Advances in Pure Mathematics, 2024, 14, 333-353
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

HB-Continuous Mappings in *L*-Topological Space

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How to cite this paper: Alharbi, N.A. (2024) *HB*-Continuous Mappings in *L*-Topological Space. *Advances in Pure Mathematics*, **14**, 333-353.
https://doi.org/10.4236/apm.2024.145019

Received: March 27, 2024
Accepted: May 18, 2024
Published: May 21, 2024

Abstract

In this paper, we introduce and study the notion of *HB*-closed sets in *L*-topological space. Then, *HB*-convergence theory for *L*-molecular nets and *L*-ideals is established in terms of *HB*-closedness. Finally, we give a new definition of fuzzy *H*-continuous [1] which is called *HB*-continuity on the basis of the notion of *H*-bounded *L*-subsets in *L*-topological space. Then we give characterizations and properties by making use of *HB*-converges theory of *L*-molecular nets and *L*-ideals.

Keywords

L-Topological Space, *HB*-Closed Set, *H*-Bounded Set, *HB*-Continuous Mappings, *HB*-Convergence, *L*-Molecular Nets, *L*-Ideals

1. Introduction

Continuity and its weaker forms constitute an important and intensely investigated area in the field of general topological spaces. In 1975 Long and Hamlett [2] introduced the notion of *H*-continuity and it has been further investigated by many authors including Noiri [3]. In 1993 Moony [4] studied the notion of *H*-bounded sets and some new characterizations and properties of *H*-bounded sets are examined. In 1995 Dang and Behers [1] extended the notion of *H*-continuity to fuzzy topology, and introduced the notion of fuzzy *H*-continuous functions using the fuzzy compactness given by Mukherjee and Sinha [5]. However, the fuzzy compactness has some shortcomings, such as the Tychonoff product theorem does not hold, and it contradicts some kinds of separation axioms. Hence, the notion of fuzzy *H*-continuous functions in [1] is unsatisfactory. In this paper, we first define the concept of *HB*-closed sets by means of the concept of almost *N*-boundedness (*H*-bounded *L*-subsets). Then by making use of *HB*-closed sets we introduce and study the *HB*-convergence theory of *L*-molecular nets and *L*-ideals. Finally, we give a new definition of fuzzy *H*-continuous [1] which calls *HB*-continuity on the basis of the notions of *HB*-closedness in *L*-topological space. In section 3, we introduce the concepts of *HB*-closure (*HB*-interior) operator and *HB*-closed (*HB*-open) sets in *L*-topological spaces and their various properties are given. And with the help of these notions we introduce and study the concept of *HB*-limit point of *L*-molecular nets and *L*-ideals. In section 4, we introduce and study the concept *HB*-continuous by means of *HB*-closed set and we present its properties and study the relationship between it and *L*-continuous, *H*-continuous mappings. Finally, in section 5, some new interesting characterizations of *HB*-continuous mappings by *HB*-limit points of *L*-molecular nets and *L*-ideals are established.

2. Preliminaries

This paper $L = L(\leq, \vee, \wedge, ')$ denotes a completely distributive lattice with the smallest element 0 and the largest element 1 ($0 \neq 1$) and with an order reversing involution on it. An $\alpha \in L$ is called a molecule of *L* if $\alpha \neq 0$ and $\alpha \leq \nu \vee \gamma$ implies $\alpha \leq \nu$ or $\alpha \leq \gamma$ for all $\nu, \gamma \in L$. The set of all molecules of *L* is denoted by $M(L)$. Let *X* be a nonempty set. L^X denotes the family of all mappings from *X* to *L*. The elements of L^X are called *L*-subsets on *X*. L^X can be made into a lattice by inducing the order and involution from *L*. We denote the smallest element and the largest element of L^X by 0_X and 1_X, respectively. If $\alpha \in L$, then the constant mapping $\underline{\alpha} : X \to \{\alpha\}$ is *L*-subset [6]. An *L*-point (or molecule on L^X), denoted by x_α, $\alpha \in M(L)$ is a *L*-subset which is defined by $x_\alpha(y) = \begin{cases} \alpha : x = y \\ 0 : x \neq y \end{cases}$.

The family of all molecules L^X is denoted by $M(L^X)$ [7]. For $\Psi \subset L^X$, we define $2^{(\Psi)}$ by the set $\{\omega \subset \Psi : \omega \text{ is finite subfamily of } \Psi\}$. An *L*-topology on *X* is a subfamily τ of L^X closed under arbitrary unions and finite intersections. The pair (L^X, τ) is called an *L*-topological space (or *L*-ts, for short) [8]. If (L^X, τ) is an *L*-ts, then for each $\eta \in L^X$, $cl(\eta)$, $\text{int}(\eta)$ and η' will denote the closure, interior and complement of η. A mapping $f : L^X \to L^Y$ is said to be an *L*-valued Zadeh function induced by a mapping $f : X \to Y$, iff $f(\mu)(y) = \vee\{\mu(x) : f(x) = y\}$ for every $\mu \in L^X$ and every $y \in Y$ [7]. An *L*-ts (L^X, τ) is called fully stratified if for each $\alpha \in L$, $\underline{\alpha} \in \tau$ [9]. If (L^X, τ) is an *L*-ts, then the family of all crisp open sets in τ is denoted by $[\tau]$ *i.e.*, $(X, [\tau])$ is a crisp topological space [10].

Definition 2.1 [11]**:** If (L^X, τ) is *L*-ts, then $\mu \in L^X$ is called regular open set iff $\mu = \text{int}(cl(\mu))$. The family of all regular open sets is denoted by $RO(L^X, \tau)$. The complement of the regular open set is called the regular closed set and satisfy $\mu = cl(\text{int}(\mu))$. The family of all regular closed sets is denoted by $RC(L^X, \tau)$.

Definition 2.2 [11]**:** The *L*-valued Zadeh mapping $f_L : (L^X, \tau) \to (L^Y, \Delta)$

is called:

(i) Almost L-continuous iff $f_L^{-1}(\eta) \in \tau'$ for each $\eta \in RC(L^Y, \Delta)$.

(ii) Weakly L-continuous iff $f_L^{-1}(\eta) \leq \text{int}\left(f_L^{-1}(cl(\eta))\right)$ for each $\eta \in \Delta$.

Definition 2.3 [12]**:** Let $f_L : (L^X, \tau) \to (L^Y, \Delta)$ be an L-valued Zadeh mapping and $A \subseteq X$, then $f_L|_A : L^A \to L^Y$ is defined as follows:

$(f_L|_A)(\mu) = f(\mu) \wedge 1_A = f(\mu^*)$, for each $\mu \in L^A$ and call $f_L|_A$ the restriction of f on A. Where μ^* denote the extension of μ in L^X, that is for each $x \in X$,

$$\mu^*(x) = \begin{cases} \mu(x) & : x \in A \\ 0 & : x \notin A \end{cases}$$

Definition 2.4 [13]**:** Let (L^X, τ) be an L-ts and $x_\alpha \in M(L^X)$. Then:

(i) $\eta \in \tau'$ is called a remote neighborhood (R-nbd, for short) of x_α if $x_\alpha \notin \eta$. The set of all R-nbds of x_α is called remoted neighborhood system and is denoted by R_{x_α}.

(ii) $\lambda \in L^X$ is called an $*$-remoted neighborhood (R^*-nbd, for short) of x_α if there exists $\mu \in R_{x_\alpha}$ such that $\lambda \leq \mu$. The set of all R^*-nbds of x_α is called $*$-remoted neighborhood system and is denoted by $R^*_{x_\alpha}$.

Definition 2.5 [14]**:** Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. Then $\Psi \subset \tau'$ is called an:

(i) α-remoted neighborhood family of μ, briefly α-RF of μ, if for each L-point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\lambda \in R_{x_\alpha}$.

(ii) $\bar{\alpha}$-remoted neighborhood family of μ, briefly $\bar{\alpha}$-RF of μ, if there exists $\gamma \in \beta^*(\alpha)$ such that Ψ is an γ-RF of μ, where $\beta^*(\alpha) = \beta(\alpha) \cap M(L)$, and $\beta(\alpha)$ denotes the union of all the minimal sets relative to α.

Definition 2.6 [11]**:** Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. Then $\Psi \subset \tau'$ is called an:

(i) Almost α-$*$-remoted neighborhood family of μ, (or briefly, almost α-R^*F) of μ, if for each L-point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\text{int}(\lambda) \in R^*_{x_\alpha}$.

(ii) Almost $\bar{\alpha}$-$*$-remoted neighborhood family of μ, (or briefly almost $\bar{\alpha}$-R^*F) of μ, if there exists $\gamma \in \beta^*(\alpha)$ such that Ψ is an almost γ-R^*F of μ.

Definition 2.7 [15]**:** Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. Then $\Psi \subset RC(L^X, \tau)$ is called an α-regular closed remoted neighborhood family of μ, briefly α-RCRF of μ, if for each L-point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\lambda \in R_{x_\alpha}$.

Definition 2.8 [16]**:** Let (L^X, τ) be an L-ts and $\mu \in L^X$. Then $x_\alpha \in M(L^X)$ is called θ-adherent point of μ and write $x_\alpha \in \theta.cl(\mu)$ iff $\mu \not\leq \text{int}(\lambda)$ for each $\lambda \in R_{x_\alpha}$. If $\mu = \theta.cl(\mu)$, then μ is called θ-closed L-subset. The family of all θ-closed L-subset of X is denoted by $\theta C\left(L^X,\tau\right)$ and its complement is called the family of all θ-open L-subset and denoted by $\theta O\left(L^X,\tau\right)$.

Definition 2.9 [11]**:** Let $\left(L^X,\tau\right)$ be an L-ts, $\mu\in L^X$. Then μ is called almost N-compact (or H-compact) set in $\left(L^X,\tau\right)$ if for each $\alpha\in M\left(L\right)$ and every α-RF Ψ of μ there is $\Psi_\circ\in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\bar{\alpha}$-R^*F of μ.

If 1_X is H-compact set, then $\left(L^X,\tau\right)$ is called H-compact space.

Theorem 2.10 [11]**:** Suppose that $f_L:\left(L^X,\tau\right)\to\left(L^Y,\Delta\right)$ is an L-almost continuous and $\mu\in L^X$ is an H-compact L-subset in $\left(L^X,\tau\right)$, then $f_L\left(\mu\right)$ is an H-compact L-subset in $\left(L^Y,\Delta\right)$.

Definition 2.11 [17]**:** An L-ts $\left(L^X,\tau\right)$ is said to be:

(i) LT_1-space iff for any $x_\alpha,y_\gamma\in M\left(L^X\right)$, $x\neq y$ there is $\lambda\in R_{x_\alpha}$ such that $y_\gamma\in\lambda$.

(ii) LT_2-space iff for any $x_\alpha,y_\gamma\in M\left(L^X\right)$, $x\neq y$ there is $\lambda\in R_{x_\alpha}$, $\eta\in R_{y_\gamma}$ such that $\lambda\vee\eta=1_X$.

(iii) $LT_{2\frac{1}{2}}$-space iff for any $x_\alpha,y_\gamma\in M\left(L^X\right)$, $x\neq y$ there is $\lambda\in R_{x_\alpha}$, $\eta\in R_{y_\gamma}$ such that $\operatorname{int}\left(\lambda\right)\vee\operatorname{int}\left(\eta\right)=1_X$.

(iv) LR_2-space (regular space) iff for all $\alpha\in M\left(L\right)$, $x\in X$ and for each $\lambda\in R_{x_\alpha}$ there is $\eta\in R_{x_\alpha}$, $\rho\in\tau'$ such that $\eta\vee\rho=1_X$ and $\lambda\wedge\rho=0_X$.

(v) LT_3-space iff it is LR_2-space and LT_1-space.

Theorem 2.12 [14]**:** Let $\left(L^X,\tau\right)$ be an L-ts and every H-compact set in fully stratified and $LT_{2\frac{1}{2}}$-space, then it is θ-closed L-subset.

Theorem 2.13 [11]**:** An L-ts $\left(L^X,\tau\right)$ is LR_2-space iff for any $\mu\in L^X$, $cl\left(\mu\right)=\theta.cl\left(\mu\right)$.

Proof. Let $\left(L^X,\tau\right)$ be an LR_2-space. For any $\mu\in L^X$ it is always true that $cl\left(\mu\right)\leq\theta.cl\left(\mu\right)$. Now, let $x_\alpha\in M\left(L^X\right)$ such that $x_\alpha\notin cl\left(\mu\right)$ and let $\lambda\in R_{x_\alpha}$, since $\left(L^X,\tau\right)$ is LR_2-space, there is $\eta\in R_{x_\alpha}$ such that $\lambda\leq\operatorname{int}\left(\eta\right)$. Now $x_\alpha\notin cl\left(\mu\right)$ implies that $\mu\leq\lambda$ for each $\lambda\in R_{x_\alpha}$ which implies that $\mu\leq\operatorname{int}\left(\eta\right)$ which implies that $x_\alpha\notin\theta.cl\left(\mu\right)$. Thus $\theta.cl\left(\mu\right)\leq cl\left(\mu\right)$. Hence $cl\left(\mu\right)=\theta.cl\left(\mu\right)$. Conversely, let $x_\alpha\in M\left(L^X\right)$ and $\lambda\in R_{x_\alpha}$. Then $cl\left(\lambda\right)\in R_{x_\alpha}$ and so $x_\alpha\notin cl\left(\lambda\right)=\theta.cl\left(\lambda\right)$. Hence there is $\eta\in R_{x_\alpha}$ such that $\lambda\leq\operatorname{int}\left(\eta\right)$. Thus $\left(L^X,\tau\right)$ is LR_2-space.

Corollary 2.14 [11]**:** If $\left(L^X,\tau\right)$ is LR_2-space, then closed L-subset is θ-closed L-subset and hence $\theta.cl\left(\mu\right)$ is θ-closed for any $\mu\in L^X$.

Definition 2.15 [13]**:** Let $\left(D,\leq\right)$ be a directed set. Then the mapping $S:D\to L^X$ and denoted by $S=\left\{\mu_n:n\in D\right\}$ is called a net of L-subsets in X. Specially, the mapping $S:D\to M\left(L^X\right)$ is said to be a molecular net in L^X. If $\mu\in L^X$ and for each $n\in D$, $S\in\mu$ then S is called a net in μ.

Definition 2.16 [13]**:** Let $\left(L^X,\tau\right)$ be an L-ts and $S=\left\{S\left(n\right):n\in D\right\}$ be a molecular net in L^X. S is called a molecular α-net ($\alpha \in M(L)$), if for each $\gamma \in \beta^*(\alpha)$ there exists $n \in D$ such that $\vee(S(m)) \geq \gamma$ whenever $m \geq n$, where $\vee(S(m))$ is the height of the molecular $S(m)$.

Definition 2.17 [13]**:** Let $S = \{S(n): n \in D\}$ and $T = \{T(m): m \in E\}$ be a be molecular nets in (L^X, τ). Then T is said to be a molecular subnet of S if there is a mapping $f: E \to D$ that satisfies the following conditions:

(i) $T = S \circ f$

(ii) For each $n \in D$ there is $m \in E$ such that $f(l) \geq n$ for each $l \in E$, $l \geq m$.

Definition 2.18 [7]**:** Let (L^X, τ) be an L-ts and S be a molecular net in (L^X, τ). Then $x_\alpha \in M(L^X)$ is called:

(i) a θ-limit point of S, (or S θ-converges to x_α) in symbols $S \xrightarrow{\theta} x_\alpha$ if for each $\mu \in R_{x_\alpha}$ there is a $n \in D$ such for each $m \in D$ and $m \geq n$ we have $S(m) \notin \text{int}(\mu)$. The union of all θ-limit points of S are denoted by $\theta.\lim(S)$.

(ii) a θ-cluster (θ-adherent) point of S, in symbols $S \overset{\theta}{\propto} x_\alpha$ if for each $\mu \in R_{x_\alpha}$ and for each $n \in D$ there is a $m \in D$ such that $m \geq n$ and $S(m) \notin \text{int}(\mu)$. The union of all θ-cluster points of S is denoted by $\theta.adh(S)$.

Theorem 2.19 [13]**:** Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then $x_\alpha \in \theta..cl(\mu)$ iff there exists a molecular net S in μ such that S is θ-converges to x_α.

Theorem 2.20 [15]**:** Assume that $S = \{S(n): n \in D\}$ is a molecular net in an L-ts (L^X, τ) and $x_\alpha \in M(L^X)$. Then $S \overset{\theta}{\propto} x_\alpha$ iff there exists a subnet T of S such that $T \xrightarrow{\theta} x_\alpha$.

Theorem 2.21 [14]**:** Let (L^X, τ) be an L-ts and $\mu \in L^X$. Then μ is H-compact set iff each α-net S contained in μ has a θ-cluster point in μ with height α for any $\alpha \in M(L)$.

Definition 2.22 [18]**:** The nonempty family $I \subset L^X$ is called an ideal if the following conditions are satisfied, for each $\mu_1, \mu_2 \in L^X$

(i) $1_X \notin I$

(ii) If $\mu_1 \leq \mu_2$ and $\mu_2 \in I$, then $\mu_1 \in I$.

(iii) If $\mu_1, \mu_2 \in I$, then $\mu_1 \vee \mu_2 \in I$.

Theorem 2.23 [19]**:** Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then $x_\alpha \in \theta..cl(\mu)$ iff there exists an ideal I in L^X such that I is θ-converges to x_α and $\mu \notin I$.

Definition 2.24 [20]**:** An L-mapping $f_L: (L^X, \tau) \to (L^Y, \Delta)$ is called H-continuous if $f_L^{-1}(\eta) \in \tau'$ for each $\eta \in L^Y$ is closed and almost N-compact.

3. *H*-Closure and *H*-Interior Operators in *L*-Topological Space

In this section, we introduce the concepts of H-Closure operator and H-interior operator by using an almost N-bounded (or H-bounded) set and discuss their properties.

Definition 3.1: Let (L^X,τ) be an *L*-ts, $\mu\in L^X$. Then μ is called almost *N*-bounded (or *H*-bounded) set in (L^X,τ) if for each $\alpha\in M(L)$ and every α-RF Ψ of 1_X, there is $\Psi_\circ\in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\bar{\alpha}$-R^*F of μ.

If 1_X is *H*-bounded set, then (L^X,τ) is called *H*-bounded space.

Theorem 3.2: Suppose that $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is an *L*-almost continuous and $\mu\in L^X$ is an *H*-bounded *L*-subset in (L^X,τ), then $f_L(\mu)$ is an *H*-bounded *L*-subset in (L^Y,Δ).

Proof. Let μ be an *H*-bounded in L^X and let $\Psi\subseteq\Delta'$ be an α-RF of 1_Y ($\alpha\in M(L)$), then $\{cl(\text{int}(\lambda)):\lambda\in\Psi\}\subset RC(L^Y,\Delta)$ is an α-RCRF of 1_Y. We now will show that $Q=\{f_L^{-1}(cl(\text{int}(\lambda))):\lambda\in\Psi\}$ is an α-RF of 1_X. In fact, since f_L is an *L*-almost continuous and $cl(\text{int}(\lambda))\in RC(L^Y,\Delta)$ then $f_L^{-1}(cl(\text{int}(\lambda)))\in\tau'$. According to the definition, Ψ there exists $\lambda\in\Psi$ such that $cl(\text{int}(\lambda))\in R_{f_L(x_\alpha)}$, *i.e.*, $f_L(x_\alpha)\notin cl(\text{int}(\lambda))$ hence $x_\alpha\notin f_L^{-1}(cl(\text{int}(\lambda)))$ for every $x\in X$. This means that Q is an α-RF of 1_X. Since μ is an *H*-bounded set, there exists $\Psi_\circ\in 2^{(\Psi)}$ such that $\{f_L^{-1}(cl(\text{int}(\lambda))):\lambda\in\Psi_\circ\}\in 2^{(\Psi)}$ is an almost $\bar{\alpha}$-R^*F of μ. Thus for some $\gamma\in\beta^*(\alpha)$ and for each $x_\gamma\in\mu$ there exists $\lambda\in\Psi_\circ$ such that $\text{int}(f_L^{-1}(cl(\text{int}(\lambda))))\in R^*_{x_\gamma}$. Since f_L is an *L*-almost continuous then it is *L*-weakly continuous and since $\text{int}(\lambda)\in\Delta$ then $f_L^{-1}(\text{int}(\lambda))\le\text{int}(f_L^{-1}(cl(\text{int}(\lambda))))$ and so $x_\alpha\notin f_L^{-1}(\text{int}(\lambda))$. Consequently, there exists $x_\gamma\in\mu$ and $\lambda\in\Psi_\circ$ satisfying $\text{int}(\lambda)\in R^*_{f_L(x_\gamma)}$ and $y_\gamma=f_L(x_\gamma)$ for each $y_\gamma\in f_L(\mu)$. Thus, $\Psi_\circ\in 2^{(\Psi)}$ is an almost $\bar{\alpha}$-R^*F of $f_L(\mu)$. By Definition 3.1, we have $f_L(\mu)$ an *H*-bounded *L*-subset in (L^Y,Δ).

Theorem 3.3: Let (L^X,τ) be an *L*-ts and let $\mu\in L^X$. Then the following statements are true:

(i) If μ is *H*-compact set, then μ is *H*-bounded set.

(ii) If μ is *H*-bounded set and $\eta\le\mu$, then η is *H*-bounded set.

(iii) If μ is *H*-compact set and $\eta\le\mu$, then η is *H*-bounded set.

Proof. (i) Let μ be an *H*-compact set and let $\Psi=\{\rho_i:i\in I\}\subset\tau'$ be an α-RF of 1_X and so Ψ is α-RF of μ. Since μ is *H*-compact set, then there exists $\Psi_\circ=\{\rho_i:i=1,2,\cdots,m\}\in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\bar{\alpha}$-R^*F of μ. Thus μ is *H*-bounded set.

(ii) Let μ be an *H*-bounded set and $\eta\le\mu$. let $\Psi=\{\rho_i:i\in I\}\subset\tau'$ be an α-RF of 1_X. Since μ is *H*-bounded set, then there exists $\Psi_\circ=\{\rho_i:i=1,2,\cdots,m\}\in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\bar{\alpha}$-R^*F of μ, thus there exists $\gamma\in\beta^*(\alpha)$ such that $\Psi_\circ$ is an almost γ-R^*F of μ. Hence $\forall\, x_\gamma\in\mu$, $\exists\,\lambda\in\Psi_\circ$ such that $\text{int}(\lambda)\in R^*_{x_\gamma}$. Since $\eta\le\mu$, then $\forall\, x_\gamma\in\eta\le\mu$, $\exists\,\lambda\in\Psi_\circ$ such that $\text{int}(\lambda)\in R^*_{x_\gamma}$. Hence $\Psi_\circ$ is an almost γ-R^*F of η and so $\Psi_\circ$ is an almost $\bar{\alpha}$ - R^*F of η. Thus η is *H*-bounded set.

(iii) Let μ be an *H*-compact set and $\eta \le \mu$. let $\Psi \subset \tau'$ be an α-RF of 1_X and so α-RF of μ. Since μ is *H*-compact set, then there exists $\Psi_\circ \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\bar{\alpha}$ - R^*F of μ, since $\eta \le \mu$, then $\Psi_\circ$ is an almost $\bar{\alpha}$ - R^*F of η. Thus η is *H*-bounded set.

Theorem 3.4: Let (L^X, τ) be an *L*-ts, $\alpha \in M(L)$ and $\mu \in L^X$. Then μ is *H*-bounded iff for each molecular α-net *S* contained in μ has θ-cluster point in 1_X with height α.

Proof. Let μ be an *H*-bounded set and $S = \{S(n): n \in D\}$ be an molecular α-net in μ. If *S* does not have any θ-cluster point in 1_X with height α. Then for all $x_\alpha \in M(L^X)$, x_α is not θ-cluster point of *S* and so there exists $\lambda_x \in R_{x_\alpha}$ and $n_x \in D$ such that $S(n) \in \operatorname{int}(\lambda_x)$ for every $n \in D$ and $n \ge n_x$. Put $\Psi = \{\lambda_x : x \in X \text{ and } \alpha \in M(L)\}$, then Ψ is an α-RF of 1_X. According to the hypothesis, Ψ has a finite family $\Psi_\circ = \{\lambda_{x^i} : i = 1, 2, \cdots, k\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\bar{\alpha}$ - R^*F of μ, that is for some $\gamma \in \beta^*(\alpha)$ and each $y_\gamma \in \mu$ there exists $\lambda_{x_i} \in \Psi_\circ$ ($i \le k$) such that $\operatorname{int}(\lambda_{x_i}) \in R^*_{y_\gamma}$. Put $\lambda = \overset{k}{\underset{i=1}{\wedge}} \lambda_{x_i}$, for each $y_\gamma \in \mu$, we have $\overset{k}{\underset{i=1}{\wedge}} \operatorname{int}(\lambda_{x_i}) = \operatorname{int}\left(\overset{k}{\underset{i=1}{\wedge}} \lambda_{x_i}\right) = \operatorname{int}(\lambda)$, thus $\operatorname{int}(\lambda) \in R^*_{y_\gamma}$. Since *D* is a directed set, then there is $n_\circ \in D$ such that $n_\circ \ge n_{x_i}$, $i = 1, 2, \cdots, k$ and $S(n) \in \operatorname{int}(\lambda_{x_i})$, $i = 1, 2, \cdots, k$ whenever $n \ge n_\circ$ and so $S(n) \in \operatorname{int}(\lambda)$. This shows that for each $y_\gamma \in \mu$, $\vee(S(n)) \not\ge \gamma$ whenever $n \ge n_\circ$. This contradicts the hypothesis that *S* is a molecular α-net. Therefore, *S* has at least a θ-cluster point in 1_X with height α.

Conversely, assume that each molecular α-net *S* contained in μ has an θ-cluster point in 1_X with height α and Ψ is an α-RF of 1_X. If for each $\Psi_\circ \in 2^{(\Psi)}$ such that $\Psi_\circ$ is not almost $\bar{\alpha}$ - R^*F of μ, that is, for each $\gamma \in \beta^*(\alpha)$ there exists $(\gamma, \Psi_\circ) \in \beta^*(\alpha) \times 2^{(\Psi)}$ there exists molecule $x_{(\gamma, \Psi_\circ)} \in \mu$ such that for each $\lambda \in \Psi_\circ$, $\operatorname{int}(\lambda) \notin R_{x_{(\gamma, \Psi_\circ)}}$. Put $D = \beta^*(\alpha) \times 2^{(\Psi)}$ and defined the order as follows: $(\gamma_1, \Psi^1_\circ) \ge (\gamma_2, \Psi^2_\circ)$ iff $\gamma_1 \ge \gamma_2$ and $\Psi^1_\circ \supset \Psi^2_\circ$. Then $S = \{S_{(\gamma, \Psi_\circ)} = x_{(\gamma, \Psi_\circ)} \in \mu : (\gamma, \Psi_\circ) \in D\}$ is an molecular α-net in μ. Since Ψ is an α-RF of 1_X, then there exists $\rho \in \Psi$ such that $\rho \in R_{y_\alpha}$ and hence $\operatorname{int}(\rho) \in R^*_{x_\alpha}$. Because $\{\rho\} \in 2^{(\Psi)}$. We take any $\gamma_1 \in \beta^*(\alpha)$, $x_{(\gamma, \Psi_\circ)} \in \operatorname{int}(\rho)$ whenever $(\gamma, \Psi_\circ) \ge (\gamma_1, \rho)$. Therefore $S_{(\gamma, \Psi_\circ)} \in \operatorname{int}(\rho)$, which contradicts to the hypothesis. Therefore there exists $\Psi_\circ \in 2^{(\Psi)}$ such that $\Psi_\circ$ is almost $\bar{\alpha}$ - R^*F of μ and hence μ is *H*-bounded.

Theorem 3.5: If (L^X, τ) fully stratified and $LT_{2\frac{1}{2}}$-space, then $\mu \in L^X$ is *H*-compact set iff μ is θ-closed and *H*-bounded set.

Proof. If $\mu \in L^X$ is *H*-compact set, then by Theorem 2.12 we have μ is θ-closed and by Theorem 3.3 (i) we have μ is *H*-bounded. Conversely, let μ be an θ-closed and *H*-bounded set and let *S* be an α-net in μ. Since μ is

H-bounded, then by Theorem 3.4 we have S has θ-cluster point, say x_α in 1_X with height α. By Theorem 2.20, then there is a subnet T of S such that T θ-converges to x_α and so $x_\alpha \in \theta.cl(\mu)$ by Theorem 2.19. Since μ is θ-closed, then $\mu = \theta.cl(\mu)$ and so $x_\alpha \in \mu$, then by Theorem 2.21 we have μ is H-compact set.

Theorem 3.6: If (L^X, τ) is LR_2-space, then $\mu \in L^X$ is H-bounded set iff $\theta.cl(\mu)$ is H-bounded set.

Proof. If $\theta.cl(\mu)$ is H-bounded set, then μ is H-bounded set by Theorem 3.3 (ii). Conversely, suppose that μ is H-bounded and $\Psi = \{\eta_{x_j} : j \in J\}$ is an α-RF of 1_X. Then for each $x \in X$ there is $\eta_{x_j} \in \Psi$ such that $\eta_{x_j} \in R_{x_\alpha}$. Since (L^X, τ) is LR_2-space, then there is $\lambda \in R_{x_\alpha}$ there is $\lambda_{x_j} \in R_{x_\alpha}$ and there is $\rho_{x_j} \in \tau'$ such that $\lambda_{x_j} \vee \rho_{x_j} = 1_X$ and. $\rho_{x_j} \wedge \eta_{x_j} = 0_X$. Then the family $\{\lambda_{x_j} : x_\alpha \in M(L^X)\}$ is an α-RF of 1_X. Since μ is H-bounded, then exists finite subset $J_\circ$ of J such that $\{\lambda_{x_j} : j \in J_\circ\}$ is an almost $\bar{\alpha}$-R^*F of μ. Since $\lambda_{x_j} \vee \rho_{x_j} = 1_X$, $x_\alpha \notin \lambda_{x_j}$, then $x_\alpha \in \rho_{x_j}$. Since $\rho_{x_j} \wedge \eta_{x_j} = 0_X$, then $\{\eta_{x_j} : j \in J_\circ\}$ is an almost $\bar{\alpha}$-R^*F of ρ_{x_j}. Therefore $\mu \le \rho_{x_j}$ for $J \in J_\circ$. Since $\rho_{x_j} \in \tau'$, and (L^X, τ) is LR_2-space, then by Theorem 2.13, we have $cl(\rho_{x_j}) = \theta.cl(\rho_{x_j})$ and so $\{\eta_{x_j} : j \in J_\circ\}$ is an almost $\bar{\alpha}$-R^*F of $\theta.cl(\rho_{x_j})$ and since $\theta.cl(\mu) \le \theta.cl(\rho_{x_j})$, then $\{\eta_{x_j} : j \in J_\circ\}$ is an almost $\bar{\alpha}$-R^*F of $\theta.cl(\mu)$. Hence $\theta.cl(\mu)$ is H-bounded set.

Theorem 3.7: If (L^X, τ) is LT_3-space, then $\mu \in L^X$ is H-bounded set iff μ is L-subset of H-compact set.

Proof. If μ is H-bounded, then by Theorem 3.6 and corollary 2.14, we have $\theta.cl(\mu)$ is θ-closed and H-bounded set, hence by Theorem 3.5, we have $\theta.cl(\mu)$ is H-compact set. Conversely, If μ is L-subset of H-compact set, then by Theorem 3.3 (iii), we have μ is H-bounded set.

Definition 3.8: Let (L^X, τ) be an L-ts and $x_\alpha \in M(L^X)$. If $\mu \in L^X$ is closed and H-bounded set, then μ is called HB-remoted neighborhood of x_α (HBR-nbd, for short) of x_α if $x_\alpha \notin \mu$. The set of all HBR-nbds of x_α is denoted by HBR_{x_α}

We note that $HBR_{x_\alpha} \subseteq R_{x_\alpha}$, $\forall\ x_\alpha \in M(L^X)$

The following example shows that the converse is not true in general

Example 3.9: Let $X = \{x\}$, $L = [0,1]$, and let $\tau = \{0_X, x_3, x_{.7}, 1_X\}$. Then (L^X, τ) is L-ts. We have $R_{x_1} = \{0_X, x_{.3}, x_{.7}\}$. Now, we show that $x_{.7} \in L^X$ is not H-bounded set.

Let $\Psi = \{x_{.7}, 1_X\} \subseteq \tau'$, then Ψ is .8-RF of 1_X. But for each $\gamma \in \beta^*(.8) = (0,2]$, any finite subfamily $\Psi_\circ \in 2^{(\Psi)}$ is not almost γ-R^*F of $x_{.7}$. Thus $\Psi_\circ$ is not almost $\overline{.8}$-R^*F of $x_{.7}$. Thus $x_{.7}$ is not H-bounded set and so $x_{.7} \notin HBR_{x_\alpha}$. Hence $R_{x_{.7}} \not\subseteq HBR_{x_{.7}}$.

Definition 3.10: Let (L^X, τ) be an *L*-ts and $\mu \in L^X$. Then $x_\alpha \in M(L^X)$ is called an *H*-bounded adherent point of μ and write $x_\alpha \in HB.cl(\mu)$ iff $\mu \not\leq \lambda$ for each $\lambda \in HBR_{x_\alpha}$. If $\mu = HB.cl(\mu)$, then μ is called *HB*-closed *L*-subset. The family of all *HB*-closed *L*-subsets is denoted by $HBC(L^X, \tau)$ and its complement is called the family of all *HB*-open *L*-subsets and denoted by $HBO(L^X, \tau)$.

Theorem 3.11: Let (L^X, τ) be an *L*-ts and let $\mu \in L^X$. Then the following statements are true:

(i) $\mu \leq cl(\mu) \leq HB.cl(\mu)$.

(ii) If $\eta \in L^X$ and $\mu \leq \eta$ then $HB.cl(\mu) \leq HB.cl(\eta)$.

(iii) $HB.cl(HB.cl(\mu)) = HB.cl(\mu)$.

(iv) $HB.cl(\mu) = \wedge\{\eta \in L^X : \eta \in HBC.(L^X, \tau), \mu \leq \eta\}$.

Proof. (i) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \notin HB.cl(\mu)$, then there exists $\lambda \in HBR_{x_\alpha}$ such that $\mu \leq \lambda$. Since $HBR_{x_\alpha} \subseteq R_{x_\alpha}$ and so $\lambda \in R_{x_\alpha}$ and hence $x_\alpha \notin cl(\mu)$. Thus $cl(\mu) \leq HB.cl(\mu)$.

(ii) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \notin HB.cl(\eta)$, then there exists $\lambda \in HBR_{x_\alpha}$ such that $\eta \leq \lambda$. Since $\mu \leq \eta$, then $\mu \leq \lambda$ and so $x_\alpha \notin HB.cl(\mu)$. Thus $HBcl(\mu) \leq HB.cl(\eta)$.

(iii) Suppose $x_\alpha \in M(L^X)$ such that $x_\alpha \in HB.cl(HB.cl(\mu))$. According to Definition 3.10, we have $HB.cl(\mu) \not\leq \lambda$ for each $\lambda \in HBR_{x_\alpha}$. Hence, there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in HB.cl(\mu)$ with $y_\gamma \notin \lambda$ and so $\mu \not\leq \lambda$, that is, $x_\alpha \in HB.cl(\mu)$. This shows that $HB.cl(HB.cl(\mu)) \leq HB.cl(\mu)$. On the other hand, $\mu \leq HB.cl(\mu)$ follows from (i) and so $HB.cl(\mu) \leq HB.cl(HB.cl(\mu))$. Therefore, $HB.cl(HB.cl(\mu)) = HB.cl(\mu)$.

(iv) On account of (i) and (iii). $HB.cl(\mu)$ is an *HB*-closed set containing μ, and so $HB.cl(\mu) \geq \wedge\{\eta \in L^X : \eta \in HBC.(L^X, \tau), \mu \leq \eta\}$. Conversely, in case $x_\alpha \in M(L^X)$ sand $x_\alpha \in HB.cl(\mu)$, then $\mu \not\leq \lambda$ for each $\lambda \in HBR_{x_\alpha}$. Hence, if η is an *HB*-closed set containing μ, then $\eta \not\leq \lambda$, and then $x_\alpha \in HB.cl(\eta) = \eta$.

This implies that $HB.cl(\mu) \leq \wedge\{\eta \in L^X : \eta \in HBC.(L^X, \tau), \mu \leq \eta\}$. Hence

$$HB.cl(\mu) = \wedge\{\eta \in L^X : \eta \in HBC.(L^X, \tau), \mu \leq \eta\}$$

From Theorem 3.11, one can see that every *HB*-closed *L*-subset is a closed *L*-subset, but the inverse is not true since every closed *L*-subset is not *H*-bounded set in general as the following example shows.

Example 3.12: By Example 3.9, let $\eta \in L^X$ be an *L*-subset, where $\eta = x_{.7}$, then η is closed *L*-subset because $\tau' = \{0_X, x_{.7}, x_{.3}, 1_X\}$. But $x_{.7} \in L^X$ is not *H*-bounded set.

Theorem 3.13: Let (L^X, τ) be an *L*-ts. The following statements hold:

(i) $0_X, 1_X \in HBC\left(L^X, \tau\right)$.

(ii) If $\mu_1, \mu_2, \cdots, \mu_n \in HBC\left(L^X, \tau\right)$, then $\bigvee_{i=1}^{n} \mu_i \in HBC\left(L^X, \tau\right)$.

(iii) If $\{\mu_i : i \in I\} \subseteq HBC\left(L^X, \tau\right)$, then $\bigwedge_{i\in I} \mu_i \in HBC\left(L^X, \tau\right)$.

(iv) Every H-bounded and closed set is HB-closed.

(v) $\mu \in L^X$ is HB-closed iff there exists $\lambda \in HBR_{x_\alpha}$ such that $\mu \le \lambda$ for each $x_\alpha \in M\left(L^X\right)$ with $x_\alpha \notin \mu$

Proof. (i) Obvious.

(ii) Let $\mu_1, \mu_2, \cdots, \mu_n \in HBC\left(L^X, \tau\right)$ and $x_\alpha \in M\left(L^X\right)$ such that $x_\alpha \in HB.cl\left(\bigvee_{i=1}^{n} \mu_i\right)$, then for each $\lambda \in HBR_{x_\alpha}$ we have $\bigvee_{i=1}^{n} \mu_i \not\le \lambda$ and so $\mu_i \not\le \lambda$ for some $i = 1, 2, \cdots, n$. Hence $x_\alpha \in HB.cl(\mu_i)$ for some $i = 1, 2, \cdots, n$. Since μ_i is HB-closed set, then $HB.cl(\mu_i) \le \mu_i$ for some $i = 1, 2, \cdots, n$ and so $x_\alpha \in \mu_i$ for some $i = 1, 2, \cdots, n$ and hence $x_\alpha \in \bigvee_{i=1}^{n} \mu_i$. Thus $HB.cl\left(\bigvee_{i=1}^{n} \mu_i\right) \le \bigvee_{i=1}^{n} \mu_i$ $(*)$

Conversely, since $\mu_i \le HB.cl(\mu_i)$ then $\bigvee_{i=1}^{n} \mu_i \le HB.cl\left(\bigvee_{i=1}^{n} \mu_i\right)$ $(**)$. Hence from $(*)$ and $(**)$ we have $HB.cl\left(\bigvee_{i=1}^{n} \mu_i\right) = \bigvee_{i=1}^{n} \mu_i$. Thus $\bigvee_{i=1}^{n} \mu_i \in HBC\left(L^X, \tau\right)$.

(iii) Let $\mu_1, \mu_2, \cdots, \mu_n \in HBC\left(L^X, \tau\right)$ and $x_\alpha \in M\left(L^X\right)$ such that $x_\alpha \in HB.cl\left(\bigwedge_{i\in I} \mu_i\right)$, then for each $\lambda \in HBR_{x_\alpha}$ we have $\bigwedge_{i\in I} \mu_i \not\le \lambda$ and so $\mu_i \not\le \lambda$ for each $i \in I$. Hence $x_\alpha \in HB.cl(\mu_i)$ for each $i \in I$. Since μ_i is HB-closed set, then $HB.cl(\mu_i) \le \mu_i$ for each $i \in I$ and so $x_\alpha \in \mu_i$ for each $i \in I$ and hence $x_\alpha \in \bigwedge_{i\in I} \mu_i$. Thus $HB.cl\left(\bigwedge_{i\in I} \mu_i\right) \le \bigwedge_{i\in I} \mu_i$ $(*)$.

Conversely, since $\mu_i \le HB.cl(\mu_i)$ then $\bigwedge_{i\in I} \mu_i \le HB.cl\left(\bigwedge_{i\in I} \mu_i\right)$ $(**)$. Hence from $(*)$ and $(**)$ we have $HB.cl\left(\bigwedge_{i\in I} \mu_i\right) = \bigwedge_{i\in I} \mu_i$. Thus $\bigwedge_{i\in I} \mu_i \in HBC\left(L^X, \tau\right)$.

(iv) Let $\mu \in L^X$ be an H-bounded and closed set and let $x_\alpha \in M\left(L^X\right)$ such that $x_\alpha \notin \mu$, since μ is H-bounded and closed set, then $\mu \in HBR_{x_\alpha}$, since $\mu \le \mu$ then $x_\alpha \notin HB.cl(\mu)$ and so $HB.cl(\mu) \le \mu$. Therefore μ is HB-closed set.

(v) Suppose that μ is HB-closed set, $x_\alpha \in M\left(L^X\right)$ and $x_\alpha \notin \mu$. By Definition 3.9, there exists $\lambda \in HBR_{x_\alpha}$ with $\mu \le \lambda$. Conversely, provided that the condition is satisfied. If μ is not HB-closed set, then there exists $x_\alpha \in M\left(L^X\right)$ such that $x_\alpha \in HB.cl(\mu)$ and $x_\alpha \notin \mu$. Hence $\mu \not\le \lambda$ for each $\lambda \in HBR_{x_\alpha}$. It conflicts with the hypothesis, and so μ is HB-closed set.

Theorem 3.14: Let $\left(L^X, \tau\right)$ be an L-ts and $\mu \in L^X$. Then $\mu \in HBC\left(L^X, \tau\right)$ iff $\mu \in HBR_{x_\alpha}$ for each $x_\alpha \notin \mu$.

Proof. It follows directly from Theorem 3.13 (v).

Theorem 3.15: Let $\left(L^X, \tau\right)$ be an L-ts and $\mu \in L^X$. Then the mapping $HB.cl : L^X \to L^X$ is called closure operator of HB-boundedness iff it satisfies:

(i) $HB.cl(0_X) = 0_X$.

(ii) $\mu \le HB.cl(\mu)$.

(iii) $HB.cl(\mu \vee \eta) = HB.cl(\mu) \vee HB.cl(\eta)$.

(iv) $HB.cl(HB.cl(\mu)) = HB.cl(\mu)$.

A closure operator of *HB*-boundedness $HB.cl$ generates *L*-topology $\tau_{HB.cl}$ on L^X as: $\tau_{HB.cl} = \{\mu \in L^X : HB.cl(\mu') = \mu'\}$.

Proof. It follows directly from Theorems 3.11 and 3.13.

Theorem 3.16: Let (L^X, τ) be an *L*-ts. Then:

(i) $\tau_{HB} \le \tau$.

(ii) If (L^X, τ) is *H*-bounded space, then $\tau = \tau_{HB}$.

Proof. (i) Let $\mu \in \tau_{HB}$, then $HB.cl(\mu') \le \mu'$. Since $cl(\mu') \le HB.cl(\mu')$, hence

$cl(\mu') \le \mu'$ and so $\mu \in \tau$.

(ii) We note that $\tau_{HB} \le \tau$ from (i). Now, let $\mu \in \tau$ then $\mu' \in \tau'$. Since 1_X is *H*-bounded and $\mu' \le 1_X$, then μ' is *H*-bounded (By Theorem 3.3 (ii)) and by Theorem 3.13 (iv) we have μ' is *HB*-closed set and so $\mu' \in \tau_{HB}$. Thus $\tau = \tau_{HB}$.

Definition 3.17. Let (L^X, τ) be an *L*-ts, $\mu \in L^X$ and

$HB.\text{int}(\mu) = \vee\{\rho \in L^X : \rho \in HBO(L^X, \tau), \rho \le \mu\}$. We say that $HB.\text{int}(\mu)$ is the *HB*-interior of μ.

The following Theorem shows the relationships between *HB*-closure operator and *HB*-interior operator.

Theorem 3.18: Let (L^X, τ) be an *L*-ts and $\mu \in L^X$. Then the following are true:

(i) μ is *HB*-open iff $\mu = HB.\text{int}(\mu)$.

(ii) $(HB.cl(\mu))' = HB.\text{int}(\mu')$ and $(HB.\text{int}(\mu))' = HB.cl(\mu')$.

(iii) $HB.cl(\mu) = (HB.\text{int}(\mu'))'$ and $HB.\text{int}(\mu) = (HB.cl(\mu'))'$.

(iv) $HB.\text{int}(\mu) \le \text{int}(\mu) \le \mu$.

(v) If $\eta \in L^X$ and $\mu \le \eta$ then $HB.\text{int}(\mu) \le HB.\text{int}(\eta)$.

(vi) $HB.\text{int}(HB.\text{int}(\mu)) = HB.\text{int}(\mu)$.

Proof. (i) Let $\mu \in L^X$ be an *HB*-open set, then

$HB.\text{int}(\mu) = \vee\{\rho \in L^X : \rho \in HBO(L^X, \tau), \rho \le \mu\} = \mu$ and so $\mu = HB.\text{int}(\mu)$.

Conversely, let $\mu = HB.\text{int}(\mu)$, since

$HB.\text{int}(\mu) = \vee\{\rho \in L^X : \rho \in HBO(L^X, \tau), \rho \le \mu\}$. Therefore μ is *HB*-open set.

(ii) It follows directly from Definition 3.17 and Theorem 3.11 (iv).

(iii) It follows directly from (ii)

(iv) It follows directly from (ii) and Theorems 3.11 (i)

(v) It follows directly from (ii) and Theorem 3.11 (ii)

(vi) It follows directly from (ii) and Theorem 3.11 (iii)

Theorem 3.19: Let (L^X, τ) be an *L*-ts. The following statements hold::

(i) $0_X, 1_X \in HBO(L^X, \tau)$.

(ii) If $\mu_1, \mu_2, \cdots, \mu_n \in HBO\left(L^X, \tau\right)$, then $\overset{n}{\underset{i=1}{\wedge}} \mu_i \in HBO\left(L^X, \tau\right)$.

(iii) If $\{\mu_i : i \in I\} \subseteq HBO\left(L^X, \tau\right)$, then $\underset{i\in I}{\vee} \mu_i \in HBO\left(L^X, \tau\right)$.

Definition 3.20: Let $\left(L^X, \tau\right)$ be an *L*-ts and *S* be a molecular net in L^X. Then $x_\alpha \in M\left(L^X\right)$ is called

(i) limit point of *S* [13], (or *S* converges to x_α) in symbol $S \to x_\alpha$ if for every $\mu \in R_{x_\alpha}$ there is $n \in D$ such for each $m \in D$ and $m \ge n$ we have $S(m) \notin \mu$. The union of all limit points of *S* is denoted by $\lim(S)$.

(ii) *H*-bounded limit point of *S*, (or *S HB*-converges to x_α) in symbol $S \overset{HB}{\to} x_\alpha$ if for every $\mu \in HBR_{x_\alpha}$ there is an $n \in D$ such that $m \in D$ and $m \ge n$, we have $S(m) \notin \mu$. The union of all *HB*-limit points of *S* is denoted by $HB.\lim(S)$.

Theorem 3.21: Suppose that *S* is a molecular net in $\left(L^X, \tau\right)$, $\mu \in L^X$ and $x_\alpha \in M\left(L^X\right)$. Then the following statements hold:

(i) If $S \to x_\alpha$, then $S \overset{HB}{\to} x_\alpha$.

(ii) $x_\alpha \in HB.\lim(S)$ iff $S \overset{HB}{\to} x_\alpha$.

(iii) $\lim(S) \le HB.\lim(S)$.

(iv) $x_\alpha \in HB..cl(\mu)$ (resp. $x_\alpha \in .cl(\mu)$), iff there exists a molecular net *S* in μ such that *S* is *HB*-converges (resp. converges) to x_α.

(v) $HB.\lim(S)$ is *HB*-closed set in L^X.

Proof. (i) Let $S \to x_\alpha$ and let $\lambda \in HBR_{x_\alpha}$. Since $HBR_{x_\alpha} \subseteq R_{x_\alpha}$, then $\lambda \in R_{x_\alpha}$ Since $S \to x_\alpha$, then for every $\mu \in R_{x_\alpha}$ there is $n \in D$ such for each $m \in D$ and $m \ge n$, we have $S(m) \notin \lambda$. Thus $S \overset{HB}{\to} x_\alpha$.

(ii) Let $x_\alpha \in HB.\lim(S)$ and let $\lambda \in HBR_{x_\alpha}$. Since $x_\alpha \notin \lambda$, then $HB.\lim(S) \notin \lambda$. Therefore there exists $y_\gamma \in M\left(L^X\right)$ such that $y_\gamma \in HB.\lim(S)$ and $y_\gamma \notin \lambda$. Then $\lambda \in HBR_{y_\gamma}$ and so there is $n \in D$ much for each $m \in D$ and $m \ge n$ we have $S(m) \notin \lambda$, but since $\lambda \in HBR_{x_\alpha}$ so $S \overset{HB}{\to} x_\alpha$. Conversely, let $S \overset{HB}{\to} x_\alpha$, then by Definition 3.20 (ii) we have $x_\alpha \in HB.\lim(S)$

(iii) Let $x_\alpha \in \lim(S)$ and let $\eta \in HBR_{x_\alpha}$. Since $HBR_{x_\alpha} \subseteq R_{x_\alpha}$, then $\eta \in R_{x_\alpha}$. And since $x_\alpha \in \lim(S)$, then for each $\lambda \in R_{x_\alpha}$ there is $n \in D$ such for each $m \in D$ and $m \ge n$, we have $S(m) \notin \lambda$ and so $S(m) \notin \eta$. Hence $x_\alpha \in HB.\lim(S)$. So $\lim(S) \le HB.\lim(S)$.

(iv) Let $x_\alpha \in M\left(L^X\right)$ such that $x_\alpha \in HB.cl(\mu)$, then $\mu \not\le \lambda$ for each $\lambda \in HBR_{x_\alpha}$. Since $\mu \not\le \lambda$, then there exists $\alpha(\mu,\lambda) \in M(L)$ such that $x_{\alpha(\mu,\lambda)} \in \mu$ with $x_{\alpha(\mu,\lambda)} \notin \lambda$. Since the pair $\left(HBR_{x_\alpha}, \ge\right)$ is a directed set and so we can define a molecular net $S : HBR_{x_\alpha} \to M\left(L^X\right)$ as follows $S(\lambda) = x_{\alpha(\mu,\lambda)}$ for each $\lambda \in HBR_{x_\alpha}$ Hence *S* is a molecular net in μ. Now let $\eta \in HBR_{x_\alpha}$

such that $\lambda \le \eta$, so we have there exists $S(\eta) = x_{\alpha(\mu,\eta)} \notin \eta$ and so $S(\eta) = x_{\alpha(\mu,\eta)} \notin \lambda$. Hence *S* is *HB*-converges to x_α.

Conversely, let *S* be a molecular net in μ such that *S* is *HB*-converges to x_α then for each $\lambda \in HBR_{x_\alpha}$ there is $n \in D$ such for each $m \in D$ and $m \ge n$, we have $S(m) \notin \lambda$. Since $S(n) \in \mu$ for each $n \in D$, $m \in D$. So $S(m) \in \mu$ and $\mu \ge S(m) > \lambda$ hence $\mu \not\le \lambda$ for each $\lambda \in HBR_{x_\alpha}$. This means that $x_\alpha \in HB.cl(\mu)$.

(v) Let $x_\alpha \in HB.cl(HB.\lim(S))$, then $HB.\lim(S) \not\le \lambda$ for each $\lambda \in HBR_{x_\alpha}$ and then there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in HB.\lim(S)$ and $y_\gamma \notin \lambda$. Then for each $\mu \in HBR_{y_\gamma}$, there is $n \in D$ much for each $m \in D$ and $m \ge n$ we have $S(m) \notin \mu$ and so $S(m) \notin \lambda$. Hence $x_\alpha \in HB.\lim(S)$. Thus $HB.cl(HB.\lim(S)) \le HB.\lim(S)$ and so $HB.\lim(S)$ is *HB*-closed set.

Definition 3.22: Let (L^X, τ) be an *L*-ts and *I* be an ideal in L^X. Then $x_\alpha \in M(L^X)$ is called:

(i) limit point of *I* [18], (or *I* converges to x_α) in symbol $I \to x_\alpha$ if $R_{x_\alpha} \subseteq I$. The union of all limit points of *I* is denoted by $\lim(I)$.

(ii) *H*-bounded limit point of *I*, (or *I HB*-converges to x_α) in symbol $I \xrightarrow{HB} x_\alpha$ if $HBR_{x_\alpha} \subseteq I$. The union of all *HB*-limit points of *I* is denoted by $HB.\lim(I)$.

Theorem 3.23: Suppose that *I* is an ideal in (L^X, τ), $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then the following statements hold:

(i) If $I \to x_\alpha$, then $I \xrightarrow{HB} x_\alpha$.

(ii) $x_\alpha \in HB.\lim(I)$ iff $I \xrightarrow{HB} x_\alpha$.

(iii) $\lim(I) \le HB.\lim(I)$.

(iv) $x_\alpha \in HB..cl(\mu)$ iff there exists an ideal *I* in L^X such that $I \xrightarrow{HB} x_\alpha$ and $\mu \notin I$

(v) $HB.\lim(I)$ is *HB*-closed set in L^X.

Proof. (i) Let $I \to x_\alpha$ then $R_{x_\alpha} \subseteq I$. Since $HBR_{x_\alpha} \subseteq R_{x_\alpha}$, then $HBR_{x_\alpha} \subseteq I$. Thus $I \xrightarrow{HB} x_\alpha$.

(ii) Let $x_\alpha \in HB.\lim(I)$ and let $\lambda \in HBR_{x_\alpha}$. Since $x_\alpha \notin \lambda$ and $x_\alpha \in HB.\lim(I)$, then $HB.\lim(I) \notin \lambda$. Therefore there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in HB.\lim(I)$ and $y_\gamma \notin \lambda$. Then $\lambda \in HBR_{y_\gamma}$ and so $HBR_{x_\alpha} \subseteq HBR_{y_\gamma} \subseteq I$ hence $HBR_{x_\alpha} \subseteq I$. Thus $I \xrightarrow{HB} x_\alpha$. Conversely, let $I \xrightarrow{HB} x_\alpha$, then by Definition 3.22 (ii) we have $x_\alpha \in HB.\lim(I)$.

(iii) Let $x_\alpha \in \lim(I)$ and let $\eta \in HBR_{x_\alpha}$. since $x_\alpha \in \lim(I)$, so for each $\lambda \in R_{x_\alpha}$, $\lambda \in I$ and since $\eta \in HBR_{x_\alpha}$ so $\eta \in R_{x_\alpha}$. Hence $x_\alpha \in HB.\lim(I)$. So $\lim(I) \le HB.\lim(I)$.

(iv) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \in HB.cl(\mu)$. The family

$I=\{\rho\in L^X:\exists\,\lambda\in HBR_{x_\alpha}\ni\rho\le\lambda\}$ is an ideal in L^X. Now we show that $\mu\notin I$. Since $x_\alpha\in HB.cl(\mu)$, then for each $\lambda\in HBR_{x_\alpha}$, $\mu\not\le\lambda$. So By definition of I we have $\mu\notin I$. Finally, we show that $I\overset{HB}{\to}x_\alpha$. Let $\lambda\in HBR_{x_\alpha}$, since $\lambda\le\lambda$, then $\lambda\in I$. So $HBR_{x_\alpha}\subseteq I$. Thus $I\overset{HB}{\to}x_\alpha$.

Conversely, let I be an ideal in L^X such that $I\overset{HB}{\to}x_\alpha$ and $\mu\notin I$. Then for each $\lambda\in HBR_{x_\alpha}$, $\lambda\in I$. Since $\lambda\in I$, $\mu\notin I$, then $\mu\not\le\lambda$ and so $x_\alpha\in HB..cl(\mu)$.

(v) Let $x_\alpha\in HB..cl(HB.\lim(I))$, then $HB.\lim(I)\not\le\lambda$ for each $\lambda\in HBR_{x_\alpha}$ and then there exists $y_\gamma\in M(L^X)$ such that $y_\gamma\in HB.\lim(I)$ and $y_\gamma\notin\lambda$. Since $\lambda\in HBR_{y_\gamma}$ and $I\overset{HB}{\to}y_\gamma$ then $\eta\in I$ for each $\eta\in HBR_{x_\alpha}$. Since $y_\gamma\notin\lambda$ then $\lambda\in I$. But $\lambda\in HBR_{x_\alpha}$ and so $x_\alpha\in HB.\lim(I)$. Thus $HB..cl(HB.\lim(I))\le HB.\lim(I)$ and so $HB.\lim(I)$ is *HB*-closed set.

4. *HB*-Continuous Mappings in *L*-Topological Space

In this section we first define *HB*-continuous mappings in *L*-topological space and then investigate some of its characterizations,

Definition 4.1: An *L*-mapping $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is called :

(i) *HB*-continuous at $x_\alpha\in M(L^X)$ if $f_L^{-1}(\eta)\in R_{x_\alpha}$ for each $\eta\in HBR_{f_L(x_\alpha)}$

(ii) *HB*-continuous if $f_L^{-1}(\eta)\in\tau$ for each $\eta\in L^X$ is closed and *H*-bounded.

Theorem 4.2: Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be an *L*-continuous mapping. Then the following properties are equivalent :

(i) f_L is *HB*-continuous.

(ii) f_L is *HB*-continuous at x_α for each $x_\alpha\in M(L^X)$.

(iii) If $\eta\in\Delta$ and η' is *H*-bounded, then $f_L^{-1}(\eta)\in\tau$.

(iv) If $\eta\in L^Y$ is *H*-bounded, then $f_L^{-1}(\eta)\in\tau'$.

Proof. (i)$\Rightarrow$(ii): Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be an *HB*-continuous and $x_\alpha\in M(L^X)$, $\eta\in HBR_{f_L(x_\alpha)}$ then $f_L^{-1}(\eta)\in\tau'$. Since $f_L(x_\alpha)\notin\eta$, then $x_\alpha\notin f_L^{-1}(\eta)$

And so $f_L^{-1}(\eta)\in R_{x_\alpha}$. Thus f_L is *HB*-continuous at x_α for each $x_\alpha\in M(L^X)$.

(ii)$\Rightarrow$(i): Let f_L be an *HB*-continuous at x_α for each $x_\alpha\in M(L^X)$. If f_L is not *HB*-continuous, then there is $\eta\in L^Y$ is *H*-bounded and closed such that $f_L^{-1}(\eta)\notin\tau'$, *i.e.*, $cl(f_L^{-1}(\eta))\not\le f_L^{-1}(\eta)$. Then there exists $x_\alpha\in M(L^X)$ such that $x_\alpha\in cl(f_L^{-1}(\eta))$ and $x_\alpha\notin f_L^{-1}(\eta)$ implies that $f_L(x_\alpha)\notin\eta$, since η is closed and *H*-bounded, then $\eta\in HBR_{f_L(x_\alpha)}$. But $f_L^{-1}(\eta)\notin R_{x_\alpha}$, this contradiction. Thus f_L is *HB*-continuous mapping.

(i)$\Rightarrow$(iii): Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be an *HB*-continuous and $\eta\in\Delta$ such that η' is *H*-bounded and so η' is *H*-bounded and closed. By (i), we have

$f_L^{-1}(\eta') \in \tau'$. Since $f_L^{-1}(\eta') = \left(f_L^{-1}(\eta)\right)'$, then $f_L^{-1}(\eta) \in \tau$.

(iii) $\Rightarrow$ (i): Let $\eta \in L^Y$ be an *H*-bounded and closed, then $\eta' \in \Delta$. By (iii), we have $f_L^{-1}(\eta') \in \tau$, thus $f_L^{-1}(\eta) = \left(f_L^{-1}(\eta')\right)'$, then $f_L^{-1}(\eta) \in \tau'$. Hence f_L is *HB*-continuous mapping.

(iv) $\Rightarrow$ (iii): Let $\eta \in \Delta$ and η' be an *H*-bounded. By (iv), we have $f_L^{-1}(\eta) \in \tau'$. Thus $f_L^{-1}(\eta) = \left(f_L^{-1}(\eta')\right)' \in \tau$.

(iv) $\Rightarrow$ (ii): Let $\eta \in HBR_{f_L(x_\alpha)}$ and $x_\alpha \in M\left(L^X\right)$. Then η is closed and *H*-bounded set, $f_L\left(x_\alpha\right) \notin \eta$ and so $x_\alpha \notin f_L^{-1}(\eta)$. By (iv), we have $f_L^{-1}(\eta) \in \tau'$ and $x_\alpha \notin f_L^{-1}(\eta)$ hence $f_L^{-1}(\eta) \in R_{x_\alpha}$. Thus f_L is *HB*-continuous mapping at x_α for each $x_\alpha \in M\left(L^X\right)$.

(iv) $\Rightarrow$ (i): Let $\eta \in L^Y$ be a closed and *H*-bounded set. By (iv), we have $f_L^{-1}(\eta) \in \tau'$. Thus f_L is *HB*-continuous mapping.

Theorem 4.3: Let $f_L : \left(L^X, \tau\right) \to \left(L^Y, \Delta\right)$ be an *L*-surjective mapping. Then the following conditions are equivalent:

(i) f_L is *HB*-continuous mapping.

(ii) For each $\mu \in L^X$, $f_L\left(cl(\mu)\right) \le HB.cl\left(f_L(\mu)\right)$,

(iii) For each $\eta \in L^Y$, $cl\left(f_L^{-1}(\eta)\right) \le f_L^{-1}\left(HB.cl(\eta)\right)$,

(iv) For each $\eta \in L^Y$, $f_L^{-1}\left(HB.\mathrm{int}(\eta)\right) \le \mathrm{int}\left(f_L^{-1}(\eta)\right)$,

(v) For each *HB*-open *L*-subset ρ in L^Y, then $f_L^{-1}(\rho)$ is open *L*-subset in L^X,

(vi) For each *HB*-closed *L*-subset λ in L^Y, then $f_L^{-1}(\lambda)$ is closed *L*-subset in L^X.

Proof. (i) $\Rightarrow$ (ii): Let $\mu \in L^X$ and $x_\alpha \in M\left(L^X\right)$ such that $x_\alpha \in cl(\mu)$. Then $f_L\left(x_\alpha\right) \in f_L\left(cl(\mu)\right)$. Let $\eta \in HBR_{f_L(x_\alpha)}$. So by (i) and by Theorem 4.3, we have $f_L^{-1}(\eta) \in R_{x_\alpha}$. Since $x_\alpha \in cl(\mu)$, then $\mu \not\le f_L^{-1}(\eta)$. Since f_L is *L*-surjective then $f_L(\mu) \not\le \eta$ and $\eta \in HBR_{f_L(x_\alpha)}$ so $f_L\left(x_\alpha\right) \in HB.cl\left(f_L(\mu)\right)$. Hence $f_L\left(cl(\mu)\right) \le HB.cl\left(f_L(\mu)\right)$.

(ii) $\Rightarrow$ (iii): Let $\eta \in L^Y$. Then $f_L^{-1}(\eta) \in L^X$. By (ii) we have
$f_L\left(cl\left(f_L^{-1}(\eta)\right)\right) \le HB.cl\left(f_L\left(f_L^{-1}(\eta)\right)\right) \le HB.cl(\eta)$. So
$f_L\left(cl\left(f_L^{-1}(\eta)\right)\right) \le HB.cl(\eta)$. Thus $f_L^{-1} f_L\left(cl\left(f_L^{-1}(\eta)\right)\right) \le f_L^{-1}\left(HB.cl(\eta)\right)$. Since $cl\left(f_L^{-1}(\eta)\right) \le f_L^{-1} f_L\left(cl\left(f_L^{-1}(\eta)\right)\right)$, then $cl\left(f_L^{-1}(\eta)\right) \le f_L^{-1}\left(HB.cl(\eta)\right)$.

(iii) $\Rightarrow$ (iv): Let $\eta \in L^Y$. By (iii), we have $cl\left(f_L^{-1}(\eta')\right) \le f_L^{-1}\left(HB.cl(\eta')\right)$ Since $cl\left(f_L^{-1}(\eta')\right) = \left(\mathrm{int}\left(f_L^{-1}(\eta)\right)\right)'$ and $f_L^{-1}\left(HB.cl(\eta')\right) = \left(f_L^{-1}\left(HB.\mathrm{int}(\eta)\right)\right)'$. So $\left(\mathrm{int}\left(f_L^{-1}(\eta)\right)\right)' \le \left(f_L^{-1}\left(HB.\mathrm{int}(\eta)\right)\right)'$. Thus $f_L^{-1}\left(HB.\mathrm{int}(\eta)\right) \le \mathrm{int}\left(f_L^{-1}(\eta)\right)$.

(iv) $\Rightarrow$ (v): Let ρ be an *HB*-open *L*-subset in L^Y. Then
$f_L^{-1}(\rho) = f_L^{-1}\left(HB.\mathrm{int}(\rho)\right)$ and by (iv), we have
$f_L^{-1}\left(HB.\mathrm{int}(\rho)\right) \le \mathrm{int}\left(f_L^{-1}(\rho)\right)$, so $f_L^{-1}(\rho) \le \mathrm{int}\left(f_L^{-1}(\rho)\right)$. Thus $f_L^{-1}(\rho) \in \tau$.

(v) ⇒ (vi): Let λ be an *HB*-closed *L*-subset in L^Y. By (v), we have $f_L^{-1}(\lambda') \in \tau$. Then $\left(f_L^{-1}(\lambda)\right)' = f_L^{-1}(\lambda') \in \tau$ and so $f_L^{-1}(\lambda) \in \tau'$.

(vi) ⇒ (i): Let $\eta \in L^Y$ be an closed and *H*-bounded set, then η is *HB*-closed *L*-subset in L^Y. By (vi), we have $f_L^{-1}(\eta) \in \tau'$. Thus f_L is *HB*-continuous mapping.

Theorem 4.4: If $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is *HB*-continuous mapping, then $f_L : (L^X, \tau) \to \left(L^{f(X)}, \Delta_{f(X)}\right)$ is *HB*-continuous mapping.

Proof. Let $\eta \in \Delta_{f(X)}$ such that $1_{f(X)} \setminus \eta$ is *H*-bounded set, then $1_{f(X)} \setminus \eta$ is *H*-bounded and closed in $\left(L^{f(X)}, \Delta_{f(X)}\right)$. Therefore $\rho = 1_Y \setminus \left(1_{f(X)} \setminus \eta\right) \in \Delta$ and ρ' is *H*-bounded in (L^Y, Δ). Since $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is *HB*-continuous mapping, the by Theorem 4.2 (iii), we have $f_L^{-1}(\rho) \in \tau$, thus

$f_L^{-1}(\rho) = f_L^{-1}\left(1_Y \setminus \left(1_{f(X)} \setminus \eta\right)\right) = 1_X \setminus \left(f_L^{-1}\left(1_{f(X)} \setminus \eta\right)\right) = 1_X \setminus \left(1_X \setminus f_L^{-1}(\eta)\right) = f_L^{-1}(\eta)$.

Hence $f_L^{-1}(\eta) \in \tau$ consequently, $f_L : (L^X, \tau) \to \left(L^{f(X)}, \Delta_{f(X)}\right)$ is *HB*-continuous mapping.

Theorem 4.5: If $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is *HB*-continuous mapping and $A \subseteq X$ then $f_L|_A : (L^A, \tau_A) \to (L^Y, \Delta)$ is *HB*-continuous mapping.

Proof. Let $\eta \in L^Y$ be an *H*-bounded and closed set. Since $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is *HB*-continuous mapping, then $f_L^{-1}(\eta) \in \tau'$ and since $\left(f_L|_A\right)^{-1}(\eta) = f_L^{-1}(\eta) \wedge 1_A \in \tau'_A$. Hence $f_L|_A$ is *HB*-continuous mapping.

Theorem 4.6: Every $f_L : (L^X, \tau) \to (L^Y, \Delta)$ *L*-continuous mapping is *HB*-continuous mapping.

Proof. Let $f_L : (L^X, \tau) \to (L^Y, \Delta)$ be an *L*-continuous and let $\eta \in L^Y$ be an closed and *H*-bounded set, then $f_L^{-1}(\eta) \in \tau'$. Thus f_L is *HB*-continuous mapping.

The following example shows that the converse is not true in general.

Example 4.7: Let $\{I_j : j \in J\}$ be the usual interval base of the relative *L*-topology on $L = I = [0,1]$ induced by the set of real numbers. Define a *L*-topology τ on $[0,1]$ generated by the base consisting of, 0_X, 1_X and $\{I_{j_k} : j \in J \text{ and } k \in (0,1)\}$ where

$$I_{j_K}(x) = \begin{cases} k : x \in I \\ 0 : x \notin I \end{cases}$$

Let Δ be the *L*-topology on *I* such that the complements of any number of Δ is countable *L*-subset in *I* (*i.e.*, the support of the *L*-subset is countable). Let $f_L : (L^X, \tau) \to (L^Y, \Delta)$ be a function defined by $f(x) = x$, for all $x \in I$. Then it can be see that f_L is *HB*-continuous but not *L*-continuous mapping.

Theorem 4.8: A mapping $f_L : (L^X, \tau) \to (L^Y, \Delta_{HB})$ is *L*-continuous mapping iff it is *HB*-continuous mapping.

Proof. Since $\Delta'_{HB} \le \Delta'$, then necessity is evident. Now, we suppose that f_L is *HB*-continuous and $\eta \in \Delta'_{HB}$. Then by Theorem 4.3 (iii) we have $f_L^{-1}(\eta) = f_L^{-1}(HB.cl(\eta)) \ge cl\left(f_L^{-1}(\eta)\right)$ and so $f_L^{-1}(\eta) \in \tau'$. Thus f_L is

L-continuous mapping.

Theorem 4.9: Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be an L-mapping and (L^Y,Δ) is H-bounded space. Then f_L is L-continuous mapping iff f_L is HB-continuous mapping.

Proof. By Theorem 4.6 we need only to investigate the sufficiency. Let $\eta\in\Delta'$. Since (L^Y,Δ) is H-bounded space then by Theorem 3.2(ii), we have η is H-bounded set and so η is HB-closed L-subset. By HB-continuity of f_L, we have $f_L^{-1}(\eta)\in\tau'$. Hence f_L is L-continuous mapping.

Theorem 4.10: If f_L is HB-continuous, then f_L is H-continuous mapping.

Proof. Follows from the fact that every H-compact set is H-bounded set.

Theorem 4.11: Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be an L-mapping and (L^Y,Δ) be LT_3-space. Then f_L is H-continuous iff f_L is HB-continuous mapping.

Proof. Let f_L be an HB-continuous mapping and let $\eta\in L^Y$ be a closed and H-compact, then by Theorem 3.3 (i), we have η is H-bounded and closed. Since f_L is HB-continuous then $f_L^{-1}(\eta)\in\tau'$. Thus f_L is H-continuous.

Conversely, let f_L be an H-continuous and let $\eta\in L^Y$ be a closed and H-bounded. Then η is H-compact and closed. Since f_L is H-continuous, then $f_L^{-1}(\eta)\in\tau'$. Thus f_L is HB-continuous mapping.

Remark 4.12: For an L-mapping $f_L:(L^X,\tau)\to(L^Y,\Delta)$, we obtain the following implications:

L-continuity $\Rightarrow$ HB-continuity $\Rightarrow$ H-continuity.

None of these implications are reversible. However, if it (L^Y,Δ) is H-bounded (resp. LT_3-) space, then Theorem 4.10 (resp. Theorem 4.12) implies that the concepts of L-continuity (resp. HB-continuity) and H-continuity are equivalent.

Theorem 4.13: If $f_L:(L^X,\tau_1)\to(L^Y,\tau_2)$ is L-continuous and $g_L:(L^Y,\tau_2)\to(L^Z,\tau_3)$ is HB-continuous, then $g_L\circ f_L:(L^X,\tau_1)\to(L^Z,\tau_3)$ is HB-continuous.

Proof. Let $\eta\in L^Y$ be a closed and almost N-compact. Since g_L is HB-continuous, then $g_L^{-1}(\eta)\in\tau_2'$ and since f_L is L-continuous, then $f_L^{-1}(g_L^{-1}(\eta))\in\tau_1'$ Hence $g_L\circ f_L$ HB-continuous mapping.

Theorem 4.14: If (L^X,τ) and (L^Y,Δ) are L-ts's and $1_X=1_A\vee 1_B$ such that $1_A,1_B\in\tau'$ and $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is L-mapping and $f_L|_A$, $f_L|_B$ are HB-continuous mappings, then f_L is HB-continuous mapping.

Proof. Let $\eta\in L^Y$ be an N-almost bounded and closed then

$$\left(f_L|_A\right)^{-1}(\eta)\vee\left(f_L|_B\right)^{-1}(\eta)=\left(f_L^{-1}(\eta)\wedge 1_A\right)\vee\left(f_L^{-1}(\eta)\wedge 1_B\right)$$
$$=\left(f_L^{-1}(\eta)\wedge(1_A\vee 1_B)\right)=f_L^{-1}(\eta)\wedge 1_X=f_L^{-1}(\eta)$$

Hence $f_L^{-1}(\eta)\in\tau'$. Thus f_L is HB-continuous mapping.

Theorem 4.15: If $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is HB-continuous mapping, injective, (L^Y,Δ) is LT_1-space and H-bounded, then (L^X,τ) is LT_1-space.

Proof. Let $x_\alpha,y_\gamma\in M(L^X)$ such that $x\neq y$. Since f_L is injective L-mapping, then $f_L(x_\alpha),f_L(y_\gamma)\in M(L^Y)$ and $f(x)\neq f(y)$. Since (L^Y,Δ)

is LT_1-space, then $f_L(x_\alpha), f_L(y_\gamma)$ are closed L-subsets in (L^Y,Δ). Since (L^Y,Δ) is H-bounded, then $f_L(x_\alpha), f_L(y_\gamma)$ are H-bounded L-subsets. Since $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is HB-continuous mapping, then $f_L^{-1}(f_L(x_\alpha),)=x_\alpha$ and $f_L^{-1}(f_L(y_\gamma),)=y_\gamma$ are closed L-subsets in (L^X,τ). Hence (L^X,τ) is LT_1-space.

5. Characterizations of *HB*-Continuous Mappings in *L*-Topological Space

Theorem 5.1: Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be an HB-continuous mapping and be a fully stratified $LT_{2\frac{1}{2}}$-space and LR_2-space. If $f_L(1_X)$ is contained in some H-compact set of L^Y, then f_L is L-continuous mapping.

Proof. Let $\eta\in L^Y$ be an H-compact set containing $f_L(1_X)$ and let $\rho\in\Delta'$. Since η is B-compact in (L^Y,Δ) which is fully stratified $LT_{2\frac{1}{2}}$-space and LR_2-space, so $\eta\in\Delta'$ and η is H-bounded by Theorem 3.3 (ii). Thus $\eta\wedge\rho\in\Delta'$. Hence by Theorem 3.3 (iii), we have $\eta\wedge\rho\in L^Y$ is H-bounded. Thus $\eta\wedge\rho\in L^Y$ is closed and H-bounded. By HB-continuity of f_L, then we have $f_L^{-1}(\eta\wedge\rho)\in\tau'$. But,
$f_L^{-1}(\eta\wedge\rho)=f_L^{-1}(\eta)\wedge f_L^{-1}(\rho)=f_L^{-1}(\rho)\wedge 1_X=f_L^{-1}(\rho)$. So $f_L^{-1}(\rho)\in\tau'$. Hence f_L is L-continuous mapping.

Theorem 5.2: If $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is L-closed and L-almost continuous mapping, then $f_L^{-1}:(L^Y,\Delta)\to(L^X,\tau)$ is HB-continuous mapping.

Proof. Let $\eta\in L^X$ be an H-bounded and closed. Since f_L is L-almost continuous mapping, then by Theorem 3.2 we have is H-bounded in L^Y. Since f_L is L-closed mapping, then $f_L(\eta)\in\Delta'$. Hence by Theorem 4.3, we have f_L^{-1} is HB-continuous mapping.

Theorem 5.3: Let (L^X,τ) be an L-ts and (L^Y,Δ) be a fully stratified $LT_{2\frac{1}{2}}$-space and LR_2-space. If $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is a bijective and L-almost continuous mapping, then $f_L^{-1}:(L^Y,\Delta)\to(L^X,\tau)$ is HB-continuous mapping.

Proof. Let $\eta\in L^X$ be an H-compact. Since f_L is L-almost continuous mapping, then by Theorem 2.10, $f_L(\eta)$ is H-compact. Since (L^Y,Δ) is fully stratified $LT_{2\frac{1}{2}}$-space and LR_2-space, then $f_L(\eta)\in\Delta'$ and $f_L(\eta)$ is H-bounded. Hence by Theorem 4.2, we have f_L^{-1} is HB-continuous mapping.

Corollary 5.4: Let (L^X,τ) be an H-compact space and (L^Y,Δ) be a fully stratified $LT_{2\frac{1}{2}}$-space and LR_2-space. If $f_L:(L^X,\tau)\to(L^Y,\Delta)$ is a bijective and L-almost continuous mapping, then f_L is a homeomorphism.

Proof. Follows from Theorem 5.1 and 5.3.

Theorem 5.5: Let $f_L:(L^X,\tau)\to(L^Y,\Delta)$ be a surjective L-mapping, then the following conditions are equivalent :

(i) f_L is HB-continuous mapping.

(ii) For each $x_\alpha\in M(L^X)$ and each molecular net S in L^X,

$f_L(S) \xrightarrow{HB} f_L(x_\alpha)$ at $S \to x_\alpha$.

(iii) $f_L(\lim(S)) \le HB.\lim(f_L(S))$ for each S in L^X.

Proof: (i)$\Rightarrow$(ii): Let $x_\alpha \in M(L^X)$ and $S = \{S(n): n \in D\}$ be an molecular net in L^X which converges to x_α. Let $\eta \in HBR_{f_L(x_\alpha)}$, by (i), we have $f_L^{-1}(\eta) \in R_{x_\alpha}$. Since $S \to x_\alpha$ then there is an $n \in D$ for all $m \in D$, $m \ge n$ such that $S(m) \not\le f_L^{-1}(\eta)$ and so $f_L(S(m)) \not\le f_L f_L^{-1}(\eta) = \eta$. Thus $f_L(S(m)) \not\le \eta$. Hence $f_L(S) \xrightarrow{HB} f_L(x_\alpha)$.

(ii)$\Rightarrow$(iii): Let S be a molecular net in L^X and let $y_\alpha \in f_L(\lim(S))$, then there exists $x_\alpha \in \lim(S)$ such that $y_\alpha = f_L(x_\alpha)$. By (ii) we have $f_L(x_\alpha) \in HB.\lim(f_L(S))$. Thus $f_L(\lim(S)) \le HB.\lim(f_L(S))$ for each S in L^X.

(iii)$\Rightarrow$(i): Let $\eta \in L^Y$ be an *HB*-closed and $x_\alpha \in M(L^X)$ such that $x_\alpha \in cl(f_L^{-1}(\eta))$. By Theorem 2.19, we have molecular net S in $f_L^{-1}(\eta)$ which converges to x_α. Thus $x_\alpha \in \lim(S)$ and so $f_L(x_\alpha) \in f_L(\lim(S))$. By (iii), $f_L(x_\alpha) \in f_L(\lim(S)) \le HB.\lim(f_L(S))$ and so $f_L(S) \xrightarrow{HB} f_L(x_\alpha)$. On the other hand, since S is molecular net in $f_L^{-1}(\eta)$, then for each $n \in D$, $S(n) \in f_L^{-1}(\eta)$ and so $f_L(S(n)) \le f_L(f_L^{-1}(\eta)) = \eta$. Hence $f_L(S(n)) \le \eta$ for each $n \in D$. Thus $f_L(S)$ is molecular net in η. So we have $f_L(S) \xrightarrow{HB} f_L(x_\alpha)$ and $f_L(S)$ is molecular net in η and so $f_L(x_\alpha) \in HB.cl(\eta)$. But since η is *HB*-closed *L*-subset, so $\eta = HB.cl(\eta)$. Thus $f_L(x_\alpha) \in \eta$. Hence $x_\alpha \in f_L^{-1}(\eta)$. So $cl(f_L^{-1}(\eta)) \le f_L^{-1}(\eta)$. Hence $f_L^{-1}(\eta) \in \tau'$. Then f_L is *HB*-continuous mapping.

Theorem 5.6: If $f_L:(L^X,\tau) \to (L^Y,\Delta)$ is a surjective *L*-mapping. Then the following conditions are equivalent:

(i) f_L is *HB*-continuous mapping.

(ii) For each $x_\alpha \in M(L^X)$ and each *L*-ideal I in L^X, then $f_L(I) \xrightarrow{HB} f_L(x_\alpha)$ if $I \to x_\alpha$.

(iii) $f_L(\lim(I)) \le HB.\lim(f_L(I))$ for each I in L^X.

Proof: (i)$\Rightarrow$(ii): Let $x_\alpha \in M(L^X)$ and $I \to x_\alpha$. Let $\eta \in HBR_{f_L(x_\alpha)}$, by (i), we have $f_L^{-1}(\eta) \in R_{x_\alpha}$. Since $I \to x_\alpha$ then $f_L^{-1}(\eta) \in I$. Since $x_\alpha \notin f_L^{-1}(\eta)$, then $f_L(x_\alpha) \notin \eta$, so $\eta \in f_L(I)$. Hence $HBR_{f_L(x_\alpha)} \subseteq f_L(I)$. Thus $f_L(I) \xrightarrow{HB} f_L(x_\alpha)$.

(ii)$\Rightarrow$(iii): Let I be an *L*-ideal in L^X and let $y_\alpha \in f_L(\lim(I))$, then there exists $x_\alpha \in \lim(I)$ such that $y_\alpha = f_L(x_\alpha)$. By (ii) we have $f_L(I) \xrightarrow{HB} f_L(x_\alpha)$. So $y_\alpha = f_L(x_\alpha) \in HB.\lim(f_L(I))$. Hence $f_L(\lim(I)) \le HB.\lim(f_L(I))$ for each I in L^X.

(iii)$\Rightarrow$(i): Let $\eta \in L^Y$ be an *HB*-closed set and $x_\alpha \in M(L^X)$ such that $x_\alpha \in cl(f_L^{-1}(\eta))$. By Theorem 2.23, there exists *L*-ideal I which converges to x_α such that $f_L^{-1}(\eta) \notin I$. Moreover, $f_L(I) \le \{\rho \in L^Y : \eta \not\le \rho\}$ if $\lambda \in I$ with

$\eta \le \lambda$, then there exists $\mu \in I$ satisfy $x_\alpha \notin \mu$ such that $f_L(x_\alpha) \notin \lambda$. Since $\eta \le \lambda$, then $f_L(x_\alpha) \notin \eta$. This show that $x_\alpha \in \mu$ if $f_L(x_\alpha) \in \eta$. Thus $f_L^{-1}(\eta) \le \mu$. So $f_L^{-1}(\eta) \in I$, a contradiction. Hence $\eta \notin f_L(I)$. On the other hand, by (iii), $f_L(x_\alpha) \in f_L(\lim(I)) \le HB.\lim(f_L(I))$. Thus $f_L(I) \xrightarrow{HB} f_L(x_\alpha)$ and so $f_L(x_\alpha) \in HB.cl(\eta)$. But since η is *HB*-closed *L*-subset, so $\eta = HB.cl(\eta)$. Thus $f_L(x_\alpha) \in \eta$. Hence $x_\alpha \in f_L^{-1}(\eta)$. So $cl(f_L^{-1}(\eta)) \le f_L^{-1}(\eta)$. Hence $f_L^{-1}(\eta) \in \tau'$. Then f_L is *HB*-continuous mapping.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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