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Bose-Einstein Condensates and Atomic and Electron Lasers Including Atomic Laser in the self-Consistent Gravitation Field

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Abstract

The problem of an adequate description of the wave processes in Bose-Einstein condensates (CBE), including space-temporal evolution of CBE in the electron CBE condensate in the self-consistent electrical field and CBE atomic condensate in the self-consistent gravitational field is considered. The complete nonlocal system for the CBE evolution is delivered including particular case and analytical solutions.

Keywords

Nonlocal Physics, Transport Wave Processes in Bose-Einstein Condensate, Bose-Einstein Condensate Lasers

1. Introduction

The phenomenon of condensation of an ideal Bose gas was predicted in 1924 by Sh. Bose and A. Einstein [1] [2]. By definition, lasers produce coherent light waves, but according to the wave-corpuscle dualism underlying quantum mechanics, particles, including atoms, can be considered as waves. Based on this principle, the operation of a physical system, defined as the coherent state of propagating atoms, is defined as an atomic laser. The effect is based on the Bose-Einstein condensate, a state of matter in which a large number of particles occupy the same quantum ground state with low energy.

The first atomic laser was developed at the Massachusetts Institute of Technology in 1996 by physicist Wolfgang Ketterle and his colleagues. The Nobel Prize in Physics 2001 was awarded jointly to Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman "for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates".

Since then, several atomic laser designs have been demonstrated. These coherent "matter bundles" can be useful, for example, in the field of holography, since they are able to create holographic images with much higher resolution than with the traditional approach. But the atomic lasers developed were only able to work for a very short time. Thus, the main problem is the creation of continuous waves of matter in a Bose-Einstein condensate (CBE) at sufficiently high temperatures. Usually, extremely low temperatures are required for the formation of coherent waves of matter CBE, about one millionth of a degree above absolute zero (−273.15˚C). These lasers can create pulses of matter waves, but after sending such a pulse, it is necessary to create a new Bose-Einstein condensate (CBE) to generate a new pulse. In the past, physicists faced the same problem with optical lasers: at first they were only pulsed, and then they became continuous.

In order for these waves of matter to be used for practical purposes, it is necessary:

1) To develop an experimental technique of continuous CBE.

2) To develop a theory of atomic lasers with a continuous wave—a theory of a physical system that provides a continuous beam of coherent particles with a rest mass other than zero.

The article is devoted to the development of the nonlocal theory of coherent flows of CBE. Therefore,

1) The review of experimental achievements in the field of CBE lasers is not part of our task.

2) A review of theoretical papers based on the "corrected" Schrödinger equation does not make sense.

I recall the shortcomings of the Schrödinger model:

1) The Schrödinger equation is a postulate. Another differentiation of the function leads to other equations, for example, to equations containing the second derivative of time.

2) The Schrödinger equation does not describe dissipative processes.

3) The Schrödinger equation is not able to describe the whole complex "nucleus-electron shell".

4) The Schrödinger equation is unable to describe a spatial electron shell without the use of additional assumptions such as the Pauli principle.

5) To a large extent, the quantization result is the result of cutting infinite series and turning them into polynomials.

6) The Schrödinger equation requires setting boundary conditions for the wave function at infinity and leads to spreading of wave packets.

2. The Nonlocal Hydrodynamic Equations

The generalized hydrodynamic equations (GHE) can be obtained from the nonlocal kinetic equation in the frame of the Enskog procedure, [3] [4] [5] [6] [7]:

Continuity equation for species α

$$\frac{\partial}{\partial t}\left\{\rho_\alpha-\tau_\alpha\left[\frac{\partial\rho_\alpha}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0)\right]\right\}+\frac{\partial}{\partial\mathbf{r}}\cdot\left\{\rho_\alpha\mathbf{v}_0-\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)+\frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0\mathbf{v}_0)+\vec{\mathrm{I}}\cdot\frac{\partial p_\alpha}{\partial\mathbf{r}}-\rho_\alpha\mathbf{F}_\alpha^{(1)}-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0\times\mathbf{B}\right]\right\}=R_\alpha. \tag{2.1}$$

Continuity equation for mixture

$$\frac{\partial}{\partial t}\left\{\rho-\sum_\alpha\tau_\alpha\left[\frac{\partial\rho_\alpha}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0)\right]\right\}+\frac{\partial}{\partial\mathbf{r}}\cdot\left\{\rho\mathbf{v}_0-\sum_\alpha\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)+\frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0\mathbf{v}_0)+\vec{\mathrm{I}}\cdot\frac{\partial p_\alpha}{\partial\mathbf{r}}-\rho_\alpha\mathbf{F}_\alpha^{(1)}-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0\times\mathbf{B}\right]\right\}=0. \tag{2.2}$$

Momentum equation for species α

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho_\alpha\mathbf{v}_0-\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)+\frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{\partial p_\alpha}{\partial\mathbf{r}}-\rho_\alpha\mathbf{F}_\alpha^{(1)}\right.\right.\\&\left.\left.-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0\times\mathbf{B}\right]\right\}-\mathbf{F}_\alpha^{(1)}\left[\rho_\alpha-\tau_\alpha\left(\frac{\partial\rho_\alpha}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0)\right)\right]\\&-\frac{q_\alpha}{m_\alpha}\left\{\rho_\alpha\mathbf{v}_0-\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)+\frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{\partial p_\alpha}{\partial\mathbf{r}}-\rho_\alpha\mathbf{F}_\alpha^{(1)}\right.\right.\\&\left.\left.-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0\times\mathbf{B}\right]\right\}\times\mathbf{B}+\frac{\partial}{\partial\mathbf{r}}\cdot\left\{\rho_\alpha\mathbf{v}_0\mathbf{v}_0+p_\alpha\vec{\mathrm{I}}-\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0\mathbf{v}_0\right.\right.\\&+p_\alpha\vec{\mathrm{I}})+\frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha(\mathbf{v}_0\mathbf{v}_0)\mathbf{v}_0+2\vec{\mathrm{I}}\left(\frac{\partial}{\partial\mathbf{r}}\cdot(p_\alpha\mathbf{v}_0)\right)+\frac{\partial}{\partial\mathbf{r}}\cdot(\vec{\mathrm{I}}p_\alpha\mathbf{v}_0)\\&\left.\left.-\mathbf{F}_\alpha^{(1)}\rho_\alpha\mathbf{v}_0-\rho_\alpha\mathbf{v}_0\mathbf{F}_\alpha^{(1)}-\frac{q_\alpha}{m_\alpha}\rho_\alpha[\mathbf{v}_0\times\mathbf{B}]\mathbf{v}_0-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0[\mathbf{v}_0\times\mathbf{B}]\right]\right\}\\&=\int m_\alpha\mathbf{v}_\alpha J_\alpha^{st,el}\mathrm{d}\mathbf{v}_\alpha+\int m_\alpha\mathbf{v}_\alpha J_\alpha^{st,inel}\mathrm{d}\mathbf{v}_\alpha.\end{aligned} \tag{2.3}$$

Momentum equation for mixture

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho\mathbf{v}_0-\sum_\alpha\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)+\frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{\partial p_\alpha}{\partial\mathbf{r}}-\rho_\alpha\mathbf{F}_\alpha^{(1)}\right.\right.\\&\left.\left.-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0\times\mathbf{B}\right]\right\}-\sum_\alpha\mathbf{F}_\alpha^{(1)}\left[\rho_\alpha-\tau_\alpha\left(\frac{\partial\rho_\alpha}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0)\right)\right]\\&-\sum_\alpha\frac{q_\alpha}{m_\alpha}\left\{\rho_\alpha\mathbf{v}_0-\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)+\frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{\partial p_\alpha}{\partial\mathbf{r}}-\rho_\alpha\mathbf{F}_\alpha^{(1)}\right.\right.\\&\left.\left.-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0\times\mathbf{B}\right]\right\}\times\mathbf{B}+\frac{\partial}{\partial\mathbf{r}}\cdot\left\{\rho\mathbf{v}_0\mathbf{v}_0+p\vec{\mathrm{I}}-\sum_\alpha\tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0\mathbf{v}_0\right.\right.\\&+p_\alpha\vec{\mathrm{I}})+\frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha(\mathbf{v}_0\mathbf{v}_0)\mathbf{v}_0+2\vec{\mathrm{I}}\left(\frac{\partial}{\partial\mathbf{r}}\cdot(p_\alpha\mathbf{v}_0)\right)+\frac{\partial}{\partial\mathbf{r}}\cdot(\vec{\mathrm{I}}p_\alpha\mathbf{v}_0)\\&\left.\left.-\mathbf{F}_\alpha^{(1)}\rho_\alpha\mathbf{v}_0-\rho_\alpha\mathbf{v}_0\mathbf{F}_\alpha^{(1)}-\frac{q_\alpha}{m_\alpha}\rho_\alpha[\mathbf{v}_0\times\mathbf{B}]\mathbf{v}_0-\frac{q_\alpha}{m_\alpha}\rho_\alpha\mathbf{v}_0[\mathbf{v}_0\times\mathbf{B}]\right]\right\}=0.\end{aligned} \tag{2.4}$$

Energy equation for α species

$$\frac{\partial}{\partial t}\left\{\frac{\rho_\alpha v_0^2}{2}+\frac{3}{2}p_\alpha+\varepsilon_\alpha n_\alpha-\tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{\rho_\alpha v_0^2}{2}+\frac{3}{2}p_\alpha+\varepsilon_\alpha n_\alpha\right)\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0+\frac{5}{2}p_\alpha\mathbf{v}_0+\varepsilon_\alpha n_\alpha\mathbf{v}_0\right)-\mathbf{F}_\alpha^{(1)}\cdot\rho_\alpha\mathbf{v}_0\right]\right\}$$
$$+\frac{\partial}{\partial \mathbf{r}}\cdot\left\{\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0+\frac{5}{2}p_\alpha\mathbf{v}_0+\varepsilon_\alpha n_\alpha\mathbf{v}_0-\tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0\right.\right.\right.$$
$$\left.+\frac{5}{2}p_\alpha\mathbf{v}_0+\varepsilon_\alpha n_\alpha\mathbf{v}_0\right)+\frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0\mathbf{v}_0+\frac{7}{2}p_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{1}{2}p_\alpha v_0^2\overleftrightarrow{I}\right.$$
$$\left.+\frac{5}{2}\frac{p_\alpha^2}{\rho_\alpha}\overleftrightarrow{I}+\varepsilon_\alpha n_\alpha\mathbf{v}_0\mathbf{v}_0+\varepsilon_\alpha\frac{p_\alpha}{m_\alpha}\overleftrightarrow{I}\right)-\rho_\alpha\mathbf{F}_\alpha^{(1)}\cdot\mathbf{v}_0\mathbf{v}_0-p_\alpha\mathbf{F}_\alpha^{(1)}\cdot\overleftrightarrow{I}$$
$$-\frac{1}{2}\rho_\alpha v_0^2\mathbf{F}_\alpha^{(1)}-\frac{3}{2}\mathbf{F}_\alpha^{(1)}p_\alpha-\frac{\rho_\alpha v_0^2}{2}\frac{q_\alpha}{m_\alpha}\left[\mathbf{v}_0\times\mathbf{B}\right]-\frac{5}{2}p_\alpha\frac{q_\alpha}{m_\alpha}\left[\mathbf{v}_0\times\mathbf{B}\right]$$
$$\left.\left.-\varepsilon_\alpha n_\alpha\frac{q_\alpha}{m_\alpha}\left[\mathbf{v}_0\times\mathbf{B}\right]-\varepsilon_\alpha n_\alpha\mathbf{F}_\alpha^{(1)}\right]\right\}-\left\{\rho_\alpha\mathbf{F}_\alpha^{(1)}\cdot\mathbf{v}_0-\tau_\alpha\left[\mathbf{F}_\alpha^{(1)}\cdot\left(\frac{\partial}{\partial t}\left(\rho_\alpha\mathbf{v}_0\right)\right.\right.\right.\tag{2.5}$$
$$\left.\left.\left.+\frac{\partial}{\partial \mathbf{r}}\cdot\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{\partial}{\partial \mathbf{r}}\cdot p_\alpha\overleftrightarrow{I}-\rho_\alpha\mathbf{F}_\alpha^{(1)}-q_\alpha n_\alpha\left[\mathbf{v}_0\times\mathbf{B}\right]\right)\right]\right\}$$
$$=\int\left(\frac{m_\alpha v_\alpha^2}{2}+\varepsilon_\alpha\right)J_\alpha^{st,el}\mathrm{d}\mathbf{v}_\alpha+\int\left(\frac{m_\alpha v_\alpha^2}{2}+\varepsilon_\alpha\right)J_\alpha^{st,inel}\mathrm{d}\mathbf{v}_\alpha.$$

Energy equation for mixture

$$\frac{\partial}{\partial t}\left\{\frac{\rho v_0^2}{2}+\frac{3}{2}p+\sum_\alpha\varepsilon_\alpha n_\alpha-\sum_\alpha\tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{\rho_\alpha v_0^2}{2}+\frac{3}{2}p_\alpha+\varepsilon_\alpha n_\alpha\right)\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0+\frac{5}{2}p_\alpha\mathbf{v}_0+\varepsilon_\alpha n_\alpha\mathbf{v}_0\right)-\mathbf{F}_\alpha^{(1)}\cdot\rho_\alpha\mathbf{v}_0\right]\right\}$$
$$+\frac{\partial}{\partial \mathbf{r}}\cdot\left\{\frac{1}{2}\rho v_0^2\mathbf{v}_0+\frac{5}{2}p\mathbf{v}_0+\mathbf{v}_0\sum_\alpha\varepsilon_\alpha n_\alpha-\sum_\alpha\tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0\right.\right.\right.$$
$$\left.+\frac{5}{2}p_\alpha\mathbf{v}_0+\varepsilon_\alpha n_\alpha\mathbf{v}_0\right)+\frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{1}{2}\rho_\alpha v_0^2\mathbf{v}_0\mathbf{v}_0+\frac{7}{2}p_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{1}{2}p_\alpha v_0^2\overleftrightarrow{I}\right.$$
$$\left.+\frac{5}{2}\frac{p_\alpha^2}{\rho_\alpha}\overleftrightarrow{I}+\varepsilon_\alpha n_\alpha\mathbf{v}_0\mathbf{v}_0+\varepsilon_\alpha\frac{p_\alpha}{m_\alpha}\overleftrightarrow{I}\right)-\rho_\alpha\mathbf{F}_\alpha^{(1)}\cdot\mathbf{v}_0\mathbf{v}_0-p_\alpha\mathbf{F}_\alpha^{(1)}\cdot\overleftrightarrow{I}$$
$$-\frac{1}{2}\rho_\alpha v_0^2\mathbf{F}_\alpha^{(1)}-\frac{3}{2}\mathbf{F}_\alpha^{(1)}p_\alpha-\frac{\rho_\alpha v_0^2}{2}\frac{q_\alpha}{m_\alpha}\left[\mathbf{v}_0\times\mathbf{B}\right]-\frac{5}{2}p_\alpha\frac{q_\alpha}{m_\alpha}\left[\mathbf{v}_0\times\mathbf{B}\right]$$
$$\left.\left.-\varepsilon_\alpha n_\alpha\frac{q_\alpha}{m_\alpha}\left[\mathbf{v}_0\times\mathbf{B}\right]-\varepsilon_\alpha n_\alpha\mathbf{F}_\alpha^{(1)}\right]\right\}-\left\{\mathbf{v}_0\cdot\sum_\alpha\rho_\alpha\mathbf{F}_\alpha^{(1)}-\sum_\alpha\tau_\alpha\mathbf{F}_\alpha^{(1)}\cdot\left[\frac{\partial}{\partial t}\left(\rho_\alpha\mathbf{v}_0\right)\right.\right.\tag{2.6}$$
$$\left.+\frac{\partial}{\partial \mathbf{r}}\cdot\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\frac{\partial}{\partial \mathbf{r}}\cdot p_\alpha\overleftrightarrow{I}-\rho_\alpha\mathbf{F}_\alpha^{(1)}-q_\alpha n_\alpha\left[\mathbf{v}_0\times\mathbf{B}\right]\right]=0.$$

The force dimension, $\left[F_\alpha^{(1)}\right]=\frac{\text{cm}}{\text{s}^2}$. Here $\mathbf{F}_\alpha^{(1)}$ are the forces of the non-magnetic origin, $\mathbf{B}$ —magnetic induction, $\overleftrightarrow{I}$ —unit tensor, q_α —charge of the α —component particle, p_α —static pressure for α —component, ε_α —internal energy for the particles of α —component, $\mathbf{v}_0$ —hydrodynamic velocity for mixture, τ_α —non-local parameter.

3. System of Non-Local Equations for the Case $p_\alpha = 0$, $\mathbf{v}_0 = 0$

They often talk about a new form of matter. Exactly:

1) At a very low but finite temperature, a macroscopic number of atoms or molecules fill one energy level.

2) The gas consists of non-interacting particles.

It would seem that the existence of a finite temperature should inevitably lead to thermal chaotic motion of particles. This circumstance caused the rejection of the theory by many major theoretical physicists.

However, subsequent experiments have confirmed the possibility of the existence of such effects at the macroscopic level.

In the following we intend to consider the particular case of the basic nonlocal equations taking into account the mentioned above features of CBE.

We assume also that there is no directional motion of the CBE physical system with hydrodynamic velocity, $\mathbf{v}_0 = 0$. We find:

Continuity equation for species α

$$\frac{\partial}{\partial t}\left\{\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right\} + \frac{\partial}{\partial \mathbf{r}} \cdot \left(\tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\right) = R_\alpha. \tag{3.1}$$

Continuity equation for mixture

$$\frac{\partial}{\partial t}\left\{\rho - \sum_\alpha \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right\} + \frac{\partial}{\partial \mathbf{r}} \cdot \sum_\alpha \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)} = 0. \tag{3.2}$$

Momentum equation for species α

$$\frac{\partial}{\partial t}\left(\tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\right) - \mathbf{F}_\alpha^{(1)}\left[\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right] - \frac{q_\alpha}{m_\alpha}\left(\tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)} \times \mathbf{B}\right) = 0. \tag{3.3}$$

In the absence of an external magnetic field we have

$$\frac{\partial}{\partial t}\left(\tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\right) - \mathbf{F}_\alpha^{(1)}\left[\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right] = 0. \tag{3.4}$$

Momentum equation for mixture

$$\frac{\partial}{\partial t}\sum_\alpha \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)} - \sum_\alpha \mathbf{F}_\alpha^{(1)}\left[\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right] - \sum_\alpha \frac{q_\alpha}{m_\alpha}\tau_\alpha\left[\rho_\alpha \mathbf{F}_\alpha^{(1)} \times \mathbf{B}\right] = 0. \tag{3.5}$$

In the absence of an external magnetic field we have

$$\frac{\partial}{\partial t}\sum_\alpha \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)} - \sum_\alpha \mathbf{F}_\alpha^{(1)}\left[\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right] = 0. \tag{3.6}$$

Energy equation for α species)

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha - \tau_\alpha \frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)\right] + \frac{\partial}{\partial \mathbf{r}} \cdot \left[\tau_\alpha \varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right] - \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)} \cdot \mathbf{F}_\alpha^{(1)} \\ &= \int\left(\frac{m_\alpha v_\alpha^2}{2} + \varepsilon_\alpha\right) J_\alpha^{st,el}\,\mathrm{d}\mathbf{v}_\alpha + \int\left(\frac{m_\alpha v_\alpha^2}{2} + \varepsilon_\alpha\right) J_\alpha^{st,inel}\,\mathrm{d}\mathbf{v}_\alpha.\end{aligned} \tag{3.7}$$

But there is no dependence on velocity, then

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha - \tau_\alpha \frac{\partial}{\partial t}(\varepsilon_\alpha n_\alpha)\right] + \frac{\partial}{\partial \mathbf{r}}\cdot\left[\tau_\alpha \varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right] - \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)} = 0. \quad (3.8)$$

Energy equation for mixture

$$\frac{\partial}{\partial t}\left[\sum_\alpha \varepsilon_\alpha n_\alpha - \sum_\alpha \tau_\alpha \frac{\partial}{\partial t}(\varepsilon_\alpha n_\alpha)\right] + \frac{\partial}{\partial \mathbf{r}}\cdot\sum_\alpha \tau_\alpha \varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)} - \sum_\alpha \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)} = 0. \quad (3.9)$$

In the following we intend to consider the coherent wave processes in CBE system, where an energetic impulse is expanding with velocity v (see also (4.42)). We investigate a creation

1) The CBE electron beam in the self-consistent electrical field.

2) The CBE neutral atom beam in the self-consistent gravitational field.

4. Nonlocal Model of the Electron Bose Laser

We use the following basic equations for CBE electron beam in the self-consistent electrical field:

Continuity equation for species α

$$\frac{\partial}{\partial t}\left\{\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right\} + \frac{\partial}{\partial \mathbf{r}}\cdot\left(\tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\right) = R_\alpha. \quad (4.1)$$

BEC particles have no mutual interactions, then $R_\alpha = 0$.

For momentum equation in the absence of an external magnetic field we have

$$\frac{\partial}{\partial t}\left(\tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\right) - \mathbf{F}_\alpha^{(1)}\left[\rho_\alpha - \tau_\alpha \frac{\partial \rho_\alpha}{\partial t}\right] = 0. \quad (4.2)$$

Energy equation for α species); there is no dependence on velocity, then

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha - \tau_\alpha \frac{\partial}{\partial t}(\varepsilon_\alpha n_\alpha)\right] + \frac{\partial}{\partial \mathbf{r}}\cdot\left[\tau_\alpha \varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right] - \tau_\alpha \rho_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)} = 0. \quad (4.3)$$

We should add to the system of equations the relation defining the self-consistent electrical field $\mathbf{E}$

$$\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{E} = -\frac{1}{\varepsilon_0} e n_\alpha, \quad (4.4)$$

where n_α is numerical density, ε_0 is electrical (dimensionless) constant, e is absolute electron charge, $m_\alpha = m$. The connection between $\mathbf{E}$ and $\mathbf{F}_\alpha^{(1)}$ is written as

$$\mathbf{F}_\alpha^{(1)} = \mathbf{E}_\alpha \frac{e}{m}. \quad (4.5)$$

Unknown values are $\tau_\alpha, \mathbf{E}_\alpha, n_\alpha, \varepsilon_\alpha$. Let us transform Equation (4.2) written as

$$\frac{\partial}{\partial t}\left(\tau_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right) - \mathbf{F}_\alpha^{(1)}\left[n_\alpha - \tau_\alpha \frac{\partial n_\alpha}{\partial t}\right] = 0. \quad (4.6)$$

We find

$$\frac{\partial}{\partial t}\left(\tau_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right)+\mathbf{F}_\alpha^{(1)}\tau_\alpha \frac{\partial n_\alpha}{\partial t}=\mathbf{F}_\alpha^{(1)} n_\alpha , \tag{4.7}$$

or

$$2\tau_\alpha \frac{\partial}{\partial t}\left(n_\alpha \mathbf{F}_\alpha^{(1)}\right)+\mathbf{F}_\alpha^{(1)} n_\alpha \frac{\partial \tau_\alpha}{\partial t}=\mathbf{F}_\alpha^{(1)} n_\alpha , \tag{4.8}$$

or

$$2\tau_\alpha \mathbf{F}_\alpha^{(1)} \frac{\partial n_\alpha}{\partial t}+2\tau_\alpha n_\alpha \frac{\partial \mathbf{F}_\alpha^{(1)}}{\partial t}+\mathbf{F}_\alpha^{(1)} n_\alpha \frac{\partial \tau_\alpha}{\partial t}=\mathbf{F}_\alpha^{(1)} n_\alpha \tag{4.9}$$

or

$$2\tau_\alpha \mathbf{F}_\alpha^{(1)} \frac{\partial \ln n_\alpha}{\partial t}+2\tau_\alpha \frac{\partial \mathbf{F}_\alpha^{(1)}}{\partial t}+\mathbf{F}_\alpha^{(1)} \frac{\partial \tau_\alpha}{\partial t}=\mathbf{F}_\alpha^{(1)} \tag{4.10}$$

or

$$2\tau_\alpha \mathbf{F}_\alpha^{(1)} \frac{\partial \ln n_\alpha}{\partial t}+\mathbf{F}_\alpha^{(1)} \frac{\partial \tau_\alpha}{\partial t}=\mathbf{F}_\alpha^{(1)}-2\tau_\alpha \frac{\partial \mathbf{F}_\alpha^{(1)}}{\partial t} . \tag{4.11}$$

We neglect the time derivative of the logarithm of the numerical density:

$$\mathbf{F}_\alpha^{(1)} \frac{\partial \tau_\alpha}{\partial t}=\mathbf{F}_\alpha^{(1)}-2\tau_\alpha \frac{\partial \mathbf{F}_\alpha^{(1)}}{\partial t} \tag{4.12}$$

or

$$\frac{\partial \tau_\alpha}{\partial t}=1-2\tau_\alpha \frac{\partial \ln \mathbf{F}_\alpha^{(1)}}{\partial t} \tag{4.13}$$

We neglect the time derivative of the logarithm of the electric field intensity.

$$\frac{\partial \tau_\alpha}{\partial t}=1 \tag{4.14}$$

or

$$\tau_\alpha = t . \tag{4.15}$$

Remark:

A non-local parameter τ_α plays the same role as the kinetic coefficients of local theories, such as, say, the coefficients of viscosity and thermal conductivity. In other words, a non-local parameter is an external parameter of the theory and the relation (4.15) is only one of the ways to approximate it.

Let's continue the transformation of Equation (4.3) using (4.15)

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha - t\frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)\right]+\frac{\partial}{\partial \mathbf{r}}\cdot\left[t\varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right]-t\rho_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)}=0 \tag{4.16}$$

or

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha\right]-\frac{\partial}{\partial t}\left[t\frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)\right]+t\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right]-t\rho_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)}=0 \tag{4.17}$$

or

$$\frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)-t\frac{\partial^2}{\partial t^2}\left(\varepsilon_\alpha n_\alpha\right)-\frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)+t\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right]-t\rho_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)}=0 \tag{4.18}$$

or

$$\frac{\partial^2}{\partial t^2}(\varepsilon_\alpha n_\alpha)-\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{F}_\alpha^{(1)}\right]+mn_\alpha \mathbf{F}_\alpha^{(1)}\cdot\mathbf{F}_\alpha^{(1)}=0 \tag{4.19}$$

From Equation (4.19) follows that $\varepsilon_\alpha n_\alpha \sim const$ if there is no influence of the external forces $\mathbf{F}_\alpha^{(1)}$. Let us transform now continuity Equation (4.1) using (4.15) and omitting the interactions between Bose particles

$$\frac{\partial}{\partial t}\left\{n_\alpha - t\frac{\partial n_\alpha}{\partial t}\right\}+\frac{\partial}{\partial \mathbf{r}}\cdot\left(tn_\alpha \mathbf{F}_\alpha^{(1)}\right)=0. \tag{4.20}$$

or

$$\frac{\partial n_\alpha}{\partial t}-\frac{\partial}{\partial t}\left\{t\frac{\partial n_\alpha}{\partial t}\right\}+\frac{\partial}{\partial \mathbf{r}}\cdot\left(tn_\alpha \mathbf{F}_\alpha^{(1)}\right)=0 \tag{4.21}$$

or

$$-t\frac{\partial^2 n_\alpha}{\partial t^2}+t\frac{\partial}{\partial \mathbf{r}}\cdot\left(n_\alpha \mathbf{F}_\alpha^{(1)}\right)=0 \tag{4.22}$$

or

$$\frac{\partial^2 n_\alpha}{\partial t^2}=\frac{\partial}{\partial \mathbf{r}}\cdot\left(n_\alpha \mathbf{F}_\alpha^{(1)}\right). \tag{4.23}$$

or using relation $\mathbf{F}_\alpha^{(1)}=\mathbf{E}_\alpha\frac{e}{m}$ we find

$$\frac{\partial^2 n_\alpha}{\partial t^2}=\frac{e}{m}\frac{\partial}{\partial \mathbf{r}}\cdot(n_\alpha \mathbf{E}_\alpha). \tag{4.24}$$

Finally we reach the following system of three equations with three dependent functions $\mathbf{E}, n_\alpha, \varepsilon_\alpha$.

$$\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{E}=-\frac{1}{\varepsilon_0}en_\alpha, \tag{4.25}$$

$$\frac{\partial^2 n_\alpha}{\partial t^2}=\frac{e}{m}\frac{\partial}{\partial \mathbf{r}}\cdot(n_\alpha \mathbf{E}_\alpha), \tag{4.26}$$

$$\frac{\partial^2}{\partial t^2}(\varepsilon_\alpha n_\alpha)-\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{E}_\alpha\frac{e}{m}\right]+mn_\alpha \mathbf{E}_\alpha\frac{e}{m}\cdot\mathbf{E}_\alpha\frac{e}{m}=0. \tag{4.27}$$

Equation (4.27) can be written in the more simple form. Really

$$\frac{\partial^2}{\partial t^2}(\varepsilon_\alpha n_\alpha)-\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{E}_\alpha\frac{e}{m}\right]+mn_\alpha \mathbf{E}_\alpha\frac{e}{m}\cdot\mathbf{E}_\alpha\frac{e}{m}=0 \tag{4.28}$$

or

$$\frac{\partial^2}{\partial t^2}(\varepsilon_\alpha n_\alpha)-n_\alpha\frac{e}{m}\mathbf{E}_\alpha\cdot\frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}}-\varepsilon_\alpha\frac{e}{m}\frac{\partial}{\partial \mathbf{r}}\cdot(n_\alpha \mathbf{E}_\alpha)+mn_\alpha \mathbf{E}_\alpha\frac{e}{m}\cdot\mathbf{E}_\alpha\frac{e}{m}=0 \tag{4.29}$$

or using (4.26)

$$\frac{\partial^2}{\partial t^2}(\varepsilon_\alpha n_\alpha)-n_\alpha\frac{e}{m}\mathbf{E}_\alpha\cdot\frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}}-\varepsilon_\alpha\frac{\partial^2 n_\alpha}{\partial t^2}+mn_\alpha \mathbf{E}_\alpha\frac{e}{m}\cdot\mathbf{E}_\alpha\frac{e}{m}=0 \tag{4.30}$$

or

$$\frac{\partial^2}{\partial t^2}\left(\varepsilon_\alpha n_\alpha\right) - n_\alpha \frac{e}{m}\mathbf{E}_\alpha \cdot \frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}} - \varepsilon_\alpha \frac{\partial^2 n_\alpha}{\partial t^2} + m n_\alpha \mathbf{E}_\alpha \frac{e}{m} \cdot \mathbf{E}_\alpha \frac{e}{m} = 0 \quad (4.31)$$

or

$$2\frac{\partial \varepsilon_\alpha}{\partial t}\frac{\partial n_\alpha}{\partial t} + n_\alpha \frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - n_\alpha \frac{e}{m}\mathbf{E}_\alpha \cdot \frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}} + m n_\alpha \mathbf{E}_\alpha \frac{e}{m} \cdot \mathbf{E}_\alpha \frac{e}{m} = 0 \quad (4.32)$$

or

$$2\frac{\partial \varepsilon_\alpha}{\partial t}\frac{\partial \ln n_\alpha}{\partial t} + \frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - \frac{e}{m}\mathbf{E}_\alpha \cdot \frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}} + m \mathbf{E}_\alpha \frac{e}{m} \cdot \mathbf{E}_\alpha \frac{e}{m} = 0. \quad (4.33)$$

As before, we neglect the time derivative of the logarithm of the numerical density.

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - \frac{e}{m}\mathbf{E}_\alpha \cdot \frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}} + m\left(\frac{e}{m}\right)^2 E_\alpha^2 = 0 \quad (4.34)$$

or

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} = \frac{e}{m}\mathbf{E}_\alpha \cdot \left[\frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}} - e\mathbf{E}_\alpha\right]. \quad (4.35)$$

Equation (4.26) also can be written in the alternative form as a part of the system of nonlocal equations

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{E} = -\frac{1}{\varepsilon_0} e n_\alpha, \quad (4.36)$$

$$\frac{\partial^2 n_\alpha}{\partial t^2} = -\frac{1}{\varepsilon_0}\frac{e^2}{m} n_\alpha^2 + \frac{e}{m}\left[\mathbf{E}_\alpha \cdot \frac{\partial}{\partial \mathbf{r}}\right] n_\alpha, \quad (4.37)$$

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} = \frac{e}{m}\mathbf{E}_\alpha \cdot \left[\frac{\partial \varepsilon_\alpha}{\partial \mathbf{r}} - e\mathbf{E}_\alpha\right]. \quad (4.38)$$

In the one-dimensional case in the Cartesian coordinate system this system is written as

$$\frac{\partial E_{\alpha x}}{\partial x} = -\frac{1}{\varepsilon_0} e n_\alpha, \quad (4.39)$$

$$\frac{\partial^2 n_\alpha}{\partial t^2} = -\frac{1}{\varepsilon_0}\frac{e^2}{m} n_\alpha^2 + \frac{e}{m} E_{\alpha x} \frac{\partial n_\alpha}{\partial x}, \quad (4.40)$$

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} = \frac{e}{m} E_{\alpha x} \left[\frac{\partial \varepsilon_\alpha}{\partial x} - e E_{\alpha x}\right]. \quad (4.41)$$

We intend to find the wave solutions of this Bose electron physical system using the relation

$$\xi = x - vt \quad (4.42)$$

We transform system of Equations (4.39)-(4.41) using (4.42)

$$\frac{\partial E_{\alpha x}}{\partial \xi} = -\frac{1}{\varepsilon_0} e n_\alpha, \quad (4.43)$$

$$v^2 \frac{\partial^2 n_\alpha}{\partial \xi^2} = -\frac{1}{\varepsilon_0}\frac{e^2}{m} n_\alpha^2 + \frac{e}{m} E_{\alpha \xi} \frac{\partial n_\alpha}{\partial \xi}, \quad (4.44)$$

$$v^2 \frac{\partial^2 \varepsilon_\alpha}{\partial \xi^2} = \frac{e}{m} E_{\alpha\xi} \left[\frac{\partial \varepsilon_\alpha}{\partial \xi} - e E_{\alpha\xi} \right]. \tag{4.45}$$

We intend writing Equations (4.43)-(4.45) in the dimensionless form using scales

$$E_{\alpha x} \to E_0;\ \ n_\alpha \to n_0;\ \ x \to v t_0;\ \ \varepsilon_\alpha \to E_0 e v t_0;\ \ \xi \to v t_0. \tag{4.46}$$

Using (4.42) we have for (4.45)

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} \frac{v}{t_0} = \frac{e}{m} E_0 \tilde{E}_{\alpha\xi} \left[\frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} - \tilde{E}_{\alpha\xi} \right] \tag{4.47}$$

Equation (4.47) allows choosing the scale for t_0 in the form

$$t_0 = \frac{mv}{eE_0}, \tag{4.48}$$

then

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} = \tilde{E}_{\alpha\xi} \left[\frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} - \tilde{E}_{\alpha\xi} \right] \tag{4.49}$$

Let us consider now Equations (4.40) using (4.42). We find

$$v^2 \frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} \frac{1}{v^2 t_0^2} = -\frac{1}{\varepsilon_0} \frac{e^2}{m} \tilde{n}_\alpha^2 n_0 + \frac{e}{m} E_0 \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \frac{1}{v t_0} \tag*{4.50)}$$

or

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\frac{1}{\varepsilon_0} \frac{e^2}{m} \tilde{n}_\alpha^2 n_0 t_0^2 + \frac{e}{m} E_0 \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \frac{1}{v} \frac{mv}{eE_0} \tag{4.51}$$

or

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\frac{1}{\varepsilon_0} \frac{e^2}{m} \tilde{n}_\alpha^2 n_0 t_0^2 + \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \tag{4.52}$$

or

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\frac{1}{\varepsilon_0} \frac{e^2}{m} \tilde{n}_\alpha^2 n_0 \left(\frac{mv}{eE_0} \right)^2 + \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \tag{4.53}$$

or

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\frac{1}{\varepsilon_0} \frac{1}{m} \tilde{n}_\alpha^2 n_0 \left(\frac{mv}{E_0} \right)^2 + \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \tag{4.54}$$

or

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\frac{1}{\varepsilon_0} \frac{mv^2}{E_0^2} \tilde{n}_\alpha^2 n_0 + \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \tag{4.55}$$

and choosing the n_0 scale

$$n_0 = \frac{\varepsilon_0 E_0^2}{mv^2}. \tag{4.56}$$

This choosing leads to equation

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\tilde{n}_\alpha^2 + \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \tag{4.57}$$

and finally for (4.39) we have

$$\frac{\partial \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}} = -\tilde{n}_\alpha . \tag{4.58}$$

Let us write down the complete system of scales

$$E_{\alpha x} \to E_0 ;\ \ n_\alpha \to n_0 ;\ \ x \to v t_0 ;\ \ \varepsilon_\alpha \to E_0 e v t_0 ;$$

$$\xi \to v t_0 ;\ \ t_0 \to \frac{mv}{eE_0} ;\ \text{or}\ \left(\xi \to \frac{mv^2}{eE_0} \right),\ \ n_0 \to \frac{\varepsilon_0 E_0^2}{mv^2} \tag{4.59}$$

Remark

1) From the scale system (4.55) only two scales are independent, namely the scale of the electrical field intensity E_0 and the phase wave velocity *v*.

2) CBE physical system is created from non-interacting particles and then in many cases low index α can be omitted.

So we have a system of dimensionless ordinary differential equations.

$$\frac{\partial \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}} = -\tilde{n}_\alpha , \tag{4.60}$$

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = -\tilde{n}_\alpha^2 + \tilde{E}_{\alpha\xi} \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} , \tag{4.61}$$

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} = \tilde{E}_{\alpha\xi} \left[\frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} - \tilde{E}_{\alpha x} \right] . \tag{4.62}$$

It is possible to simplify the system (4.60)-(4.63) and even finding the analytical solutions. Using (4.60) and (4.61) we have

$$\frac{\partial^3 \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}^3} - \tilde{E}_{\alpha x} \frac{\partial^2 \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}^2} - \left(\frac{\partial \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}} \right)^2 = 0 \tag{4.63}$$

Let us consider identity

$$\frac{\partial}{\partial \tilde{\xi}} \frac{\partial \tilde{E}_{\alpha\xi}^2}{\partial \tilde{\xi}} \equiv 2 \tilde{E}_{\alpha\xi} \frac{\partial^2 \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}^2} + 2 \left(\frac{\partial \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}} \right)^2 . \tag{4.64}$$

Using (4.64) we simplify equation (4.63)

$$\frac{\partial^3 \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}^3} - \frac{1}{2} \frac{\partial^2 \tilde{E}_{\alpha\xi}^2}{\partial \tilde{\xi}^2} = 0 \tag{4.65}$$

and after integration

$$\frac{\partial \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}} - \frac{1}{2} \tilde{E}_{\alpha\xi}^2 = \tilde{A} \tilde{\xi} + \tilde{B} . \tag{4.66}$$

with dimensionless constants $\tilde{A}$ and $\tilde{B}$. Using (4.60) we reach the analytical solution

$$\tilde{n}_\alpha + \frac{1}{2}\tilde{E}_{\alpha\xi}^2 + \tilde{A}\tilde{\xi} + \tilde{B} = 0\,. \tag{4.67}$$

In the dimension form the first two terms in the left hand side of Equation (4.67) takes the form

$$n_\alpha \frac{mv^2}{\varepsilon_0 E_0^2} + \frac{1}{2}\frac{E_\alpha^2}{E_0^2} = \frac{1}{\varepsilon_0 E_0^2}\left(\frac{\varepsilon_0 E_\alpha^2}{2} + mn_\alpha v^2\right). \tag{4.68}$$

From (4.67) follows

$$\tilde{B} = -\left(\frac{1}{2}\tilde{E}_{\alpha\xi}^2 + \tilde{n}_\alpha\right)_{\tilde{\xi}=0} \tag{4.69}$$

or

$$\tilde{B} = -\frac{1}{\varepsilon_0 E_0^2}\left(\frac{\varepsilon_0 E_\alpha^2}{2} + mn_\alpha v^2\right)_{\tilde{x}=\tilde{t}} \tag{4.70}$$

or

$$\tilde{B} = \frac{B}{\varepsilon_0 E_0^2} \tag{4.71}$$

and

$$B = -\left(\frac{\varepsilon_0 E_\alpha^2}{2} + mn_\alpha v^2\right)_{\tilde{x}=\tilde{t}} \tag{4.72}$$

A term $\frac{1}{2}\tilde{E}_{\alpha\xi}^2$ according to the Umov-Poynting theorem is a dimensionless density of electromagnetic energy in the absence of a magnetic field. It is known that in Maxwell theory Umov-Poynting vector represents the directional energy flux (the energy transfer per unit area per unit time). Two terms in the round bracket of relation (4.72) correspond to the electric and kinetic densities of the BEC physical object.

Let us introduce $\frac{\varepsilon_0 E_\alpha^2}{2} + mn_\alpha v^2 = \Sigma_{total}$ as the total energy density. Then

$$B = -\Sigma_{total,\tilde{\xi}=0} \tag{4.73}$$

From (4.63) follows

$$\tilde{A} = \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{\xi}}\left(\tilde{n}_\alpha + \frac{1}{2}\tilde{E}_{\alpha\xi}^2\right)\right]_{\tilde{\xi}=0} \tag{4.74}$$

Equation (4.66) is independent ordinary differential equation of the first order and can be easily numerically integrated using for example the Maple possibilities.

We reach the following system of equations

$$\frac{\partial \tilde{E}_{\alpha\xi}}{\partial \tilde{\xi}} - \frac{1}{2}\tilde{E}_{\alpha\xi}^2 = A\tilde{\xi} + B\,, \tag{4.75}$$

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} = \tilde{E}_{\alpha\xi}\left[\frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} - \tilde{E}_{\alpha x}\right]. \tag{4.76}$$

Using (4.63) we indicate conditions and particular cases in the constant choice:

$$-\tilde{A}\tilde{\xi}-\tilde{B}\geq\frac{1}{2}\tilde{E}_{\alpha\xi}^{2}; \tag{4.77}$$

$$\tilde{\xi}=0\rightarrow-\tilde{B}\geq\frac{1}{2}\tilde{E}_{\alpha\xi}^{2}; \tag{4.78}$$

$$\tilde{\xi}>0\rightarrow-\tilde{A}\tilde{\xi}-\tilde{B}\geq\frac{1}{2}\tilde{E}_{\alpha\xi}^{2}. \tag{4.79}$$

Let us show a numerical result for the case $\tilde{A}=\tilde{B}=-1$ obtained with the Maple help (see **Figure 1**).

Obviously in this case we reach nonnegative numerical density for the wave $\tilde{\xi}=\tilde{x}-\tilde{t}$. This typical example shows that the wave solution exists in the bounded $\tilde{\xi}$ domain. This fact allows estimating the analytical solution for the energy Equation (4.60). As a result we obtain the better understanding the solution behavior.

Let us consider in the energy equation $\tilde{E}_{\alpha\xi}$ as an average constant $\tilde{E}_{av}$ in the domain of the wave regime existing and transform the corresponding linear equation.

$$\frac{\partial^2\tilde{\varepsilon}_\alpha}{\partial\tilde{\xi}^2}=\tilde{E}_{av}\left[\frac{\partial\tilde{\varepsilon}_\alpha}{\partial\tilde{\xi}}-\tilde{E}_{av}\right]. \tag{4.80}$$

We introduce the dependent variable

$$y=\tilde{\varepsilon}_\alpha+\tilde{E}_{av}\tilde{\xi} \tag{4.81}$$

and transform (4.80)

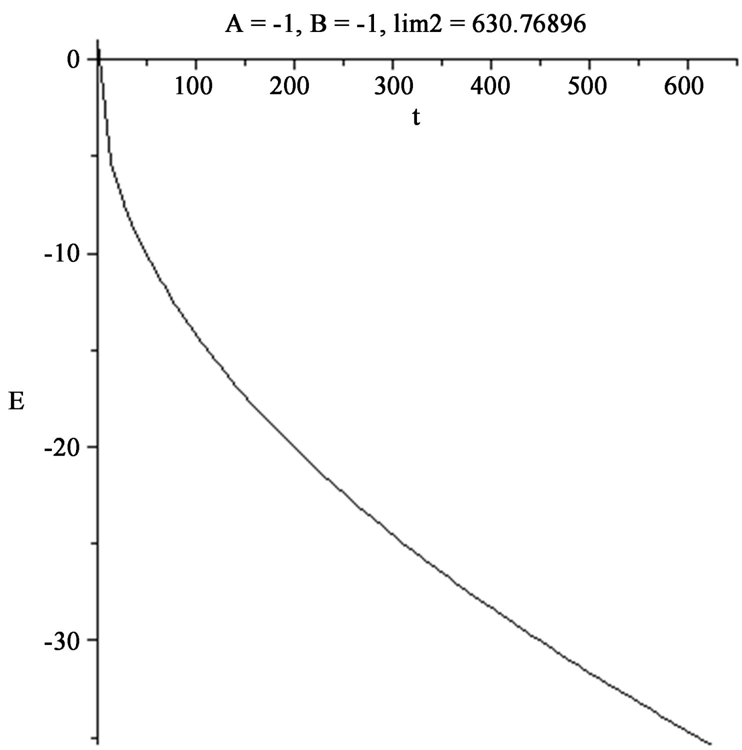

Figure 1. The dimensionless electrical intensity $\tilde{E}_{\alpha\xi}(\tilde{\xi})$.

$$\frac{\partial^2 y}{\partial \tilde{\xi}^2} = \tilde{E}_{av} \frac{\partial y}{\partial \tilde{\xi}} \tag{4.82}$$

or

$$\frac{\partial y}{\partial \tilde{\xi}} = \tilde{E}_{av} y + C \,. \tag{4.83}$$

We introduce a new variable

$$y = z - \frac{C}{\tilde{E}_{av}} \,. \tag{4.84}$$

and write

$$\frac{\partial z}{\partial \tilde{\xi}} = \tilde{E}_{av} z \,. \tag{4.85}$$

After integration we obtain

$$z = \mathrm{e}^{\tilde{E}_{av}\tilde{\xi} + C_1} \tag{4.86}$$

or after substitutions

$$y + \frac{C}{\tilde{E}_{av}} = \mathrm{e}^{\tilde{E}_{av}\tilde{\xi} + C_1} \tag{4.87}$$

or

$$\tilde{\varepsilon}_{\alpha} + \tilde{E}_{av}\tilde{\xi} + \frac{C}{\tilde{E}_{av}} = C_2 \mathrm{e}^{\tilde{E}_{av}\tilde{\xi}} \,, \tag{4.88}$$

or

$$\tilde{\varepsilon}_{\alpha} = C_2 \mathrm{e}^{\tilde{E}_{av}\tilde{\xi}} - \tilde{E}_{av}\tilde{\xi} - \frac{C}{\tilde{E}_{av}} \,, \tag{4.89}$$

$$\tilde{\varepsilon}_{\alpha} = C_2 \mathrm{e}^{\tilde{E}_{av}(\tilde{x} - \tilde{t})} - \tilde{E}_{av}\left(\tilde{x} - \tilde{t}\right) - \frac{C}{\tilde{E}_{av}} \,. \tag{4.90}$$

From estimation (4.90) follows that

$$\tilde{\varepsilon}_{\alpha} = C_2 - \frac{C}{\tilde{E}_{av}} \tag{4.91}$$

if $\tilde{x} = \tilde{t}$. Estimations (4.90) and (4.91) can be useful in applications.

5. Nonlocal Model of the Gravitational Bose Laser

Next, we intend to construct a theory of atomic Bose lasers in a self-consistent gravitational field. In general terms, the construction of the theory is similar to the theory of section 4. However, the difference in both results and mathematical transformations are very significant. The mentioned system of equations has the form

$$\mathbf{g}_{\alpha} = -\frac{\partial}{\partial \mathbf{r}} \Psi_{\alpha} \,, \tag{5.1}$$

$$\frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} \Psi_{\alpha} = 4\pi G m n_{\alpha} \,, \tag{5.2}$$

Continuity equation for species α

$$\frac{\partial}{\partial t}\left\{n_\alpha - \tau_\alpha \frac{\partial n_\alpha}{\partial t}\right\} + \frac{\partial}{\partial \mathbf{r}}\cdot\left(\tau_\alpha n_\alpha \mathbf{g}_\alpha\right) = 0. \tag{5.3}$$

Momentum equation for species α

$$\frac{\partial}{\partial t}\left(\tau_\alpha n_\alpha \mathbf{g}_\alpha\right) - \mathbf{g}_\alpha\left[n_\alpha - \tau_\alpha \frac{\partial n_\alpha}{\partial t}\right] = 0. \tag{5.4}$$

Energy equation for α species

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha - \tau_\alpha \frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)\right] + \frac{\partial}{\partial \mathbf{r}}\cdot\left[\tau_\alpha \varepsilon_\alpha n_\alpha \mathbf{g}_\alpha\right] - \tau_\alpha \rho_\alpha \mathbf{g}_\alpha \cdot \mathbf{g}_\alpha = 0 \tag{5.5}$$

Unknown values are

$$: \tau_\alpha, \mathbf{g}_\alpha, \Psi_\alpha, n_\alpha, \varepsilon_\alpha . \tag{5.6}$$

Let us transform Equation (5.4)

$$\frac{\partial}{\partial t}\left(\tau_\alpha n_\alpha \mathbf{g}_\alpha\right) + \mathbf{g}_\alpha \tau_\alpha \frac{\partial n_\alpha}{\partial t} = \mathbf{g}_\alpha n_\alpha \tag{5.7}$$

or

$$2\tau_\alpha \mathbf{g}_\alpha \frac{\partial n_\alpha}{\partial t} + n_\alpha \frac{\partial}{\partial t}\left(\tau_\alpha \mathbf{g}_\alpha\right) = \mathbf{g}_\alpha n_\alpha \tag{5.8}$$

or

$$2\tau_\alpha \mathbf{g}_\alpha \frac{\partial \ln n_\alpha}{\partial t} + \frac{\partial}{\partial t}\left(\tau_\alpha \mathbf{g}_\alpha\right) = \mathbf{g}_\alpha \tag{5.9}$$

We neglect the time derivative of the logarithm of the numerical density.

$$\frac{\partial}{\partial t}\left(\tau_\alpha \mathbf{g}_\alpha\right) = \mathbf{g}_\alpha \tag{5.10}$$

or

$$\tau_\alpha \frac{\partial \mathbf{g}_\alpha}{\partial t} + \mathbf{g}_\alpha \frac{\partial \tau_\alpha}{\partial t} = \mathbf{g}_\alpha \tag{5.11}$$

or

$$\tau_\alpha \frac{\partial \ln \mathbf{g}_\alpha}{\partial t} + \frac{\partial \tau_\alpha}{\partial t} = 1, \tag{5.12}$$

We neglect the time derivative of the logarithm of the gravitational acceleration.

Then as before we can use approximation:

$$\tau_\alpha = t . \tag{5.13}$$

Let ‘s continue the transformation of the equations. Using (5.13) we fined

$$\mathbf{g}_\alpha = -\frac{\partial}{\partial \mathbf{r}}\Psi_\alpha, \tag{5.14}$$

$$\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\Psi_\alpha = 4\pi G m n_\alpha, \tag{5.15}$$

$$\frac{\partial}{\partial t}\left\{n_\alpha - t\frac{\partial n_\alpha}{\partial t}\right\} + t\frac{\partial}{\partial \mathbf{r}}\cdot\left(n_\alpha \mathbf{g}_\alpha\right) = 0. \tag{5.16}$$

$$\frac{\partial}{\partial t}\left(t n_\alpha \mathbf{g}_\alpha\right) - \mathbf{g}_\alpha\left[n_\alpha - t\frac{\partial n_\alpha}{\partial t}\right] = 0. \tag{5.17}$$

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha - t\frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)\right] + t\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{g}_\alpha\right] - t\rho_\alpha \mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0 \tag{5.18}$$

Let us transform (5.16)

$$-t\frac{\partial^2 n_\alpha}{\partial t^2} + t\frac{\partial}{\partial \mathbf{r}}\cdot\left(n_\alpha \mathbf{g}_\alpha\right) = 0. \tag{5.19}$$

or

$$\frac{\partial^2 n_\alpha}{\partial t^2} = \frac{\partial}{\partial \mathbf{r}}\cdot\left(n_\alpha \mathbf{g}_\alpha\right). \tag{5.20}$$

Let us return now to the energy Equation (5.18)

$$\frac{\partial}{\partial t}\left[\varepsilon_\alpha n_\alpha\right] - \frac{\partial}{\partial t}\left[t\frac{\partial}{\partial t}\left(\varepsilon_\alpha n_\alpha\right)\right] + t\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{g}_\alpha\right] - t\rho_\alpha \mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0 \tag{5.21}$$

or

$$-t\frac{\partial^2}{\partial t^2}\left(\varepsilon_\alpha n_\alpha\right) + t\frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{g}_\alpha\right] - t\rho_\alpha \mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0 \tag{5.22}$$

or

$$\frac{\partial^2}{\partial t^2}\left(\varepsilon_\alpha n_\alpha\right) - \frac{\partial}{\partial \mathbf{r}}\cdot\left[\varepsilon_\alpha n_\alpha \mathbf{g}_\alpha\right] + \rho_\alpha \mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0\,. \tag{5.23}$$

or

$$\begin{aligned}&\varepsilon_\alpha\frac{\partial^2 n_\alpha}{\partial t^2} + n_\alpha\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} + 2\frac{\partial n_\alpha}{\partial t}\frac{\partial \varepsilon_\alpha}{\partial t} - \varepsilon_\alpha\frac{\partial}{\partial \mathbf{r}}\cdot\left[n_\alpha \mathbf{g}_\alpha\right]\\&-n_\alpha\left[\mathbf{g}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}}\right]\varepsilon_\alpha + \rho_\alpha \mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0\end{aligned} \tag{5.24}$$

or using (5.20)

$$n_\alpha\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} + 2\frac{\partial n_\alpha}{\partial t}\frac{\partial \varepsilon_\alpha}{\partial t} - n_\alpha\left[\mathbf{g}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}}\right]\varepsilon_\alpha + m n_\alpha \mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0 \tag{5.25}$$

or

$$2\frac{\partial \ln n_\alpha}{\partial t}\frac{\partial \varepsilon_\alpha}{\partial t} + \frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - \left[\mathbf{g}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}}\right]\varepsilon_\alpha + m\mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0\,. \tag{5.26}$$

As before, we neglect the time derivative of the logarithm of the numerical density n_α. Then:

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - \left[\mathbf{g}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}}\right]\varepsilon_\alpha + m\mathbf{g}_\alpha\cdot\mathbf{g}_\alpha = 0\,. \tag{5.28}$$

We obtain the transformed system of equations

$$\mathbf{g}_\alpha = -\frac{\partial}{\partial \mathbf{r}}\Psi_\alpha, \tag{5.29}$$

$$\frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} \Psi_\alpha = 4\pi G m n_\alpha , \tag{5.30}$$

$$\frac{\partial^2 n_\alpha}{\partial t^2} = \frac{\partial}{\partial \mathbf{r}} \cdot \left(n_\alpha \mathbf{g}_\alpha \right). \tag{5.31}$$

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - \left[\mathbf{g}_\alpha \cdot \frac{\partial}{\partial \mathbf{r}} \right] \varepsilon_\alpha + m \mathbf{g}_\alpha \cdot \mathbf{g}_\alpha = 0 . \tag{5.32}$$

In the one-dimensional case in the Cartesian coordinate system we have

$$g_\alpha = -\frac{\partial}{\partial x} \Psi_\alpha , \tag{5.33}$$

$$\frac{\partial^2}{\partial x^2} \Psi_\alpha = 4\pi G m n_\alpha , \tag{5.34}$$

$$\frac{\partial^2 n_\alpha}{\partial t^2} = \frac{\partial}{\partial x} \left(n_\alpha g_\alpha \right). \tag{5.35}$$

$$\frac{\partial^2 \varepsilon_\alpha}{\partial t^2} - g_\alpha \frac{\partial \varepsilon_\alpha}{\partial x} + m g_\alpha^2 = 0 , \tag{5.36}$$

where the measured value of the gravitational constant G is known with some certainty to four significant digits. In SI units, its value is approximately 6.674 × 10^{-11} $\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$.

We intend finding the wave solution of this system of equations using the variable $\xi = x - vt$. We obtain from (5.33)-(5.36)

$$g_\alpha = -\frac{\partial}{\partial \xi} \Psi_\alpha , \tag{5.37}$$

$$\frac{\partial^2}{\partial \xi^2} \Psi_\alpha = 4\pi G m n_\alpha , \tag{5.38}$$

$$v^2 \frac{\partial^2 n_\alpha}{\partial \xi^2} = \frac{\partial}{\partial \xi} \left(n_\alpha g_\alpha \right). \tag{5.39}$$

$$v^2 \frac{\partial^2 \varepsilon_\alpha}{\partial \xi^2} - g_\alpha \frac{\partial \varepsilon_\alpha}{\partial \xi} + m g_\alpha^2 = 0 , \tag{5.40}$$

Equations (5.37)-(5.40) can be written in the dimensionless form using the scales

$$\Psi_\infty \to \Psi_0 ; \quad n_\alpha \to n_0 ; \quad x \to v t_0 ; \quad g \to g_0 ; \quad \varepsilon_\alpha \to g_0 v t_0 ; \quad \xi \to v t_0 .$$

We fined from (5.37)

$$g_0 \tilde{g}_\alpha = -\frac{\partial}{\partial \tilde{\xi}} \tilde{\Psi}_\alpha \frac{\Psi_0}{v t_0} , \tag{5.41}$$

Then

$$\Psi_0 = v t_0 g_0 \tag{5.42}$$

and

$$\tilde{g}_\alpha = -\frac{\partial}{\partial \tilde{\xi}} \tilde{\Psi}_\alpha . \tag{5.43}$$

Equation (5.38) takes the form

$$\frac{\partial^2}{\partial \xi^2}\Psi_\alpha = 4\pi G m n_\alpha n_0 v^2 t_0^2 \frac{1}{v t_0 g_0}, \tag{5.44}$$

then

$$n_0 = \frac{g_0}{4\pi G m v t_0} \tag{5.45}$$

and

$$\frac{\partial^2}{\partial \tilde{\xi}^2}\tilde{\Psi}_\alpha = \tilde{n}_\alpha . \tag{5.46}$$

Let us consider now momentum Equation (5.39)

$$v^2 \frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2 v t_0} = g_0 \frac{\partial}{\partial \tilde{\xi}}\left(\tilde{n}_\alpha \tilde{g}_\alpha\right). \tag{5.47}$$

Then

$$g_0 = \frac{v}{t_0} \tag{5.48}$$

and

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = \frac{\partial}{\partial \tilde{\xi}}\left(\tilde{n}_\alpha \tilde{g}_\alpha\right). \tag{5.49}$$

We transform now the last equation in the system

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} = \tilde{g}_\alpha \left[\frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} - \tilde{g}_\alpha\right], \tag{5.50}$$

So, we have a system of dimensionless ordinary differential equations.

$$\tilde{g}_\alpha = -\frac{\partial}{\partial \tilde{\xi}}\tilde{\Psi}_\alpha, \tag{5.51}$$

$$\frac{\partial^2}{\partial \tilde{\xi}^2}\tilde{\Psi}_\alpha = \tilde{n}_\alpha, \tag{5.52}$$

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} = \frac{\partial}{\partial \tilde{\xi}}\left(\tilde{n}_\alpha \tilde{g}_\alpha\right)., \tag{5.53}$$

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} = \tilde{g}_\alpha \left[\frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} - \tilde{g}_\alpha\right] \tag{5.54}$$

with the complete scale system

$$\Psi_\alpha \to v^2;\ n_\alpha \to \frac{1}{4\pi G m t_0^2};\ x \to v t_0;\ g \to \frac{v}{t_0};\ \varepsilon_\alpha \to v^2;\ \xi \to v t_0 \tag{5.55}$$

with two independent scales v, t_0. The system of Equations (5.51)-(5.54) has an analytical solution.

After differentiating the left and right sides of Equation (5.51)

$$\frac{\partial}{\partial \tilde{\xi}}\tilde{g}_\alpha = -\frac{\partial^2}{\partial \tilde{\xi}^2}\tilde{\Psi}_\alpha \tag{5.56}$$

and comparing with (5.52) we find

$$n_\alpha = -\frac{\partial}{\partial \tilde{\xi}} \tilde{g}_\alpha . \tag{5.57}$$

Let us transform (5.53) using (5.57)

$$\frac{\partial^2 \tilde{n}_\alpha}{\partial \tilde{\xi}^2} + \frac{\partial}{\partial \tilde{\xi}} \left(\tilde{n}_\alpha \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} \right) = 0 . \tag{5.58}$$

or

$$\frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} + \tilde{n}_\alpha \frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} = A \tag{5.59}$$

or

$$\frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}} + \frac{1}{2} \frac{\partial \tilde{n}_\alpha^2}{\partial \tilde{\xi}} = A \tag{5.60}$$

or

$$\tilde{n}_\alpha + \frac{1}{2} \tilde{n}_\alpha^2 = A\tilde{\xi} + B \tag{5.61}$$

or

$$\tilde{n}_\alpha^2 + 2\tilde{n}_\alpha - 2A\tilde{\xi} - 2B = 0 \tag{5.62}$$

From the algebraic Equation (5.62) follows

$$\tilde{n}_{\alpha;1,2} = -1 \pm \sqrt{1 + 2\left(A\tilde{\xi} + B\right)} \tag{5.63}$$

with the conditions

$$A\tilde{\xi} + B > 0 \tag{5.64}$$

$$\tilde{n}_\alpha = \sqrt{1 + 2\left(A\tilde{\xi} + B\right)} - 1 . \tag{5.65}$$

Let us write down other form of the system using (5.65) and (5.57) written in the form

$$\sqrt{1 + 2\left(A\tilde{\xi} + B\right)} - 1 = -\frac{\partial}{\partial \tilde{\xi}} \tilde{g}_\alpha . \tag{5.66}$$

We have

$$\frac{\partial^2 \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}^2} - \tilde{g}_\alpha \frac{\partial \tilde{\varepsilon}_\alpha}{\partial \tilde{\xi}} + \tilde{g}_\alpha^2 = 0 , \tag{5.67}$$

$$\frac{\partial}{\partial \tilde{\xi}} \tilde{g}_\alpha + \sqrt{1 + 2\left(A\tilde{\xi} + B\right)} - 1 = 0 . \tag{5.68}$$

As we see from (5.67) the energy $\tilde{\varepsilon}_\alpha \sim const$ if the dimensionless gravitational acceleration $\tilde{g}_\alpha = 0$. One obtains in the Maple realization (see **Figure 2**).

```
dsolve[interactive]({
diff(V(t),t$2)-G(t)*diff(V(t),t)+G(t)^2=0,

diff(G(t),t)+(1+2(A*t+B))^0.5-1=0,
V(0)=0,D(V)(0)=0,G(0)=1
});
>
```

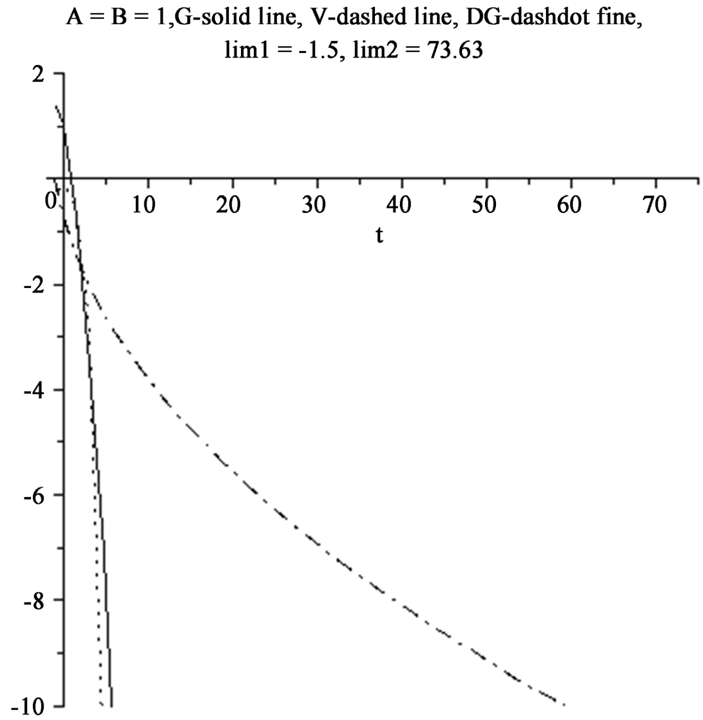

Figure 2. Dependences $\tilde{g}(\tilde{\xi})$, $\tilde{\varepsilon}(\tilde{\xi})$, $\frac{\partial \tilde{g}}{\partial \tilde{\xi}} = -\tilde{n}$; $A = B = 1$; $\tilde{g}(\tilde{\xi}) \leftrightarrow G$, $\tilde{\varepsilon}(\tilde{\xi}) \leftrightarrow V$, $\frac{\partial \tilde{g}}{\partial \tilde{\xi}} \leftrightarrow DG$.

That's convention in the Schrödinger theory that a free, unbound electron has zero energy. This means that we need to add energy to make the bound state free, which corresponds to raising its energy to zero. In this theory we take a convention that energy $\tilde{\varepsilon}(0) = 0$. It means in other words that in Boson wave $\tilde{\varepsilon}(\tilde{\xi}) = 0$ if $\tilde{x} = \tilde{t}$.

If A and B are small values we find

$$\tilde{n}_\alpha = A\tilde{\xi} + B \tag{5.69}$$

or

$$\tilde{n}_\alpha = A(\tilde{x} - \tilde{t}) + B \tag{5.70}$$

or for the wave with $\tilde{\xi} \geq 0$

$$\tilde{n}_\alpha = \left[\frac{\partial \tilde{n}_\alpha}{\partial \tilde{\xi}}\right]_{\tilde{\xi}=0} (\tilde{x} - \tilde{t}) + \tilde{n}_{\alpha,\tilde{\xi}=0}. \tag{5.72}$$

Using (5.68) we fined for this case

$$A\tilde{\xi} + B = -\frac{\partial}{\partial \tilde{\xi}} \tilde{g}_\alpha. \tag{5.73}$$

and

$$\frac{1}{2} A\tilde{\xi}^2 + B\tilde{\xi} + C = -\tilde{g}_\alpha. \tag{5.74}$$

As we see the energetic impulse expanding is convoying by the self-consistent gravitational wave.

6. The CBE Theory from the Position of Nonlocal Physics

We intend to investigate the pressure p evolution for the CBE case. Then we should suppose that: hydrodynamic velocity $\mathbf{v}_0 = 0$, the external forces are absent, stationary case.

In the local case we should add the condition $\tau_\alpha = 0$. In the local case the left hand side of the energy Equation (2.6) turns into identical zero.

In nonlocal case we find

$$\frac{\partial}{\partial \mathbf{r}} \cdot \sum_\alpha \tau_\alpha \frac{\partial}{\partial \mathbf{r}} \cdot \left(\frac{5}{2} \frac{p_\alpha}{n_\alpha} + \varepsilon_\alpha \right) \frac{p_\alpha}{m_\alpha} \vec{\mathrm{I}} = 0. \tag{6.1}$$

For the one component system one obtains

$$\frac{\partial}{\partial \mathbf{r}} \cdot \tau \frac{\partial}{\partial \mathbf{r}} \cdot \left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) \frac{p}{m} \vec{\mathrm{I}} = 0. \tag{6.2}$$

Let be $\tau = const$, then

$$\frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} \left[\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) \frac{p}{m} \right] \vec{\mathrm{I}} = 0. \tag{6.3}$$

For the 1D case

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) \frac{p}{m} \right] = 0 \tag{6.4}$$

and after integration we reach algebraic relation

$$\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) p = Ax + B. \tag{6.5}$$

Obviously

$$A = \frac{\mathrm{d}}{\mathrm{d}x} \left[\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) p \right] \tag{6.6}$$

and

$$B = \left(\frac{5}{2} \frac{p}{n} + \varepsilon \right)_{x=0} p_{x=0} \tag{6.7}$$

Relation (6.5) written as

$$\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) p = \frac{\mathrm{d}}{\mathrm{d}x} \left[\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) p \right] x + \left(\frac{5}{2} \frac{p}{n} + \varepsilon \right)_{x=0} p_{x=0} \tag{6.8}$$

can be rewritten as

$$p = x \frac{\mathrm{d}}{\mathrm{d}x} \ln \left[\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right) p \right] + \frac{\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right)_{x=0} p_{x=0}}{\left(\frac{5}{2} \frac{p}{n} + \varepsilon \right)} \tag{6.9}$$

or

$$p = \frac{\mathrm{d}\ln\left[\left(\frac{5}{2}\frac{p}{n}+\varepsilon\right)p\right]}{\mathrm{d}\ln x} + \frac{\left(\frac{5}{2}\frac{p}{n}+\varepsilon\right)_{x=0} p_{x=0}}{\left(\frac{5}{2}\frac{p}{n}+\varepsilon\right)}. \tag{6.10}$$

Omitting the small first term in the right hand side of relation (6.10) we find

$$p_{Bose} \cong p_{x=0} \frac{\left(\frac{5}{2}\frac{p}{n}+\varepsilon\right)_{x=0}}{\frac{5}{2}\frac{p}{n}+\varepsilon} \tag{6.11}$$

or

$$p_{Bose} \cong p_{x=0} \frac{\frac{5}{2}k_B T_0 + \varepsilon_0}{\frac{5}{2}k_B T + \varepsilon}. \tag{6.12}$$

From relation (6.10) follows:

1) The CBE pressure is damping very slowly with the distance growing.

2) The CBE pressure is damping with the temperature growing as $1/T$.

Individual atoms of an ideal gas have only the kinetic energy. In the general case particles possess rotational or vibrational degrees of freedom, and can be electronically excited to higher energies. Therefore, the internal energy ε of an ideal gas depends solely on its temperature and numerical density of gas particles.

The critical temperature in the CBE theory is written usually as

$$T_c = 3.3125 \frac{\hbar^2 n^{2/3}}{m k_B}, \tag{6.13}$$

where n is the particle density, $\hbar$ the reduced Planck constant and m the mass per boson. In literature you can find other estimations of T_c. Obviously

$$p_c = 3.3125 \frac{\hbar^2 n^{5/3}}{m}, \tag{6.14}$$

The T_c and p_c existing defines the Cauchy condition $p_{x=0}$ in (6.12).

7. Conclusion

The nonlocal theory of the wave processes in Bose-Einstein condensate is created. Adequate description of the wave processes in Bose-Einstein condensates (CBE) leads to the theory of CBE lasers. Space-temporal evolution of CBE in the electron CBE condensate in the self-consistent electrical field and CBE atomic condensate in the self-consistent gravitational field is considered. The complete nonlocal system for the CBE evolution is delivered including particular cases and analytical solutions. The stable wave regime exists in the bounded domain of the wave independent variable. The complete system of equations is delivered. For the 1D case the analytical solutions are obtained.

The operation of Bose-effect lasers requires very low temperatures. In other words, a) electron and gravitational lasers may have a natural origin in cosmic space. Effects of gas flows ejected from the upper atmospheres of stars are known in astrophysics. Different types of stars have different types of stellar winds. Stellar winds and bipolar outflows can be considered as candidates for CBE lasers. b) Electronic and gravitational lasers in the technological version can be created for use in cosmic space. c) In any case, you should use the created theory.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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