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Forecasting Crude Oil Price Volatility by Heston Model

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Abstract

Given the significance and complexity of forecasting the crude oil price volatility, this paper introduces the Heston model to predict volatility dynamics of crude oil price. The high-frequency intra-day data of the West Texas Intermediate (WTI) market serves to model the problem. Furthermore, the study used the Euler-Maruyama scheme to simulate Heston model using an error analysis. On the other hand, the study used the mean square error (MSE), the mean average error (MAE), and the root means square error (RMSE) for the forecast accuracy of the GARCH-type models. The results of the error analysis indicated that Heston's stochastic volatility model is more consistent with oil performance data than traditional GARCH-class models. The study's findings demonstrated that the Heston model is more economical in terms of setup and is capable of handling stylized facts.

Keywords

Stochastic Volatility, Euler-Maruyama Scheme, Heston Model, Crude Oil Price Volatility

1. Introduction

Globally, crude oil prices, gold prices, currency exchange rates and other macroeconomic indicators play a fundamental role in the global economy. To clarify, the crude oil price is always treated more seriously since oil is the blood in the vessels of the economy. Forecasting crude oil price volatility provides market participants with valuable information regarding market uncertainty. Overall, oil price variability implies enormous losses or gains and therefore lower incomes or higher reserves to achieve the development targets [1]. Therefore, forecasting the price of crude oil and its volatility has attracted a lot of attention from researchers, policy makers and investors.

Moreover, instabilities in the price system affect governments in many sectors, such as, planning and decision-making regarding the revenue system. Oil price uncertainties are also linked to inflation and have implications for the cost of consumer products and goods in industry. As a result, it may have a critical effect on the policy order of petroleum-dependent economies. Furthermore, the association between the price of oil and agricultural commodity prices is important for the world food policy, and the sensitivity of the different economic agents on which the main input in the production process is oil [2] [3] [4].

Numerous studies consider the modeling and forecasting the volatility of crude oil price by using improved GARCH models. Although various work has been done to find the most suitable model that offers the best performance of out-of-sample forecasts, none of the models has always outperformed the other [2]. For example, the Autoregressive Conditional Heteroscedasticity (ARCH) model, considered a powerful tool to describe conditional, and historical volatilities, was introduced by Engle [5] [6] [7]. This was followed by the introduction of the generalized autoregressive conditional heteroskedasticity (GARCH) model as an extension to the ARCH models [8]. Moreover, different scholars have developed variations of GARCH models, for example integrated GARCH (IGARCH), exponential GARCH (EGARCH), asymmetrical power GARCH (APGARCH), and fractional integrated GARCH (FIGARCH) [5] [6] [9]. These extension models were enhancing the GARCH model to capture the features of the time series data. Therefore, from an investor's perspective, the forecast nature of future returns and volatility in oil markets is critical in determining asset prices, hedging, pricing of derivatives, and risk control. Given its vital global significance and the minimization of the negative effect fluctuations of oil price, the academic literature over the past few years has focused on modeling and forecasting oil price volatility. The market for crude oil sometimes seems at times relatively quiet, while other highly volatile markets [2].

The upgraded GARCH-type models are capable of capturing the most important stylized facts [4] [5] [10]. For more information on stylised facts, please refer to the following studies [1] [2] [6] [7] [11] [12]. These models are popular for the modeling of time-variable conditional volatility as a deterministic function of lagged variance, lagged conditional squared residuals, and the past observations with the future volatility [2] [13]. In addition, these models are parametric and usually assess daily, weekly or monthly volatility using sampled data at the same rate. However, these models may not capture the fat-tail property of financial data.

Currently, stochastic volatility (SV) models are alternative existing models to accurately capture stylized facts [7]. The SV models are non-parametric and are based on the time-continuous probability process. In addition, the variance in the SV models is treated as an unobserved quantity that admits a stochastic, and logarithmic first-order autoregressive processes. In general, these models extend the geometric Brownian motion model by incorporating an additional source of uncertainty. Therefore, in describing volatility behavior, the volatility approximation is more realistic [3]. Although the (SV) model is theoretically appealing, it is difficult, because the unobserved volatility process introduces the model into a non-linear model. This process leads to the likelihood function depending on the high-dimension integrals [7] [9].

The study's new contribution to the literature is the introduction of the Heston stochastic volatility model to forecast the price volatility of crude oil. The model develops the Black and Scholes model (BSM) by incorporating the process driven by a Cox-Ingersoll-Ross (CIR) process [14] [15]. Furthermore, the model can capture the volatility clustering, the leverage effect, and the heavy-tailed nature of the return distributions. In this model, while the volatility of the variance controls the kurtosis of the underlying asset return distribution, the correlation defines its asymmetry. To conclude, the model is analytical tractable [14]. Despite the advantages of the model, the fundamental challenge is the complexity of the approximation process. Compared to models like BSM, the implementation of the Heston model involves more sophisticated calculations, and a more difficult process for the model calibration. Since the closed-form solutions for non linear SDEs are rarely available or are too hard to obtain, numerical approximations are very important. To overcome this challenge, this study employs the Euler-Maruyama scheme to approximate the price volatility model.

To sum up, the main objective of this study is to introduce the Heston stochastic volatility model in forecasting price volatility of a crude oil. Moreover, the innovations of this study are; introduction of the Heston model to measure the price volatility of crude oil, and applying the Euler-Maruyama method to simulate the Heston model. Moreover, in the accuracy analysis of the model, the study compares the Heston model performance with the improved GARCH-type models by performing error analysis. Normally, the smallest error implies a good approximation measure. For improved GARCH-type models, three error measures to evaluate the forecasting accuracy of models were employed. These are the mean squared error (MSE), the root means squared error (RMSE), and the mean absolute error (MAE).

The remaining sections of this paper are organized as follows. In Section 2, the mathematical model is presented. Moreover, Section 3 presents the Data and preliminary analysis; a numerical example is in Section 4. Finally, Section 5 concludes the study.

1.1. Mathematical Model

In this study, the crude oil market is assumed to be complete, frictionless, and continuously open over a fixed time interval [0, T]. Moreover, the financial market uncertainties are defined and modeled using a complete filtered probability space $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$. In the probability space above, Ω is the sample space, $\mathbf{F} = \mathcal{F}(t)$, $t \geq 0$, is the information available at a time *t*, and $\mathbf{P}$ is the historical probability measure.

1.2. The Heston Model

In this study, we consider that crude oil price dynamics assume a stochastic volatility model, particularly the Heston model. The introduction of a stochastic volatility model is because the volatility is not constant or deterministic instead it is a random process. Hence, the asset price at time t follows the system of stochastic differential equations (SDE) given:

$$\mathrm{d}Y(t) = Y(t)\left(\mu \mathrm{d}t + \sqrt{V(t)}\mathrm{d}W_1(t)\right), \tag{1}$$

$$\mathrm{d}V(t) = \beta\left(\theta - V(t)\right)\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W_2(t), \tag{2}$$

$$Y(0) = Y_0 \text{ and } V(0) = V_0. \tag{3}$$

In addition, β is the mean reversion rate, θ is the long-run variance, and the volatility of volatility (variance of volatility) is denoted by σ. Thus, the Heston volatility model is the system of two correlated SDEs namely the price of asset and the volatility process, see (1) and (2). In addition, to ensure that $V(t)$ is almost surely non-negative (the Feller condition has to be net), that is, $2\beta\theta > \sigma^2$. Moreover, $W_1(t)$ and $W_2(t)$ are standard Brownian motions with a nonzero correlation.

1.3. The Euler-Maruyama Method

The simplest and most useful scheme for the approximation of the numerical solution of the stochastic differential equations is the Euler-Maruyama. This is a generalization of the Euler's approach method in ordinary differential equations and stochastic differential equations [16]. Consider the SDE below:

$$\mathrm{d}Y(t) = \mu\left(t, Y(t)\right)\mathrm{d}t + \sigma\left(t, Y(t)\right)\mathrm{d}W(t), \quad Y(0) = Y_0, \tag{4}$$

where σ and μ are scalar functions, $W(t)$ is the Wiener process, and $Y(0)$ is the initial condition. The solution to (4) admits the process Y_t:

$$Y_t = Y_0 + \int_0^t \mu(s, Y_s)\mathrm{d}s + \int_0^t \sigma(s, Y_s)\mathrm{d}W_s. \tag{5}$$

To approximate the solution of the SDE on the interval $[0, T]$, time is discretized into N equal subintervals with width $\Delta t = \frac{T}{N}$. Therefore, the Euler-Maruyama approximation to the actual solution of (4) is the Markov chain X expressed in the form:

$$Y_{n+1} = Y_n + \mu(t_n, Y_n)\Delta t + \sigma(t_n, Y_n)\Delta W_n, \tag{6}$$

with Y_{n+1} standing for an approximation, $\Delta t = t_{n+1} - t_n = \frac{T}{N}$, $\Delta W = W_{n+1} - W_n$, and $n = 0, 1, 2, \cdots, N$. From the expression, ΔW is normally distributed, *i.e.*, $\Delta W \sim N(0, \Delta t)$.

Proposition 1.1 *Assume that* Y_{n+1}*, and* V_{n+1} *are discrete approximate of* Y *and* V *respectively. The corresponding continuous Euler-Maruyama method is then used to approximate the solution to the Heston model is given by*:

$$Y_{n+1} = Y_n\left(1 + \mu\Delta t + \sqrt{V_n}\Delta W_n\right),$$

$$V_{n+1} = V_n + \beta\left(\theta - V_n\right)\Delta t + \sigma\sqrt{V_n}\Delta W_n.$$

where Y_{n+1} *is the approximate price, and* V_{n+1} *is the approximate volatility.*

1.4. Numerical Approximation of the Model

Simulation methods are critical for approximating numerical solutions of SDEs. For the model efficiency, the focus is on analyzing errors measured at $t = T$ by quantity.

$$\varepsilon_{\Delta t} = \mathbf{E}\left|Y(T) - \bar{Y}(T)\right|.$$

This process determines whether the method is converging to the exact solution. The Euler-Maruyama scheme exhibits both strongly and weakly convergent properties [17] [18] [19]. The strong (convergent) error measures the error of the approximate Y sampling paths on average, the pathwise error is the random quantity that satisfies:

$$\lim_{\Delta t \to 0} \mathbf{E}\left(\left|Y(T) - \bar{Y}(T)\right|\right) = 0. \tag{7}$$

From above expression, $\mathbf{E}$ is the expected value, while $\bar{Y}(T)$ is the approximation of $Y(t)$ at time $t = T$ calculated with a constant step Δt.

Conversely, it is weakly convergent (weak error) if the random quantity satisfies:

$$\lim_{\Delta t \to 0} \left|\mathbf{E}\left[f\left(Y(T)\right)\right] - \mathbf{E}\left(f\left(\bar{Y}(T)\right)\right)\right| = 0, \tag{8}$$

for all f polynomials.

1.5. Approximation of GARCH-Type Models

In the present study, the symmetric GARCH and asymmetric GARCH (EGARCH and TGARCH) models compare with the Heston model. These models are discussed extensively in the few studies mentioned here [2] [7] [9] [12] [20] [21]. To assess the forecast performance of the models above, the Mean Square Error (MSE), Root Mean Square Error (RMSE), and Mean Absolute Error (MAE) are used. These error measures are the most well-known for testing model accuracy [2]. The model with the smallest error is referred to be the best one.

1.5.1. The Means Squared Error (MSE)

The MSE is regularly used to compare forecasting performance of models. It is considered the most suitable measure for determining methods that avoid significant errors. Thus, the MSE is computed by averaging the square of the difference between the original and predicted values of the data.

$$\text{RMSE} = \sum_{t}^{n} \frac{e_t^2}{n} \tag{9}$$

where, $e = Y_T - \overline{Y}$.

1.5.2. The Root Means Squared Error (RMSE)

RMSE is the standard deviation for errors that occur when a forecast is performed. This is similar to mean square error (MSE), but the root of the value is taken into consideration when determining the accuracy of the model. It is expressed by:

$$\text{RMSE} = \sqrt{\sum_{t}^{n} \frac{e_t^2}{n}} \tag{10}$$

1.5.3. The Mean Absolute Error (MAE)

It is expressed as:

$$\text{MAE} = \sum_{t=1}^{n} \frac{\left|\frac{e_t}{Y_t}\right|}{n} \tag{11}$$

where $\left|\frac{e_t}{Y_t}\right|$ is the absolute error calculated on the values adjusted for a specific forecasting method.

2. Data Description and Preliminary Analysis

To carry out the numerical illustration, this study employs daily spot prices of West Texas Intermediate (WTI) observations. The data is taken from the U.S. Energy Information Administration (EIA). In brief, the WTI crude oil is a specific grade of crude oil and one of the main three benchmarks in oil pricing, along with Brent and Dubai Crude.

The sampling period covers the period from January 4, 2009, to December 31, 2019, and consists 2761 observations. Since the logarithmic returns are analytically tractable when sub-period returns are related together to form returns over long intervals, then the crude oil price of each series is transformed into logarithmic returns. Study focuses on the logarithmic behavior of the crude oil returns. Furthermore, simulations using the Heston model and the application of the Euler-Maruyama numerical method to approximate the model are performed.

2.1. Preliminary Analysis

The preliminary analysis reveals significant differences in price movements during the period under consideration. Similarly, there are signs of positive asymmetry in the statistical distribution of crude oil prices, implying extreme of the right tail. Also, the returns are non-Normal, which proved significantly asymmetrical positive and excess kurtosis, as is expected from daily returns.

For the kurtosis, the price of crude oil is leptokurtic, which implies the fat tails than the normal distribution, see **Table 1**. Particularly, **Table 1** presents the following; descriptive statistics, the unit root test, the normality test, and the ARCH effect test for crude oil. The results indicate that the mean returns are measured by standard deviation. In addition, returns show leptokurtic features (*i.e.*, fat-tailed) and therefore, the variance of oil prices for crude primarily reflects irregular but extreme differences. Similarly, returns are skewed positively, implying that the series has a longer right tail than the left tail.

Moreover, the unit root test, and the Augmented Dickey-Fuller (ADF) tests examine the presence of a unit root [22] [23]. The method tests the hypotheses H0: A series is non-static versus H1: A series is stationary. The results of unit root test for the data, altogether at levels 0.01 and 0.05 reject the null hypothesis that there is a unit root in the returns.

Furthermore, the ARCH effect test looks at the existence of a heteroscedasticity of crude oil data. The ARCH effect test reveals the presence of strong conditional heteroscedasticity for the crude oil prices, which is common in financial data. Therefore, there is sufficient evidence to reject the null hypothesis that there is no ARCH effect. Likewise, the Jarque-Bera (JB) statistic using kurtosis and skewness test information illustrates non-normality. Therefore, it leads to the rejection of the null hypothesis.

2.2. Volatility and Approximation of Model Parameters

Considering Y_t as the price of crude oil on day *t*, then daily spot price return transformation is:

$$S_t = \log\left(\frac{Y_t}{Y_{t-1}}\right), \tag{12}$$

where S_t is the daily return on crude oil, Y_t is the current day crude oil price, and Y_{t-1} is the crude oil price of the previous day. Also, the daily square returns are taken as an approximation of the real volatilities. Using the transformation results the annual and daily volatilities are determined with their respective variances as presented in **Table 2**. Therefore, we find the daily and annual volatility of crude oil prices (σ) with the respective variances (σ^2) for each year by using the Parkinson's extreme values method [24].

3. Numerical Illustrations

The daily observations of crude oil return for the period from 02.01.2019 to 31.12.2019 amount to 250 days are used in this study. From these observations, we compute the model parameters (see **Table 3**).

Figures 1-3 show the spot prices, returns, and volatility of crude oil respectively. The results reveal an asymmetrical pattern in crude oil price behavior and returns. It implies that the performance patterns suggest evidence of volatility clustering. Thus, periods of relatively low volatility are the response to the periods of high volatility. The unusual peaks in these figures are evidence of major unstable trends in crude oil price returns.

Table 1. Basic statistics for the crude oil data.

Parameter	Value
A: Descriptive statistics	
Maximum	113
Minimum	26.19
Mean	71.5394 (3.87E−05)
Stand.dev.	21.51654 (0.0098)
Skewness	0.1319
Kurtosis	4.0524
B: Normality test	
Jarque-Bera	2506.982***
C: Unit root test	
ADF	−74.32***
PP	−73.54***
D: ARCH effects	
LM(10)	35.27

Note: ***indicate statistical significance at 0.05 level.

Table 2. Crude oil volatilities, WTI 2009-2019.

Time interval	Number of Trading days	Minimum Price	Maximum price	Annual Volatility	Annual Variance	Daily Volatility	Daily Variance
2.1.2009-31.12.2009	251	34.03	81.02	0.2320	0.054	0.00021	4.573E−08
4.1.2010-31.12.2010	252	64.78	91.47	0.1280	0.016	0.00006	4.228E−09
3.1.2011-29.12.2011	252	75.40	113.40	0.0226	0.001	0.00009	8.076E−09
3.1.2012-31.12.2012	252	75.40	109.39	0.1316	0.017	0.00069	4.717E−09
2.1.2013-27.12.2013	252	86.55	110.61	0.0792	0.006	0.00025	6.184E−10
3.1.2014-31.12.2014	249	53.45	107.96	0.1145	0.132	0.00053	2.773E−09
2.1.2015-30.12.2015	252	34.55	61.35	0.2030	0.041	0.00016	2.674E−08
4.1.2016-30.12.2016	252	26.19	54.01	0.2111	0.045	0.00018	3.126E−08
3.1.2017-29.12.2017	250	42.48	60.45	0.1071	0.012	0.00005	2.106E−09
2.1.2018-31.12.2018	249	44.48	77.41	0.1367	0.019	0.00008	5.638E−09
4.1.2019-09.12.2019	250	46.31	66.23	0.1483	0.022	0.00009	8.757E−09

Table 3. Computed parameters.

Parameter	Symbol	Value
Initial price	Y_0	46.31
Initial volatility	V_0	2.3×10^{-4}
Vol-volatility	σ	9.0×10^{-5}
long-run variance	θ	8.8×10^{-9}
Reversion rate	β	2.95×10^{-3}
Mean log-return	μ	4.94×10^{-4}

Particularly, Figure 1 shows the change in the spot price for the crude oil. Precisely, it shows that over the 2014-2016, crude oil prices declined sharply, reflecting increased uncertainty in the crude oil market. Moreover, Figure 2 illustrates the dynamics of crude oil returns. It implies that a number of global supply and demand tremors cause significant fluctuations in crude oil markets. Generally speaking, crude oil prices tend to reflect a high degree of uncertainty over time.

3.1. Simulation Results

This section starts by focusing on the logarithmic behavior of crude oil prices and conducting simulations using Heston's stochastic volatility model. First of all, the January 2009 to January 2019 observations were used for parameters computation. Moreover, to forecast, the data from 02.01.2019 to 31.12.2019 were used. That is, for the parameters estimation the interval $t=1$ to N was used while for forecasting $N+1$ was used. This study uses Euler-Maruyama's numerical method to simulate the Heston model.

In addition, the following parameters were also used in the simulations: the initial time "$t=0$", the terminal time "$T=1$", the number of discretization between "$n=100$", $\Delta t=0.01$ (the uniform mesh size), the number of paths) "$N=1000$". Figure 3 displays the results of the crude oil volatility simulation. The simulated results suggest that crude oil prices tend to be asymmetrical with lower kurtosis.

3.2. Error Analysis

Normally, for the efficacy of the model, the emphasis is on error analysis. We start by defining the error by

$$\mathbf{E}\left|Y(T)-\bar{Y}(T)\right|.$$

To test the convergence strength for the Euler-Maruyama, we use $N=2^9$ dicretized Brownian paths over $[0,1]$, and five different stepsizes: $\Delta t=2^{p-1}dt$ for $1\le p\le 5$. Results for this process are shown in Table 5. These results are compared with the results obtained from GARCH-type models as shown in Table 5. To conclude, the model with the lowest error implies a more accurate estimate of the model.

Table 4, and Table 5 present error analysis results for the Euler-Maruyama scheme and GARCH-type models respectively. Particularly, Table 4 shows the error analysis results for the Heston model. Likewise, the error analysis for the GARCH-type models is presented in Table 5 as shown. The results in Table 5 show that the GARCH model offers the best accurate forecast than the two improved GARCH-type models (EGARCH and TGARCH). In general, results in Table 4, and Table 5, reveal the small errors in Heston model approximation than the improved GARCH models. Basing on results, it is evident that the Heston model can estimate better crude oil price volatility than the counterpart improved GARCH-type models.

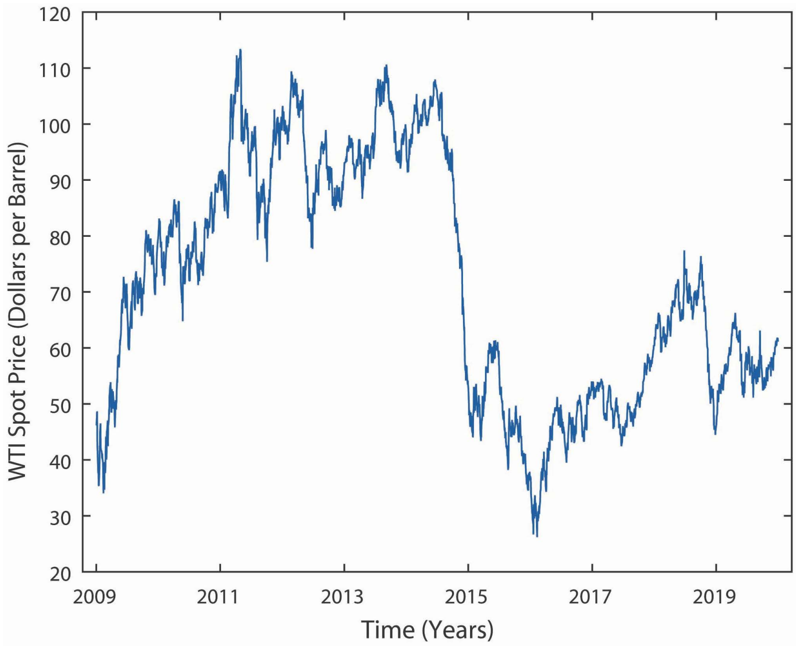

Figure 1. Crude oil Spot price.

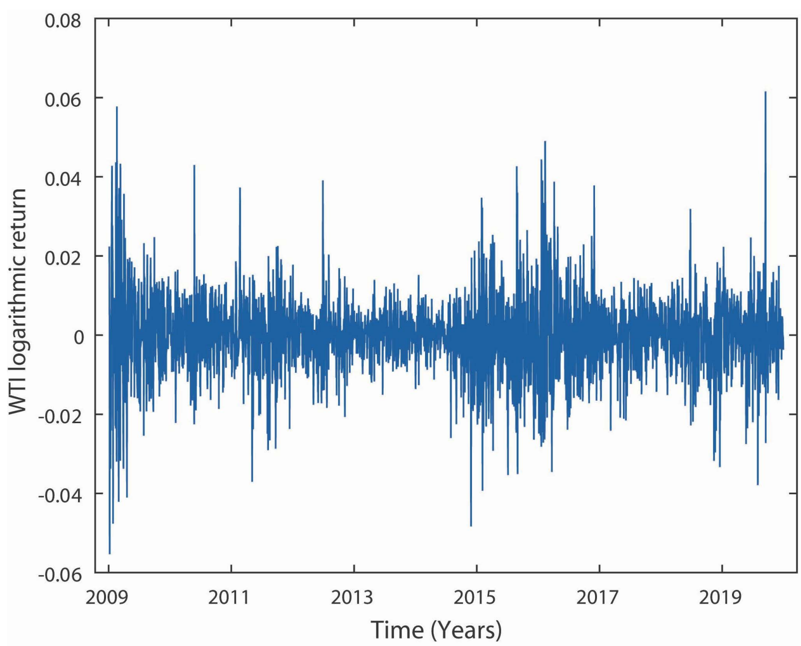

Figure 2. Crude oil logarithmic returns.

4. Conclusion

This study introduces the Heston stochastic volatility model to model the crude oil price volatility. It applied the Euler-Maruyama scheme to approximate the Heston stochastic volatility model. Moreover, the differences in the trend of prices for the covered period are revealed by simulation results. Furthermore, results show that the Heston model fits data well compared to counterpart improved GARCH-type models in the approximation of the price volatility.

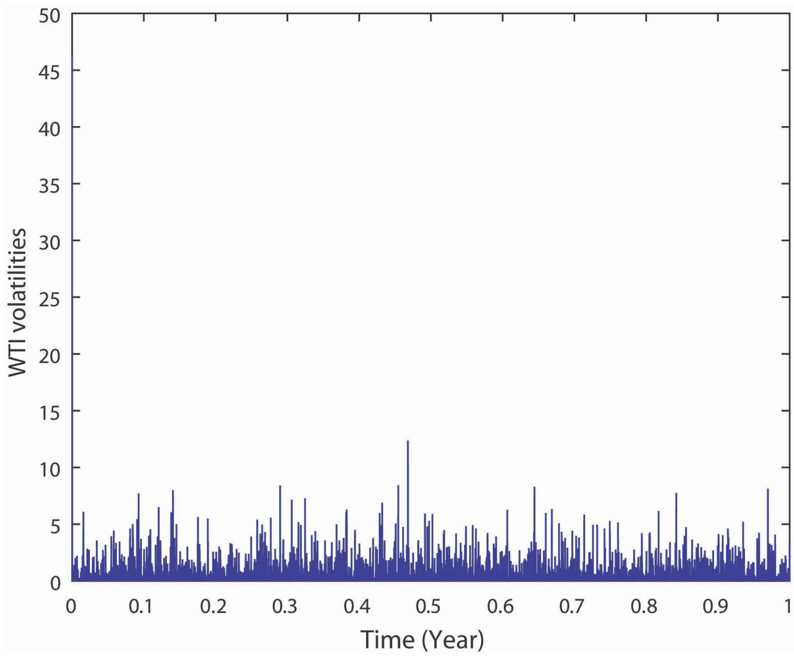

Figure 3. Crude oil volatilities.

Table 4. The error analysis, the Euler-Maruyama scheme.

Step size (Δt)	Error
0.0010	0.000564
0.1010	0.010153
0.2010	0.011524
0.3010	0.010771
0.4010	0.012742
0.5010	0.013142
0.6010	0.012421
0.7010	0.012101
0.8010	0.010621
0.9010	0.011283

Table 5. The Error analysis, the GARCH-type models.

Model	MSE	RMSE	MAE
GARCH	0.01578	0.01681	0.01198
EGARCH	0.01572	0.01695	0.01656
TGARCH	0.01587	0.01957	0.01803

Because of the strategic role of crude oil price volatility and its effects to all countries globally, it is essential to employ different forecasting methods. In the future study, we will work on jump-diffusion models to investigate the behavior of crude oil price volatility.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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