The Application of the Improved Variable Separation Method in the (2+1)-Dimensional Modified Dispersive Water-Wave System

PAN Shuxia[1], SUN Wangjie[2]
1. Department of Basics, Jilin Curative of Institute, Jilin, China
2. Deptment of Science, Jilin Institute of Chemical Technology, Jilin, China
e-mail: pan4652405@163.com, sunwj405@163.com

Abstract: Starting from the backlund transformation and using Cole-Hopf transformation, we reduce the (2+1)-dimensional modified dispersive water-wave system to a simple linear evolution equation with two arbitrary functions of $\{x,t\}$ and $\{y,t\}$ in this paper. And we can obtain some new solutions of the original equations by investigating the simple linear evolution equation which include the solutions obtained by the variable separation approach.

Keywords: bäcklund transformation; cole-Hopf transformation; the (2+1)-dimensional modified dispersive water-wave system; variable separation approach

1. Introduction

Many dynamic problems in physics and other fields are usually characterized by nonlinear evolution partial differential equations which are often called governing equations. To understand the physical mechanism of these problems one has to study the solutions to the associated governing equations. Looking for analytical solutions to nonlinear physical models has long been a major concern for both mathematicians and physicists since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications Much work has been done over the last years on the subject of obtaining special solutions of a nonlinear partial differential equation (PDE). Some of the most important methods are the inverse scattering transformation (IST), bilinear method, symmetry reductions, backlund and Darboux transformations and so on[4, 6, 7, 11, 20, 21]. In comparison with the linear case, it is known that IST is an extension of the Fourier transformation in the nonlinear case. In addition to the fourier transformation, there is another powerful tool called the variable separation method in the linear case. Recently, two kinds of "variable separating" procedure have been established. The first method is called the *formal variable separation approach* (FVSA) [14], or equivalently the symmetry constraints or nonlinearization of the Lax pairs [2,3,8]. The independent variable of a reduced filed in FVSA have not totally been separated though the reduced field satisfies some lower-dimensional equations. The second type of variable separation method has been established for some types of nonlinear models like the DS equation, the NNV equation[12,13,15,18,19,22,23,25], and a non-integrable (2+1)-dimensional Kdv equation[24].For the DS equation, the NNV ansatz, some special types of exact solutions can be obtained from two (1+1)-dimensional variable separated fields.

In this paper, starting from the backlund transformation and using Cole-hopf transformation, we reduce the (2+1)-dimensional modified dispersive water-wave system [1,5] to a simple linear evolution equation with two arbitrary functions of {x,t} and {y,t}. To find out some special solutions of this equation, similar to Lou's variable separation approach, we look for the solutions in the form

$$f = a_0 + a_1 p(x,t) + a_2 q(y,t) + a_3 p(x,t) q(y,t),$$

and

$$f = p(x,y) + q(y,t).$$

By doing so, we can avoid solving bilinear equation or higher multi-linear equation of the original models and obtain some new variable separation solutions.

2. New Solutions

Let us consider the (2+1)-dimensional modified dispersive water-wave system

$$u_{yt} + u_{xxy} - 2v_{xx} - \left(u^2\right)_{xy} = 0, \qquad (1)$$

$$v_t - v_{xx} - 2\left(uv\right)_x = 0, \qquad (2)$$

which was used to model nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow water of uniform depth, and can also be derived from the celebrated Kadomtsev-Petviashvili (KP) equation by the symmetry constraint[16,17]. It is worth while mentioning that the system has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics, etc. So a good understanding of more solutions (1) and (2) is very helpful, especially for coastal and civil engineers to apply the nonlinear water model in a harbor and coastal design. Meanwhile, finding more types of solutions of system (1) and (2) is of fundamental interest in fluid dynamics.

To solve the (2+1)-dimensional modified dispersive water-wave system, we take the following backlund transformations of Eqs. (1) and (2):

$$u = \frac{f_x}{f} + u_0\left(x,t\right), v = \frac{f_{xy}}{f} - \frac{f_x f_y}{f^2}, \qquad (3')$$

which can be obtained from the standard Painleve truncation expansion with u0(x,t), an arbitrary function of {x,t}.

It is easy to deduce from (3') that

$$v = u_y \qquad (3)$$

which leads to a simple transformation from u to v. Substituting (3) into system (1)-(2). We can change system (1)-(2) into a single differential equation:

$$u_{yt} - u_{xxy} - \left(u^2\right)_{xy} = 0 \qquad (4)$$

Substituting the first term of (3') into (4), we have

$$\partial y\left\{\partial x\left(\frac{f_t - f_{xx} - 2u_0 f_x}{f}\right) + u_{0t} - 2u_0 u_{0xx}\right\} = 0, \qquad (5)$$

it corresponds to making a Cole-hopf transformation for $u - u_0$.

It is obvious that (5) is equivalent to

$$f_t - f_{xx} - 2u_0 f_x = \left(h_1\left(x,t\right) + h_2\left(y,t\right)\right) f, \qquad (6)$$

where $h_1(x,t)$ and $h_2(y,t)$ are arbitrary functions of the in dicated variables.

To find out some special solutions of the Eq. (6), similar to Lou's variable separation approach, we look for the solutions in the form

$$f = a_0 + a_1 p + a_2 q + a_3 pq, \qquad (7)$$

where a_0,a_1,a_2 and a_3 are arbitrary constants, p=p(x,t) and q=q(y,t) are arbitrary functions of the indicated variables.

In fact we have

Theorem 1. Suppose a_0, a_1, a_2 and a_3 are arbitrary constants, and p=p(y,t) and q=q(y,t) are arbitrary non-constant functions of the indicated variables that satisfy the following conditions:

(i) $a_0 + a_1 p + a_2 q + a_3 pq \neq 0;$

(ii) $q = q\left(y,t\right) satisfies$

$q_t = -c\left(t\right) q\left(a_0 + a_2 q\right)$

with $c(t)$ being a arbitrary function. Then

$$u = \frac{f_x}{f} + u_0\left(x,t\right)$$

and

$$v = \frac{f_{xy}}{f} - \frac{f_x f_y}{f^2}$$

are just solutions of Eqs. (1) and (2),where f is given by (7) and

$$u_0 = \frac{p_t - p_{xx} + c\left(t\right) p\left(a_0 + a_1 p\right)}{2p_x}.$$

Proof. According to the discussion,we only need to prove that there exist functions $h_1(x,t)$ and $h_2(y,t)$ such that (6),(7) hold for arbitrary functions p(x,t) and q(y,t) which satisfy the conditions (i) and (ii). In fact,

$p(x,t) \neq -\frac{a_0}{a_1}$

and

$q(x,t) \neq -\frac{a_0}{a_2},$

because p(x,t) and q(y,t) are non-constant functions. We can choose

$$h_1\left(x,t\right) = c\left(t\right)\left(a_0 + a_1 p\right), \qquad (8)$$

$$h_2\left(y,t\right) = -c\left(t\right)\left(a_0 + a_2 q\right), \qquad (9)$$

with c(t) being an arbitrary function. We take

$$u_0 = \frac{p_t - p_{xx} + c(t) p(a_0 + a_1 p)}{2p_x},$$

where p=p(x,t) and q=q(y,t) are arbitrary functions mentioned above. For such $h_1(x,t)$, $h_2(x,t)$,p,q and u_0, it si easy to verify that Eq. (6) holds by direct computation. Hence

$$u = \frac{f_x}{f} + u_0\left(x,t\right), v = \frac{f_{xy}}{f} - \frac{f_x f_y}{f^2}$$

are just solutions of Equations (1) and (2) This completes

978-1-935068-06-8 © 2009 SciRes.

the proof.

Remark 1. From the point of view of mathematics, we need to improve some restrictions on the smoothness of the functions in this paper. For convenience, we assume that all the functions have sufficient smoothness in the paper.

Taking $u = u_1(x, y, t) + u_0(y, t)$ in (4), we have

$$\partial y\left(u_{1t} - 2u_1u_{1x} - u_{1xx} - 2u_0u_{1x}\right) + u_{0yt} = 0. \quad (10)$$

And setting $u_1 = \partial x(\ln f)$, we get

$$\partial y \partial x\left(\frac{f_t - f_{xx} - 2u_0 f_x}{f}\right) + u_{0yt} = 0. \quad (11)$$

From the above discussion, we know that in order to obtain the solutions of (1)and(2), it suffices to find a solution to Eq.(11).

Theorem 2. Assume that $u_0(y,t) = u_0(y)$ and $C_1(y)$,$C_2(y)$,$C_3(y)$ are arbitrary functions. Let functions f=f(x,y,t) be given by

$$f = C_1(y)e^{-\frac{5}{4}u_0^2(y)t} + C_2(y)e^{\frac{1}{2}u_0(y)x} + C_3(y)e^{-\frac{5}{2}u_0(y)x}.$$

Then

$$u = \partial x(\ln f) + u_0, v = u_y$$

are solutions of Equation.(1) and Equation.(2)

Proof. Note that $u_0(y,t) = u_0(y)$, it follows that from (11)

$$\partial y \partial x\left(\frac{f_t - f_{xx} - 2u_0 f_x}{f}\right) = 0. \quad (12)$$

Clearly, (12) is equivalent to

$$f_t - f_{xx} - 2u_0 f_x = \left(h_1(x,t) + h_2(y,t)\right) f, \quad (13)$$

where $h_1(x,t)$ and $h_2(y,t)$ are arbitrary functions of the indicated variables.

Setting $h_1(x,t) = 0, h_2(y,t) = -\frac{5}{4}u_0^2(y)$ in (13). Our aim is to seek for the solutions of the form:

$$f = p(x,y) + q(y,t). \quad (14)$$

To this end, substituting (14) i nto (13), we derive

$$q_t - p_{xx} - 2u_0 p_x = -\frac{5}{4}u_0^2(y)\left[p(x,y) + q(y,t)\right]. \quad (15)$$

Equation (15) leads to

$$q_t = -\frac{5}{4}u_0^2(y)q(y,t),$$

$$-p_{xx} - 2u_0 p_x + \frac{5}{4}u_0^2(y)p(x,y) = 0. \quad (16)$$

Solving system (16) we derive

$$p(x,y) = C_2(y)e^{\frac{1}{2}u_0^2(y)x} + C_3(y)e^{-\frac{5}{2}u_0(y)x},$$

$$q(x,y) = C_1(y)e^{-\frac{5}{4}u_0^2(y)t} \quad (17)$$

Substituting (17) into (14) yields:

$$f = C_1(y)e^{-\frac{5}{4}u_0^2(y)t} + C_2(y)e^{\frac{1}{2}u_0(y)x} + C_3(y)e^{-\frac{5}{2}u_0(y)x}.$$

This completes the proof.

Remark 2. Obviously, the solutions obtained by Theorem 1 are similar to those obtained by using the separation variables approach in [12, 13, 15, 18, 19, 22, 23, 24, 25]. Yet, our method has some merits. Firstly, the condition to determine q is much simple. In fact, we only need to solve a Riccati equation with an arbitrary function, while in [12, 13, 15, 18, 19, 22, 23, 24, 25] the Riccati equation is one that includes three arbitrary functions, which leads to a difficult task. Secondly, we need not to dal with a complex bilinear equation or higher multi-linear equation of the original models while [12, 13, 15, 18, 19, 22, 23, 24, 25] has to. Thirdly, those solutions obtained by Theorem 2 haven't been reported in the above mentioned literature.

It is well known that there don't exist a unified method to solve the differential equation, especially for nonlinear differential equation. Since there exist infinity solutions for the solution equations, it is natural to consider that: do there exist solutions with some other forms to equations (1) and (2) and how to seek for them if they exist?

For the former, the authors obtained a class of solutions that are determined by a function F_1 in t and three functions F_2, F_3 F_4 in y in [9] through the tanh-function method. We have obtained forthrightly the same type of explicit exact solutions for the (2+1)-dimensional modified dispersive water-wave system in [10] through improving some procedures of the tanh-function method. Those solutions cannot be obtained by the variable separation approach and the method in this paper, which means that the tanh-function method is still a independent method.

3 Conclusions and Discussion

In summary, starting from the Backlund transformation and using Cole-Hopf transformation, we reduce the (2+1)-dimensional modified dispersive water-water-wave system to a simple linear evolution equation with two arbitrary functions of {x, t} and {y, t} in this paper. And we can obtain some new solutions of the original equations by investigating the simple linear evolution equation which include the solutions obtained by the variable separation approach. Meanwhile, by the results of this paper, we can obtain the solutions which describe some new natural phenomena of equations (1) and (2) by the fashions given in[12, 13, 15, 18, 19, 22, 23, 24, 25]. The method used in this paper can also be applied to deal with many (2+1)-dimensional nonlinear evolution equations, some results will be given in another paper.

Acknowledgement

This work is supported by the National Key Basic Research Development of China (Grant No. 1998030600) and the National Nature Science Foundation of China (Grant No.10072013).

References

[1] M. Boiti, Inv. Prob. 3 (1987), 37.
[2] C.W.Cao, Sci. China 33 (1990), 528.
[3] Y.Chen and Y. S. Li, Phys. Lett. A 175 (1991)22.
[4] P. A Clarkson and M. D. Kruskal, J. Math. Phys. 30 (1989), 2201.
[5] V. G. Durovsky and E.G. Konopelchenko, J. Phys.27 (1994), 4619.
[6] C. S. Gardner, J, M. Green, M. D. Kruskal and R. M. Miura, Phys. Rev. Lett. 19(1967), 1095.
[7] R. Hirota, Phys. Rev. Lett. 27 (1971), 1192.
[8] B. G. Konopelchenko, J. Sidorenko and W, Strampp, Phys, Lett. A 175 (1991), 17.
[9] D. S Li and H. Q. Zhang, Chaos, Solitons Fractals 18 (2003), 193.
[10] D. S. Li and H. Q. Zhang, The soliton-like solutions to the (2+1)-dimensional medified dispersive water-wave system, Chin, Phys. (2004), accepted for publication.
[11] S. Y. Lou, Phys. Lett. A 151 (1990), 133.
[12] S. Y. Lou, Phys. Lett. A 277 (2000), 94.
[13] S. Y. Lou, Physica Scripta 65 (2002), 7.
[14] S. Y. Lou, and L. L. Chen, J. Math. phys. 40 (1996), 6491.
[15] S. Y. Lou, J. Z. Lu, J. Phys. A: Math. Gen. 29 (1996), 4209.
[16] S. Y. Lou, and X. B. Hu, J. Math. Phys. A 38(1997), 6401.
[17] S. Y. Lou, and X. B Hu, J. Commun. Theor. Phys. 29 (1998),145.
[18] S. Y. Lou, and X. Y. Ruan, J. Phys. A: Math. Gen. 30 (2001), 305.
[19] S. Y. Lou, C. L. Chen and X. Y. Tang, J. Math. Phys .43 (2002), 4078.
[20] V. B. Matveev and M. A. Salle, Darbous Transformations and Solitons, Springer, Berlin, 1991.
[21] P. J. Olver, Application of Lie Group to Differential Equation, Springer, New, York, 1986.
[22] H.Y. Ruan and Y. X. Chen, Acta. Phys. Sin. 8(1999), 241.
[23] X. Y. Tang and S.Y. Lou, Chaos, Solitons. Fractals 14 (2002), 1451.
[24] X. Y. Tang and S. Y. Lou, Commun. Theor. Phys. 38 (2002), 1.
[25] X. Y. Tang and S. Y. Lou and Y. Zhang, Phys. Rev. E 66 (2002), 046601.

978-1-935068-06-8 © 2009 SciRes.