H+\left(\mu+\tau_c+\gamma_1+\beta_1\right) I_H^{**}\\ &+\beta_1 I_H+\kappa_1 A_H+\left(\mu+\phi_1+\phi_c\right) T_H^{**}+\gamma_1 I_H+\phi_1 T_H+\gamma_c I_c+\left(\mu+\sigma_1+\kappa_1\right) A_H^{**}\\ &+\tau_c I_H+\phi_c T_H+\omega_c T_{TB}+\beta_c L_{TB}+\left(\mu+\phi_c\right) I_c^{**}+\beta_2 S I_{TB}+\omega_2 L_{TB}+\alpha_2 T_{TB}\\ &+\left(\alpha_c+\tau_2+\sigma_2\right) I_{TB}+\left(\alpha_c+\mu+\tau_2+\sigma_2\right) I_{TB}^{**}+\tau_2 I_{TB}^{**}+\tau_2 I_{TB}\phi_2 L_{TB}+\left(\omega_2+\kappa_2\right.\\ &\left.+\alpha_2+\mu\right) T_{TB}^{**}+\gamma S I_{TB}+\left(\omega_2+\beta_c+\phi_c+\mu\right) L_{TB}^{**}\kappa_2 T_{TB}+\left(\mu+\eta_2\right) R_{TB}^{**}+\mu S^{**}\end{aligned}$$

and

$$\begin{aligned}Q &= \alpha_1 S I_H+\beta_2 S I_{TB}+\gamma_2 S I_{TB}+\mu S+\Lambda\frac{S^{**}}{S}+\eta_2 R_{TB}\frac{S^{**}}{S}+\left(\mu+\tau_c+\gamma_1+\beta_1\right) I_H\\ &+\alpha_1 S I_H^{**}+\left(\mu+\phi_1+\phi_c\right) T_H+\beta_1 I_H\frac{T_H^{**}}{T_H}+\kappa_1 A_H\frac{T_H^{**}}{T_H}+\left(\mu+\sigma_1+\kappa_1\right) A_H\\ &+\gamma_1 I_H\frac{A_H^{**}}{A_h}+\phi_1 T_H\frac{A_H^{**}}{A_H}+\gamma_c I_c\frac{A_H^{**}}{A_h}+\left(\mu+\sigma_c\right) I_c+\tau_c T_H+\omega_c T_{TB}+\beta_c L_{TB}+\alpha_c I_{TB}\\ &+\beta_2 S I_{TB}^{**}+\omega_2 L_{TB}\frac{I_{TB}^{**}}{I_{TB}}+\alpha_2 T_{TB}\frac{I_{TB}^{**}}{I_{TB}}+\left(\omega_2+\kappa_2+\alpha_2+\mu\right) T_{TB}+\tau_2 I_{TB}\frac{T_{TB}^{**}}{T_{TB}}\\ &+\phi_2 L_{TB}\frac{T_{TB}^{**}}{T_{TB}}+\left(\omega_2+\beta_c+\phi_c+\mu\right) L_{TB}+\gamma_2 S I_{TB}\frac{L_{TB}^{**}}{L_{TB}}+\left(\mu+\eta_2\right) R_{TB}+\kappa_2 T_{TB}\frac{R_{TB}^{**}}{R_{TB}}\end{aligned}$$

Therefore for global stability of the Endemic equilibrium to be attained Q must be greater than T, that is, $Q>T$ since $\frac{\mathrm{d}L}{\mathrm{d}t}=T-Q$. When $Q>T$ then $\frac{\mathrm{d}L}{\mathrm{d}t}<0$ because the basic reproduction R_0 is greater than 1 hence the co-infection of HIV-TB persists in the population.

5. Impact of HIV on TB

Considering the basic reproduction number of HIV given below

$$R_{0H} = \frac{\alpha_1 \Lambda_1}{(\mu + \beta_1)\mu} \tag{5.1}$$

making μ the subject in (5.1) above we have

$$\mu = \frac{-R_{oH}\beta_1 + \sqrt{R_{oH}^2 \beta_1^2 + 4R_{oH}\alpha_1 \Lambda_1}}{2R_{oH}} \tag{5.2}$$

which can be further simplified to get

$$\mu = -\frac{\beta_1}{2} + \frac{\sqrt{R_{oH}\beta_1^2 + 4\alpha_1 \Lambda_1}}{2\sqrt{R_{oH}}} \tag{5.3}$$

Considering the basic reproduction number of TB given by

$$R_{0T} = \frac{\beta_2 \Lambda_2}{\mu\left((\sigma_2 + \mu + \tau_2) - \tau_2 \alpha_2\right)} \tag{5.4}$$

and substituting the value of μ in Equation (5.3) in Equation (5.4) we have

$$R_{oT} = \frac{2B_1\sqrt{R_{oH}(C)}}{2A_1(C)\sqrt{R_{oH}(C)} + \sqrt{R_{oH}(C)\left(\beta_1(C)\right)^2 + A_2(C)}} \tag{5.5}$$

where

$$A_1 = -\frac{\beta_1}{2} \tag{5.6}$$

$$A_2 = 4\alpha_1 \Lambda_1 \tag{5.7}$$

$$B_1 = \beta_2 \Lambda_2 \tag{5.8}$$

$$B_2 = \tau_2 \alpha_2 \tag{5.9}$$

$$C = \frac{2A_1\sqrt{R_{oH}} - 2B_2\sqrt{R_{oH}} + 2\tau_2\sqrt{R_{oH}} + 2\sigma_2\sqrt{R_{oH}} + \sqrt{R_{oH}\beta_1^2 + A_2}}{2\sqrt{R_{oH}}} \tag{5.10}$$

6. Impact of TB on HIV

Considering the basic reproduction number of TB given below,

$$R_{0T} = \frac{\beta_2 \Lambda_2}{\mu\left((\sigma_2 + \mu + \tau_2) - \tau_2 \alpha_2\right)} \tag{6.1}$$

making μ the subject in (6.1) above we obtained

$$\mu = \frac{\sqrt{R_{0T}\left(\left(-\alpha_2\tau_2 + \tau_2 + \sigma_2\right)^2 R_{0T} + 4\beta_2\Lambda_2\right)} + \left(\alpha_2\tau_2 - \tau_2 - \sigma_2\right)R_{0T}}{2R_{0T}} \tag{6.2}$$

Now considering the basic reproduction number of HIV given below

$$R_{0H} = \frac{\alpha_1 \Lambda_1}{(\mu + \beta_1)\mu} \tag{6.3}$$

and substituting the value of μ given by Equation (6.2) in Equation (6.3) we got

$$R_{oH} = \frac{4\alpha_1\Lambda_1 R_{oT}^2}{\left(-\sqrt{R_{oT}\left(A_2^2 R_{oT} + 4\beta_2\Lambda_2\right)} + R_{oT}A_2\right)\left(-\sqrt{R_{oT}\left(A_2^2 R_{oT} + 4\beta_2\Lambda_2\right)} + R_{oT}\left(A_2 - 2\beta_1\right)\right)} \quad (6.4)$$

where

$$A_2 = -\alpha_2\tau_2 + \tau_2 + \sigma_2$$

Numerical Simulations

In the following section we explored the spread dynamics of the HIV-TB co-infection in a population. From the model parameters we have obtained the values used to draw the following graphs. The initial conditions used for the graphs were as follows: $S(0)=98999$, $I_H(0)=1000$, $T_H(0)=1000$, $A_H(0)=100$, $I_C(0)=100$, $I_{TB}(0)=1000$, $T_{TB}(0)=100$, $L_{TB}(0)=100$, $R_{TB}(0)=2000$. The graphs were drawn using maple software which utilized rungekutta order four method. The values of the parameters and their corresponding reference sources are summarized in Table 2 below.

The general population increases gradually within the given time at a constant rate. Close to this general population is the susceptible population, which drops at a small rate an d then levels off. The treatment HIV population rises from zero to around 700 and then falls to around 200 and keeps dropping in value to almost an insignificant one at the end of the period of study. The infected HIV population rises from zero to around 100 and then falls to a very small value until it goes to zero, for this period. The AIDS population is almost insignificant

Table 2. Co-infection transmission dynamics parameters.

Parameter	Interpretation	Value	Reference
Λ	Recruitment rate	1650	[24]
τ_c	Rate of progression from HIV to coinfection	0.000004	[5]
α_c	Progression rate from TB infection to coinfection	0.004	[24]
β_c	Progression rate from Latent TB to coinfection	0.00000001	[18]
ω_c	Rate of progression from TB treatment to coinfection	0.0001	[18]
ϕ_c	Rate of progression from Treatment of HIV to Coinfection	0.00000000127399	[5]
γ_c	Rate of progression from coinfection to AIDs	0.01	[5] [24]
σ_c	Death due to coinfection	0.033	[5]
ω_2	Rate of progression from Latent TB to coinfection	0.0011375	[18]

and remains quite low until the end of the study period. The infectious TB rises from zero to around eight hundred and then levels off. It lies very close to the treated population. The co-infected population is quite low from the very beginning but exits in the population until the end of this period of study. The latent population exists at low levels of about 100 people from the beginning rises to around 150 and is steady until the end of the study. The recovered population increases from zero to around 1000 and then levels off until the end of the study period as illustrated in Figure 2.

From the graph in Figure 3 above the infected HIV population rises from around 1500 to around 3000, then decreases gradually to around 500 and falls to almost zero in the duration of five hundred days. The AIDS population is around 2500 at the beginning of the study period. It then drops to around 1000, and continues dropping gradually until zero within 250 days. The treatment HIV population starts from 2500 people, rises to around 9000 and starts declining gradually to around 1000 and levels off until the end of the six hundred days of study. The latent TB population starts from zero and rises to around 1000, increases to around 1500 and then levels off until the end of this 600 days. The recovered TB population starts from around 1000, increases steadily to around 12,000 and then levels off until the end of this study period. The co-infected population starts from zero and increases to around 700 and continues to increase at a small rate to around 800 and levels off until the end of the study period.

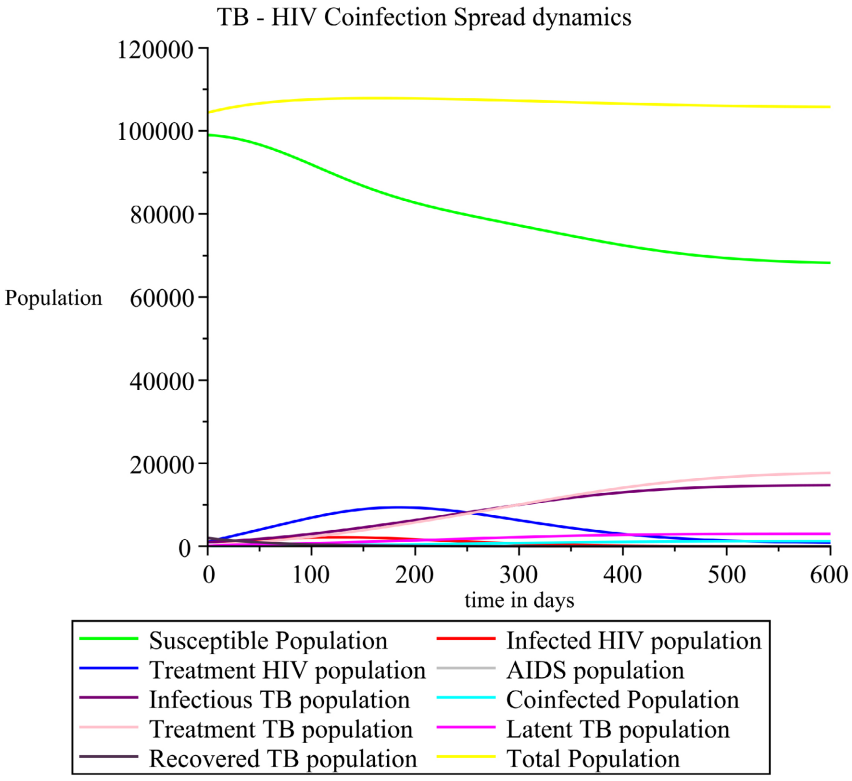

Figure 2. The general graph for HIV-TB coinfection spread dynamics.

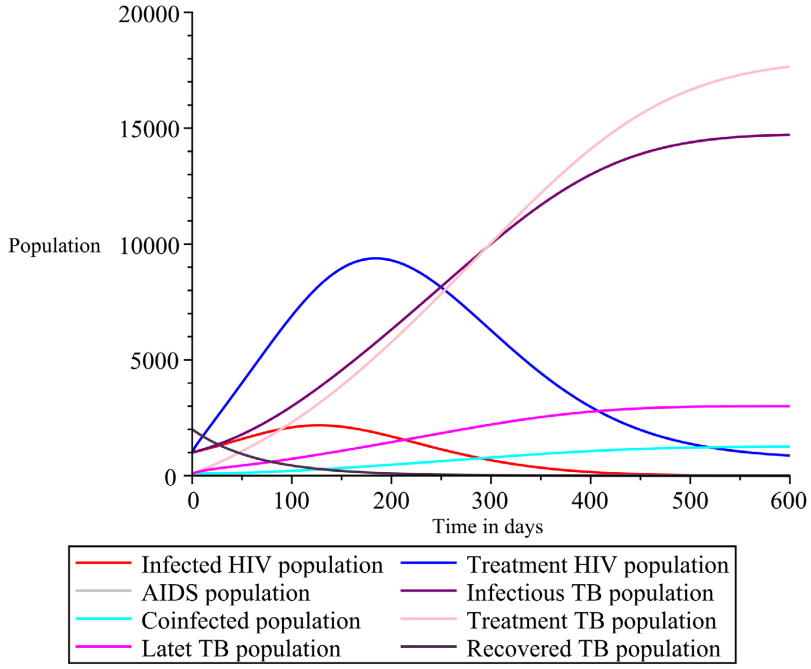

Figure 3. The general graph for HIV-TB coinfection spread dynamics for $I_H, A_H, T_H, L_{TB}, I_{TB}, I_C, T_{TB}$ and R_{TB}.

From the graph in **Figure 4** above, when "beta", the rate at which infected HIV patients go for treatment, is 0.075, the co-infected population increases steadily until the value of around 1200, in a period of 600 days. When the value of "beta" decreases to 0.0075, the co-infected population increases from around 150 to around 50 and keeps very low (to around 20) until the end of the study period. When "beta" is 0.75, the co-infected population increases from around 100 to around 1200 and levels off until the end of the 600 days.

In this **Figure 5**, we observe that when "alpha" the rate at which susceptible population acquires HIV, is 0.0000010411591, the latent population increases steadily from around 300 to around 2800 and levels off until the end of the study period. When "alpha" increases to 0.00000121411591, the latent population increases steadily from around 350 to around 500 decreases a bit to around 490. It then increases steadily to around 1100 at the end of the study period. When "alpha" increases to 0.00000141411591 this latent population increases from 100 to around 300, then decreases to around 200 and continues to decrease steadily to around 100 at the end of the study period, which is 600 days.

7. Discussion

As we noted from the numerical simulation from the model above, when the rate of infection of HIV increases, latent TB is activated and hence increasing the risk of co-infection. We also realized that when the value of β, the rate at which people infected with HIV go for treatment, increases the HIV population reduces. We also observe from the model that, as the rate of activation of latent TB to active TB increases, the infectious TB population increases.

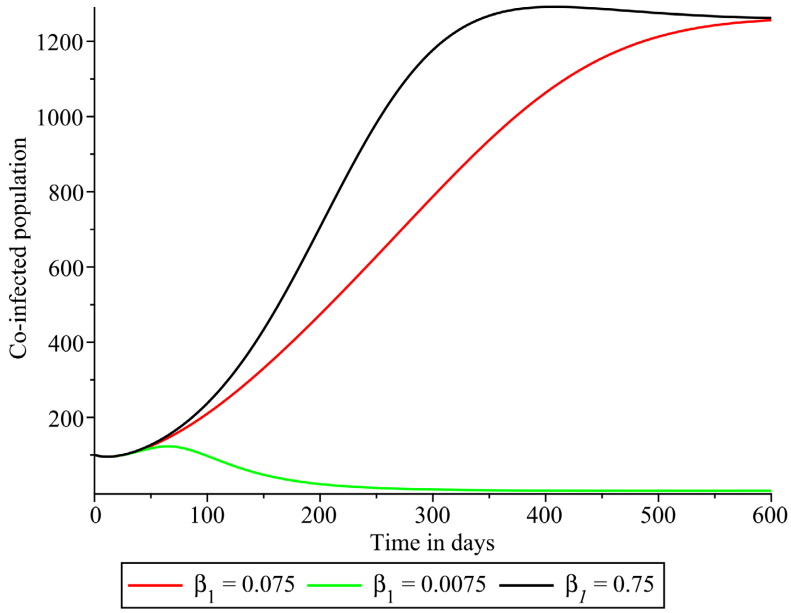

Figure 4. Infected compartment at different values of β_1.

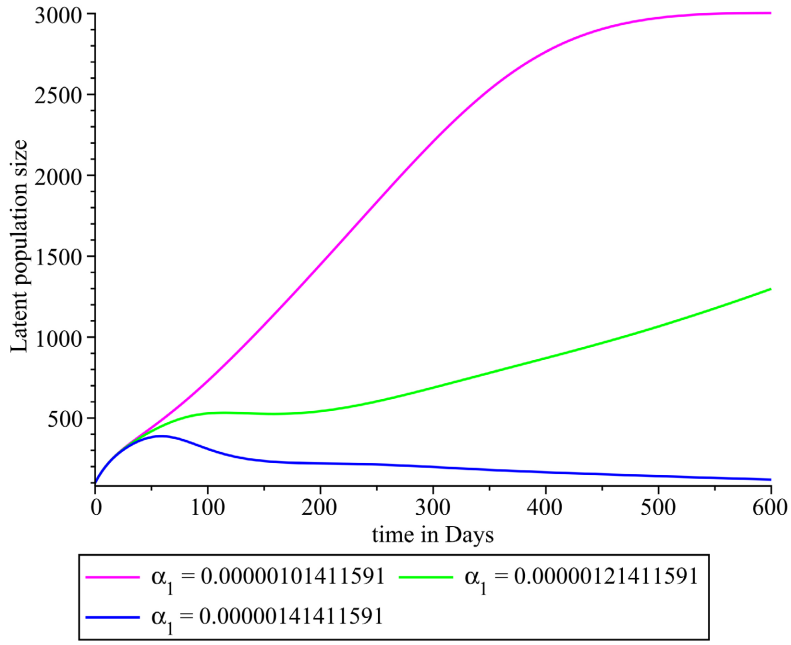

Figure 5. Latent TB population at different values of HIV infection rate (α_1).

Latent TB increases when α, the rate at which the susceptible population in the model gets infected by HIV, increases. This condition then predisposes the general susceptible population to co-infection. When β, the rate at which infected HIV patients go for treatment, increases, the HIV population reduces, hence reducing the activation of latent to active TB. The community, therefore, has a responsibility to ensure that the HIV-infected population initiates treatment for HIV, and goes for it consistently, for this will help decrease the viral load in our communities. Since the susceptible population is among the community mem-

bers, a lot of effort must be put in ensuring that this population is tested and those infected must go for treatment consistently, without fail. Inconsistent treatment will cause high levels of infectious HIV population among community members, which causes a great risk of co-infection.

8. Conclusions and Recommendations

8.1. Conclusion

A mathematical model for co-infection has been developed. The corresponding differential equations were formulated and analyzed. The positive and invariant region was found to be bounded for the co-infection model. Through numerical simulation, it was found that, when the rate of HIV treatment β increases the HIV population decreases. Consequently it was found that when this population decreases, activation of latent TB decreases. Conversely, when the rate of treatment of HIV, β, decreases the population of HIV increases and the population of latent TB decreases suggesting that the high level of HIV infections leads to activation of latent TB to active TB. Therefore, for co-infection to be dealt with effectively, those infected with HIV must be encouraged by all means to go for treatment consistently.

8.2. Recommendations

The study recommends that communities in collaboration with public health officials, must take responsibility of identifying those infected with HIV and ensuring that they go for treatment consistently. Every available opportunity must be used to inform the population about the danger imposed by coinfection. All public gatherings, including religious meetings, should be used as avenues of passing this important information. This information must be passed on to vulnerable youth who might underrate the importance of going for HIV treatment consistently, in order to reduce the levels of co-infection. This approach will reduce the impact of HIV-TB coinfection in the communities. Further research can be done on reversing the conversion of RNA of the HIV virus to DNA to reduce impact of HIV virus on the immune system of patients.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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