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The Convergence Rate of Fréchet Distribution under the Second-Order Regular Variation Condition

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Abstract

In this article we consider the asymptotic behavior of extreme distribution with the extreme value index $\gamma > 0$. The rates of uniform convergence for Fréchet distribution are constructed under the second-order regular variation condition.

Keywords

Convergence Rate, Second-Order Regular Variation Condition, Fréchet Distribution, Extreme Value Index

1. Introduction

The central limit theory focuses on the extreme behavior of sample partial sums, but in nature and human society, there are also a class of extreme risk events, such as floods, earthquakes, precipitation and economic crises. Although these events are rare, once they occur, they will bring significant losses to society. Therefore, studying the laws of extreme value occurrence is extremely important. Extreme value theory emerged in this context, as an important branch of probability theory, mainly focusing on the tail behavior of extreme value distributions. In recent years, the application range of extreme value theory has been very extensive. For example, predicting the probability of extreme events such as the above, estimating the percentile of extreme value distribution, and applying it to fields such as financial risk management, see de Haan and Ferreira (2006) [1].

Let $\{X_n, n \geq 1\}$ be independent, identically distributed (iid) random variables with common distribution function

$$F(x) = P[X_1 \leq x], \quad x \in R.$$

Denote the extreme value by

$$M_n = \bigvee_{i=1}^{n} X_i,$$

and suppose there exist normalizing constants $a_n > 0$ and $b_n \in R$ such that

$$\frac{M_n - b_n}{a_n}$$

has a nondegenerate limit distribution as $n \to \infty$, *i.e.*

$$P\left[M_n \le a_n x + b_n\right] \to G(x). \tag{1.1}$$

Fisher and Tippett (1928) [2], Gnedenko (1943) [3] proposed the extreme value distribution $G(x)$ takes the form of

$$G(x) = G_\gamma(x) = \exp\left\{-(1+\gamma x)^{-1/\gamma}\right\}, \gamma \in R, 1+\gamma x \ge 0, \tag{1.2}$$

where the parameter γ in (1.2) is called the extreme value index. This also means that F is in the domain of attraction of extreme value distribution.

Under the special case of the extreme value index $\gamma > 0$, the extreme value distribution can be written as

$$G(x) = \Phi_{1/\gamma}(x) = \begin{cases} 0, & x < 0, \\ \exp\left\{-x^{-1/\gamma}\right\}, & x \ge 0, \end{cases} \tag{1.3}$$

which is also called the Fréchet distribution, and the convergence in (1.1) can be rewritten as

$$P\left[M_n \le a_n x\right] \to \Phi_{1/\gamma}(x). \tag{1.4}$$

Based on theoretical studies, many scholars focus on the first-order asymptotic analysis in extreme events. But with the widespread application of extreme value theory, several authors discovered the first-order asymptotic results obtained by using the limits of extreme value distributions are relatively rough, and often requires a more accurate approximate representation. It is necessary to know the further expansion of first-order convergence. Therefore, research on the convergence rate of first-order asymptotic result in extreme value theory has attracted the attention of many scholars. de Haan and Peng (1997) [4] considers the convergence rate of two-dimensional extreme value distribution. In the research on the convergence speed of one-dimensional extreme value distribution, de Haan and Resnick (1996) [5] established the rates of convergence of the distribution of the extreme order statistics to its limit distribution under the second-order von Mises condition with $\gamma \in R$. Cheng and Jiang (2001) [6] focuses on the rates of the uniform convergence for distributions of extreme values ($F^n(a_n x + b_n)$ to $G_\gamma(x)$) under the condition of generalized regular variation of second-order. For the speed at which the extreme value distribution converges to its limit distribution in special distributions, Liao *et al.* (2014) [7] derived the asymptotic behavior of the distribution of the maxima for samples obeying skew-normal distribution. Peng *et al.* (2010) [8] established the limiting distribution of the extremes and the associated convergence rates for the mixed exponential distributions. Chen and Huang (2014) [9] construsted the exact uniform convergence rate of the asymmetric normal distribution of the maximum and minimum to its extreme value limit. Chen and Feng (2014) [10] considered the rates of convergence of extremes for short-tailed symmetric distribution under power normalization. Chen *et al.* (2012) [11] studied the rates of convergence of extremes for general error distribution under power normalization.

The second-order asymptotic result can provide a more accurate approximate expression, and it can characterize the speed of first-order convergence, which can provide a better guidance for the prediction, risk management, and control of extreme events, see Lin (2012) [12], Mao and Hu (2013) [13]. The focus of this paper is on rates of convergence in (1.4). We set out to explain our condition. For a nondecreasing function *f*, define the left-continuous inverse of *f* is

$$f^{\leftarrow}(t)=\inf\{x: f(x)\geq t\}.$$

Let $f=(-1/\log F)^{\leftarrow}$. Necessary and sufficient condition for the convergence for (1.4) is that *f* is regulary varying, *i.e.*

$$\lim_{t\to\infty}\frac{f(tx)}{f(t)}=x^{\gamma} \tag{1.5}$$

holds for $x>0$ and $\gamma>0$, written as $f\in RV(\gamma)$. Regarding regular variation refer to Resnick (1987) [14]. So in order to get the convergence rate of (1.4), we need to require a rate of convergence condition in (1.5). The condition as follows.

Supppose the second-order regular variation condition

$$\lim_{t\to\infty}\frac{f(tx)/f(t)-x^{\gamma}}{B(t)}=\kappa(x) \tag{1.6}$$

holds for all $x>0$, where B has constant sign near infinity and satisfying $\lim_{t\to\infty}B(t)=0$. The function $\kappa(x)$ should not be a multiple of x^{γ}. By Theorem B.3.1 of de Haan and Ferreira (2006) [1], We know that $|B|\in RV_{\rho}$ and $\kappa(x)=x^{\gamma}(x^{\rho}-1)/\rho$ for $\rho\leq 0$.

For convenience, let $G_0(x)=\exp\{-\mathrm{e}^{-x}\}$ and its derivative $G_0'(x)=G_0(x)\mathrm{e}^{-x}$, define

$$J_n(x)=G_0\left(\gamma^{-1}\log f(nx)/f(n)\right)-G_0(\log x);$$
$$J(x)=\gamma^{-1}x^{-\gamma}G_0'(\log x)\kappa(x)$$

and $a_n:=f(n)$. Moreover, for any function *g* on $(0,\infty)$, denote $g(\infty)=\lim_{t\to\infty}g(t)$ and $g(0)=\lim_{t\to 0}g(t)$ if the limits exist.

In the following, we will provide the rates of convergence in (1.4) under the second-order regular variation condition (1.6). The rest of the paper is organized as follows. In Section 2, we present the auxiliary lemmas. Theorem and its proof are given in Section 3.

2. Lemmas

Before presenting the main conclusion, we first provide the following lemmas. Recall that a measureable function *f* on R^{+} is said to be generalized regular varying with prameter $\gamma \in R$ and auxiliary *a*, denote $f \in GR(\gamma, a)$, if

$$\lim_{n\to\infty} \frac{f(tx)-f(t)}{a(t)} = \frac{x^{\gamma}-1}{\gamma}, x \in R^{+}. \quad (2.1)$$

Define

$$a^{*}(t) = \begin{cases} \gamma f(t), \gamma > 0; \\ -\gamma\left(f(\infty)-f(t)\right), \gamma < 0; \\ f(t) - \frac{1}{t}\int_0^t f(u)\mathrm{d}u, \gamma = 0 \end{cases}$$

and

$$p_n(x) = \frac{f(nx)}{f(n)} - x^{\gamma}.$$

Lemma 2.1 (cf. de Haan (1970) [15]). *if* $f \in RV(\gamma)$, *for any* $\xi, \delta \geq 0$, *there exists a* $t_0 = t_0(\xi, \delta) \geq 0$ *such that*

$$(1-\xi)\min\left\{x^{\gamma+\delta}, x^{\gamma-\delta}\right\} < \frac{f(tx)}{f(t)} < (1+\xi)\max\left\{x^{\gamma+\delta}, x^{\gamma-\delta}\right\}, \quad \forall t, \quad tx \geq t_0. \quad (2.2)$$

Lemma 2.2 (Cheng and Jiang (2001) [6], Proposition 1.2). *If* $f \in GR(\gamma, a)$, *then* $a^{*}(t) \sim a(t)$ *as* $n \to \infty$ *and for all* $\xi, \delta > 0$, *there exists* $t_0 = t_0(\xi, \delta) \geq 1$ *such that,*

$$\left|\frac{f(tx)-f(t)}{a^{*}(t)} - \frac{x^{\gamma}-1}{\gamma}\right| \leq \xi \max\left\{x^{\gamma+\delta}, x^{\gamma-\delta}\right\}, \forall t, tx \geq t_0 \quad (2.3)$$

holds for $t, tx \geq t_0$.

Lemma 2.3 (de Haan and Ferreira (2006) [1], Theorem 2.3.9). *If f satisfies the second-order condition* (1.6), *then for all* $\xi, \delta \geq 0$, *there exists* $t_0 = t_0(\xi, \delta) \geq 1$ *such that*

$$\left|\frac{\frac{f(tx)}{f(t)} - x^{\gamma}}{B(t)} - \kappa(x)\right| \leq \xi x^{\gamma+\rho} \max\left\{x^{\delta}, x^{-\delta}\right\} \quad (2.4)$$

holds for $t, tx \geq t_0$, *where*

$$B(t) = \begin{cases} \rho\left(1 - \dfrac{\lim_{s\to\infty} s^{-\gamma} f(s)}{t^{-\gamma} f(t)}\right), & \rho < 0; \\ 1 - \dfrac{\int_0^t s^{-\gamma} f(s)\mathrm{d}s}{t^{1-\gamma} f(t)}, & \rho = 0. \end{cases}$$

Proof.

$$\left|\frac{\frac{f(tx)}{f(t)}-x^{\gamma}}{B(t)}-\kappa(x)\right|=x^{\gamma}\left|\frac{\frac{f(tx)}{f(t)}x^{-\gamma}-1}{B(t)}-\frac{x^{\rho}-1}{\rho}\right|$$

$$=x^{\gamma}\left|\frac{(tx)^{-\gamma}f(tx)-t^{-\gamma}f(t)}{t^{-\gamma}f(t)B(t)}-\frac{x^{\rho}-1}{\rho}\right|.$$

Obviously, $t^{-\gamma}f(t)\in GR\left(\rho,t^{-\gamma}f(t)B(t)\right)$, then by Lemma 2.2 the lemma is complete.

Lemma 2.4. *Suppose f satisfies the second-order regular variation condition* (1.6), *then*

$$\lim_{n\to\infty}\sup_{x\in[\alpha_n,\beta_n]}x^{-(1+\gamma)}G_0(\log(x))\left(B^{-1}(n)p_n(x)-\kappa(x)\right)=0, \tag{2.5}$$

$$\lim_{n\to\infty}\sup_{x\in[\alpha_n,\beta_n]}B^{-1}(n)x^{-2\gamma}p_n^2(x)=0, \tag{2.6}$$

where $\alpha_n=-1/\log B^2(n)$, $\beta_n=1/B^2(n)$.

Proof. Note that $B(t)\in RV_{\rho}$ and $B^2(t)\in RV_{2\rho}$ with $\rho\le 0$. From (2.2) in Lemma 2.1 there exists a constant $C_0>0$ and an integer $n_0>0$ such that $B^2(n)\ge C_0n^{2\rho-1}$ for all $n\ge n_0$. Hence $n\alpha_n\ge -n\left[(2\rho-1)\log n+\log C_0\right]^{-1}\to\infty$ as $n\to\infty$. This implies that (2.4) holds for all $x\in[\alpha_n,\beta_n]$. Therefore, for any $\delta\in(0,1)$, we have

$$\begin{aligned}&\sup_{x\in[\alpha_n,\beta_n]}x^{-(1+\gamma)}G_0(\log(x))\left(B^{-1}(n)p_n(x)-\kappa(x)\right)\\&\le\xi\sup_{x\in[\alpha_n,\beta_n]}x^{-(1+\gamma)}\exp\left(-x^{-1}\right)\max\left\{x^{\gamma+\rho+\delta},x^{\gamma+\rho-\delta}\right\}\\&=\xi\sup_{x\in R^+}\exp\left(-x^{-1}\right)\max\left\{x^{\rho-1+\delta},x^{\rho-1-\delta}\right\}<\infty.\end{aligned}$$

Hence (2.5) holds. For (2.6), choosing $\delta\in(0,1/4)$,

$$\begin{aligned}B^{-1}(n)x^{-2\gamma}p_n^2(x)&=x^{-2\gamma}B(n)B^{-2}(n)p_n^2(x)\\&=x^{-2\gamma}B(n)\left(B^{-1}(n)p_n(x)-\kappa(x)+\kappa(x)\right)^2\\&\le 2x^{-2\gamma}B(n)\left(B^{-1}(n)p_n(x)-\kappa(x)\right)^2+2x^{-2\gamma}B(n)\kappa^2(x)\\&\le 2\xi B(n)\sup_{x\in[\alpha_n,\beta_n]}\max\left\{x^{2\rho+2\delta},x^{2\rho-2\delta}\right\}+2x^{-2\gamma}B(n)\kappa^2(x)\\&=Q_{n1}+Q_{n2}.\end{aligned}$$

The first part $Q_{n1}\le 2\xi B(n)\max\left\{\left(-1/\log B^2(n)\right)^{2\rho-2\delta},\left(B(n)\right)^{-4\rho-4\delta}\right\}\to 0$ as $n\to\infty$. Similarly, the second part $Q_{n2}=\sup_{x\in[\alpha_n,\beta_n]}2B(n)\left(\left(x^{\rho}-1\right)/\rho\right)^2\to 0$ as $n\to\infty$. The lemma is proved.

Lemma 2.5. *If f satisfies the second-order regular variation condition* (1.6), *then*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty]}\left|B^{-1}(n)J_n(x)-J(x)\right|=0.$$

Proof. We only prove this lemma for the case that $B(n)$ is positive, the proof of another case is similar.

$$J_n(x) = G_0\left(\frac{1}{\gamma}\log\frac{f(nx)}{f(n)}\right) - G_0(\log x)$$
$$= q_n(x) G_0'\left(\log x + \hat{\theta}_{1n} q_n(x)\right),$$

where $q_n(x) = \gamma^{-1}\log f(nx)/f(n) - \log x$ and $\hat{\theta}_{1n} \in (0,1)$. Note that for some $\hat{\theta}_{2n} \in (0,1)$,

$$q_n(x) = \gamma^{-1} x^{-\gamma} p_n(x)\left(1 + \hat{\theta}_{2n} x^{-\gamma} p_n(x)\right)^{-1}.$$

Therefore,

$$\begin{aligned}
\left|B^{-1}(n) J_n(x) - J(x)\right| &= \left|B^{-1}(n) q_n(x) G_0'\left(\log x + \hat{\theta}_{1n} q_n(x)\right) - J(x)\right| \\
&\le B^{-1}(n)\left|q_n(x) - \gamma^{-1} x^{-\gamma} p_n(x)\right| G_0'\left(\log x + \hat{\theta}_{1n} q_n(x)\right) \\
&\quad + B^{-1}(n)\gamma^{-1} x^{-\gamma} p_n(x)\left|G_0'\left(\log x + \hat{\theta}_{1n} q_n(x)\right) - G_0'(\log x)\right| \\
&\quad + \left|B^{-1}(n)\gamma^{-1} x^{-\gamma} p_n(x) G_0'(\log x) - J(x)\right| \\
&= E_{n1} + E_{n2} + E_{n3}.
\end{aligned}$$

By Lemma 2.4, we know that

$$\begin{aligned}
\sup_{x\in[\alpha_n,\beta_n]} E_{n3}(x) &= \sup_{x\in[\alpha_n,\beta_n]} \gamma^{-1} x^{-\gamma} G_0'(\log x)\left(B^{-1}(n) p_n(x) - \kappa\right)(x) \\
&= \sup_{x\in[\alpha_n,\beta_n]} \gamma^{-1} x^{-(\gamma+1)} G_0(\log x)\left(B^{-1}(n) p_n(x) - \kappa\right)(x) \\
&\to 0.
\end{aligned}$$

Letting $M = \max\left\{\sup_{x\in R} G_0'(\log x), \sup_{x\in R} G_0''(\log x)\right\}$, we have from (2.6) that

$$\begin{aligned}
\sup_{x\in[\alpha_n,\beta_n]} E_{n1}(x) &\le \sup_{x\in[\alpha_n,\beta_n]}\left|M\gamma^{-1} B^{-1}(n) x^{-\gamma} p_n(x)\left[\left(1 + \hat{\theta}_{2n} x^{-\gamma} p_n(x)\right)^{-1} - 1\right]\right| \\
&\le \sup_{x\in[\alpha_n,\beta_n]}\left|M\gamma^{-1} B^{-1}(n) x^{-2\gamma} p_n^2(x)\left(1 + \hat{\theta}_{2n} x^{-\gamma} p_n(x)\right)^{-1}\right| \\
&\to 0.
\end{aligned}$$

For the second part $E_{n2}(x)$, by mean value theorem and also from (2.6),

$$\begin{aligned}
\sup_{x\in[\alpha_n,\beta_n]} E_{n2}(x) &= \sup_{x\in[\alpha_n,\beta_n]}\left|\gamma^{-1} B^{-1}(n) x^{-\gamma} p_n(x) G_0''\left(\log x + \hat{\theta}_{3n}\hat{\theta}_{1n} q_n(x)\right)\hat{\theta}_{1n} q_n(x)\right| \\
&\le \sup_{x\in[\alpha_n,\beta_n]} M\gamma^{-2} B^{-1}(n) x^{-2\gamma} p_n^2(x)\left(1 + \hat{\theta}_{2n} x^{-\gamma} p_n(x)\right)^{-1} \\
&\to 0.
\end{aligned}$$

So we obtain

$$\lim_{n\to\infty}\sup_{x\in[\alpha_n,\beta_n]}\left|B^{-1}(n) J_n(x) - J(x)\right| = 0. \tag{2.7}$$

It remains to deal with the parts of the integral near $\pm\infty$. For $x \ge \beta_n$,

$$\begin{aligned}
&\sup_{x\in[\beta_n,\infty]}\left|B^{-1}(n) J_n(x)\right| \\
&\le B^{-1}(n)\sup_{x\in[\beta_n,\infty]}\left(\left|1 - G_0\left(\gamma^{-1}\log f(nx)/f(n)\right)\right| + \left|1 - G_0(\log x)\right|\right) \\
&\le B^{-1}(n)\left|1 - G_0\left(\gamma^{-1}\log f(n\beta_n)/f(n)\right)\right| + \left|1 - G_0(\log\beta_n)\right| \\
&\le B^{-1}(n)\left|G_0\left(\gamma^{-1}\log f(n\beta_n)/f(n)\right) - G_0(\log\beta_n)\right| + 2B^{-1}(n)\left|1 - G_0(\log\beta_n)\right|.
\end{aligned}$$

Noting $J(\infty)=0$, the first part goes to zero by (2.7). The second part goes to zero because of $1-G_0(\log\beta_n)\sim B^2(n)$ and $B(n)\to 0$ as $n\to\infty$. So we have $\lim_{n\to\infty}\sup_{x\in[\beta_n,\infty]}\left|B^{-1}(n)J_n(x)-J(x)\right|=0$. Similarly, for $x\le\alpha_n$,

$$\begin{aligned}&\sup_{x\in[0,\alpha_n]}\left|B^{-1}(n)J_n(x)\right|\\&\le B^{-1}(n)\sup_{x\in[0,\alpha_n]}\left(\left|G_0\left(\gamma^{-1}\log f(nx)/f(n)\right)\right|+\left|G_0(\log x)\right|\right)\\&\le B^{-1}(n)\left|G_0\left(\gamma^{-1}\log f(n\alpha_n)/f(n)\right)-G_0(\log\alpha_n)\right|+2B^{-1}(n)\left|G_0(\log\alpha_n)\right|\\&\to 0.\end{aligned}$$

Combing $J(0)=0$ we have $\lim_{n\to\infty}\sup_{x\in[0,\alpha_n]}\left|B^{-1}(n)J_n(x)-J(x)\right|=0$. The proof of the lemma is completed.

3. Theorem and Its Proof

Theorem 3.1. *If f satisfies the second-order regular variation condition* (1.6), *then*

$$\lim_{n\to\infty}\frac{F^n(a_nx)-\Phi_{1/\gamma}(x)}{B(n)}=-J\left(x^{1/\gamma}\right)\tag{3.1}$$

holds uniformly on $x>0$.

Proof. Let $z_n(x)=\left[-n\log F(a_nx)\right]^{-1}$, equivalently, $F^n(a_nx)=G_0(\log z_n(x))$. So we have

$$\begin{aligned}K_n(x)&:=B^{-1}(n)\left(F^n(a_nx)-\Phi_{1/\gamma}(x)\right)+J\left(x^{1/\gamma}\right)\\&=B^{-1}(n)\left(G_0(\log z_n(x)-G_0(1/\gamma\log x))+J(z_n(x))+J\left(x^{1/\gamma}\right)-J(z_n(x))\right)\\&=K_{n1}(x)+K_{n2}(x).\end{aligned}$$

In order to show (3.1), we need only to prove

$$\lim_{n\to\infty}\sup_{0<F(a_nx)<1}\left|K_{ni}(x)\right|=0\quad\text{for } i=1,2;\tag{3.2}$$

$$\lim_{n\to\infty}\sup_{F(a_nx)=0}\left|K_n(x)\right|=0;\tag{3.3}$$

$$\lim_{n\to\infty}\sup_{F(a_nx)=1}\left|K_n(x)\right|=0.\tag{3.4}$$

If $0<F(a_nx)<1$, according to the definition of the *f*, for any $\delta>0$ we have

$$\frac{f(nz_n(x))}{a_n}\le x\le\frac{f(nz_n(x)+\delta)}{a_n}.$$

Hence

$$\begin{aligned}&B^{-1}(n)\left(G_0(\log z_n(x))-G_0\left(1/\gamma\log\left(f(nz_n(x)+\delta)/a_n\right)\right)+J(z_n(x))\right)\\&\le K_{n1}(x)\le B^{-1}(n)\left(G_0(\log z_n(x))-G_0\left(1/\gamma\log\left(f(nz_n(x))/a_n\right)\right)+J(z_n(x))\right).\end{aligned}$$

Then by Lemma 2.5, we can obtain $\lim_{n\to\infty}\sup_{0<F(a_nx)<1}\left|K_{n1}(x)\right|=0$. It is obvious that $z_n(x)\to x^{1/\gamma}$. Since $J(x)$ is continuous on $(0,\infty)$ and $J(0)=J(\infty)=0$, we can also obtain $\lim_{n\to\infty}\sup_{0<F(a_nx)<1}\left|K_{n2}(x)\right|=0$.

For the situation of $F(a_n x)=0$, we have $x \le f(0)/a_n$. The left of (3.1) is

$$\begin{aligned}\lim_{n\to\infty}\sup_{F(a_nx)=0} B^{-1}(n)\Phi_{1/\gamma}(x) &\le \lim_{n\to\infty}\sup_{F(a_nx)=0} B^{-1}(n)\Phi_{1/\gamma}\left(a_n^{-1}f(0)\right)\\ &=\lim_{n\to\infty}\sup_{F(a_nx)=0}\left(B^{-1}(n)J_n(0)-J(0)\right)\\ &\to 0.\end{aligned}$$

Note that $\lim_{n\to\infty} f(0)/a_n \to 0$. For any δ, there exists n_0 such that $x \le f(0)/a_n \le \delta$ for all $n \ge n_0$. Therefore,

$$\lim_{n\to\infty}\sup_{F(a_nx)=0}\left|J\left(x^{1/\gamma}\right)\right| \le \lim_{n\to\infty}\sup_{x\le\delta}\left|J\left(x^{1/\gamma}\right)\right|=0$$

by letting $\delta\to 0$. Then,

$$\begin{aligned}&\lim_{n\to\infty}\sup_{F(a_nx)=0}\left|B^{-1}(n)\left(F^n(a_nx)-\Phi_{1/\gamma}(x)\right)+J\left(x^{1/\gamma}\right)\right|\\ &\le \lim_{n\to\infty}\sup_{F(a_nx)=0} B^{-1}(n)\Phi_{1/\gamma}(x)+\lim_{n\to\infty}\sup_{F(a_nx)=0} J\left(x^{1/\gamma}\right)\\ &\to 0.\end{aligned}$$

If $F(a_nx)=1$, then $x \ge f(\infty)/a_n$. The left of (3.1) is

$$\begin{aligned}\lim_{n\to\infty}\sup_{F(a_nx)=1}\left|B^{-1}(n)\left(1-\Phi_{1/\gamma}(x)\right)\right| &\le \lim_{n\to\infty}\sup_{F(a_nx)=1}\left|B^{-1}(n)\left(1-\Phi_{1/\gamma}\left(f(\infty)/a_n\right)\right)\right|\\ &\le \lim_{n\to\infty}\sup_{F(a_nx)=1}\left|B^{-1}(n)\left(G_0(\log\infty)-\Phi_{1/\gamma}\left(f(\infty)/a_n\right)\right)\right|\\ &=\lim_{n\to\infty}\sup_{F(a_nx)=1}\left|B^{-1}(n)J_n(\infty)-J(\infty)\right|\to 0.\end{aligned}$$

For sufficiently large number M_0, there exists n_0 such that $x \ge f(\infty)/a_n \ge M_0$ for all $n \ge n_0$. Hence

$$\lim_{n\to\infty}\sup_{F(a_nx)=1}\left|J\left(x^{1/\gamma}\right)\right| \le \lim_{n\to\infty}\sup_{x\ge M_0}\left|J\left(x^{1/\gamma}\right)\right|=0$$

by letting $M_0\to\infty$. Furthermore,

$$\lim_{n\to\infty}\sup_{F(a_nx)=1}\left|B^{-1}(n)\left(F^n(a_nx)-\Phi_{1/\gamma}(x)\right)+J\left(x^{1/\gamma}\right)\right|\to 0.$$

So we prove this theorem.

Remark 3.1. Uniform limit in Theorem 3.1 gives an Edgeworth expansion as follows:

$$\begin{aligned}F^n(a_nx)=\Phi_{1/\gamma}(x)+\frac{1}{\gamma}B(n)\left(-\log\Phi_{1/\gamma}(x)\right)^{1+\gamma}\Phi_{1/\gamma}(x)\kappa\left(-\log^{-1}\Phi_{1/\gamma}(x)\right)\\ +o\left(B(n)\right)\end{aligned}$$

holds uniformly on $x>0$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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