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New Solutions for an Elliptic Equation Method and Its Applications in Nonlinear Evolution Equations

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Abstract

In this paper, we study an elliptic equation with four distinct real roots and obtain five new solutions to this type of elliptic equation. Using these obtained new elliptic function solutions we can construct a series of explicit exact solutions for many nonlinear evolution equations. As examples, we choose combined KdV-MKdV equation, a fourth-order integrable nonlinear Schrödinger equation and generalized Dullin-Gottwald-Holm equation to demonstrate the effectiveness of these new elliptic function solutions. These new elliptic function solutions can be applied to many other nonlinear evolution equations.

Keywords

Elliptic Equation, Periodic Wave Solution, Singular Wave Solution, Combined KdV-MKdV Equation, Generalized Dullin-Gottwald-Holm Equation

1. Introduction

Nonlinear phenomena exist in many fields of natural science, such as quantum mechanics, condensed matter, optics, electromagnetism, fluid dynamics, biology, chemistry, geography, atmospheric circulation, etc., which are essentially governed by nonlinear evolution equations. As traveling waves continue to be discovered in the motion of particles, it is becoming more and more important to find new traveling wave solutions, which may help us to explore complex physical laws and reveal the mysterious nature of matter motion. It is an important and interesting task to study exact traveling wave solution for nonlinear evolution equations. Unfortunately, it is worth noting that there is no one unique method that can solve all nonlinear evolution equations, which led to appearing a number of methods, such as the Jacobi elliptic function method [1], the optional decoupling condition approach [2] the homogeneous balance method [3], improved simple equation method [4], $\frac{G'}{G}$-expansion method [5] [6] [7], Hirota bilinear form [8] [9], improved $\frac{G'}{G}$-expansion method [10], Truncation Painleve expansion method [11], homotopy perturbation method [12], variational method [13], Bäcklund transformation [14] [15].

In [16], the author considered the elliptic equation as following:

$$\varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}, \tag{1.1}$$

where $\varepsilon^2 = 1$. The author devised a new unified algebraic method, which is called fan sub-equation mapping method. The core of this method is to use solutions of a general elliptic equation to construct solutions of nonlinear evolution equations. When at least two of the five parameters of c_0, c_1, c_2, c_3, and c_4 are zero, the author got a lot of results. This method has been shown to be very efficient for constructing solutions to nonlinear evolution equations [17] [18] [19] [20] [21]. In [22], the author studied the following elliptic equations

$$\varphi'^2 = h_0 + h_1\varphi + h_2\varphi^2 + h_3\varphi^3 + h_4\varphi^4 = \left(r + p\varphi + q\varphi^2\right)^2, \tag{1.2}$$

and

$$\varphi'^2 = h_0 + h_1\varphi + h_3\varphi^3 + h_4\varphi^4 = \left(r + p\varphi + q\varphi^2\right)^2, \tag{1.3}$$

where $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 2rq + p^2$, $h_3 = 2pq$, $h_4 = q^2$. In [23], the authors discussed the elliptic equation as follows:

$$\varphi'^2 = A\varphi + B\varphi^2 + C\varphi^3 + D\varphi^4 + E = D\left(\varphi - \gamma_1\right)\left(\varphi - \overline{\gamma_1}\right)\left(\varphi - \gamma_2\right)\left(\varphi - \overline{\gamma_2}\right),$$

where $B = -A\left(2 - m^2\right)$, $C = 2A\left(1 - m^2\right)$, $D = -A\left(1 - m^2\right)$, $E = -\frac{A}{4}$, $0 < m < 1$. $\overline{\gamma_1}$ and $\overline{\gamma_2}$ are the complex conjugates of γ_1, γ_2, respectively. The authors obtained four new unbounded singular solutions for Equation (1.4). In [24], the author considered the following equation

$$\varphi'^2 = A\varphi\left(\varphi - \alpha_1\right)\left(\varphi - \alpha_2\right)\left(\varphi - \alpha_3\right), \tag{1.5}$$

where $\alpha_1 < \alpha_2 < \alpha_3$. The author got five different periodic solutions for Equation (1.5).

In this paper, we consider the elliptic equation as following:

$$\varphi'^2 = A\left(\varphi - \alpha_1\right)\left(\varphi - \alpha_2\right)\left(\varphi - \alpha_3\right)\left(\varphi - \alpha_4\right), \tag{1.6}$$

where $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. Obviously, Equation (1.6) is a generalization of equation Equation (1.5). Next, we will give three new periodic solutions and two new singular solutions for Equation (1.6). In addition, we will use these new solutions to construct traveling wave solutions for combined KdV-MKdV equation, a fourth-order integrable nonlinear Schrödinger equation and generalized Dullin-Gottwald-Holm equation.

2. New Solutions for Equation (1.6)

With the help of Maple, we obtain the following new solutions of Equation (1.6):

1) When $A>0$, there exist two unbounded singular solutions as follows:

$$\varphi(\xi)=\frac{\alpha_2(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4}, \tag{2.1}$$

and

$$\varphi(\xi)=\frac{\alpha_3(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_4(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3}, \tag{2.2}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_3-\alpha_2)(\alpha_4-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$,

$\varphi(\xi)\in(-\infty,\alpha_1]\cup[\alpha_4,+\infty)$. $sn(\cdot,\cdot)$ is an elliptic integral of the first class.

2) When $A>0$, there exists one bounded periodic solution as follows:

$$\varphi(\xi)=\frac{\alpha_1(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3},\quad \varphi(\xi)\in[\alpha_2,\alpha_3], \tag{2.3}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_3-\alpha_2)(\alpha_4-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

3) When $A<0$, there are two bounded periodic solutions as follows:

$$\varphi(\xi)=\frac{\alpha_4(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_2-\alpha_4},\varphi(\xi)\in[\alpha_2,\alpha_3], \tag{2.4}$$

and

$$\varphi(\xi)=\frac{\alpha_2(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4},\varphi(\xi)\in[\alpha_3,\alpha_4], \tag{2.5}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_4-\alpha_3)(\alpha_2-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

Interestingly, these two different periodic solutions are parametric symmetry. That is to say, the solution (2.4) are converted to (2.5) through the mutual replacement $\alpha_2 \leftrightarrow \alpha_4$ and $\alpha_1 \leftrightarrow \alpha_3$.

3. The Elliptic Equation Method and Its Apllications

3.1. The Ellipse Equation Method

For a given nonlinear evolution equation, say in two independent variables

$$Q(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, \cdots) = 0, \tag{3.1}$$

where $u_t = \frac{\partial u(x,t)}{\partial t}$, $u_x = \frac{\partial u(x,t)}{\partial x}$. We assume that the travelling wave solutions of Equation (3.1) can be expressed as follows:

$$u(x,t) = u(\xi) = \sum_{i=0}^{N} \beta_i \varphi^i(\xi), \tag{3.2}$$

where $\xi = x - ct$, c is wave speed. $\varphi(\xi)$ satisfy Equation (1.6) and the following equation:

$$\varphi''(\xi) = 2A\varphi^3(\xi) - \frac{3A}{2}\gamma_1\varphi^2(\xi) + A\gamma_2\varphi(\xi) - \frac{A}{2}\gamma_3, \tag{3.3}$$

where $\gamma_1 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_1$, $\gamma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$, $\gamma_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4$. The N is a positive integer that can be determined by balancing the liner term of highest order with the nonlinear term in Equation (3.1). With the help of Maple, substituting (3.2), Equation (1.6) and Equation (3.3) into Equation (3.1), and setting the coefficients of φ^i in the obtained system of equations to zero, we obtain a set of algebraic equations with respect to $c, A, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \cdots, \beta_N$. These equations can be solved by Grobner basis elimination method. Finally, substituting each solution of these algebraic equations into (3.2) and using the solutions of Equation (1.6), some new parameter expressions of travelling wave solutions for Equation (3.1) can be obtained.

3.2. Combined KdV-MKdV Equation

Combined KdV-MKdV equation [16] usually expressed in the following form:

$$u_t(x,t) + pu(x,t)u_x(x,t) + qu^2(x,t)u_x(x,t) + u_{xxx}(x,t) = 0, \tag{3.4}$$

where p, q are constant parameters. Equation (3.4) has been studied by some authors [16] [25] [26] [27]. In order to seek travelling wave solutions, we assume that

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{3.5}$$

where c is the wave velocity. Substituting (3.5) into Equation (3.4), we get

$$-cu'(\xi) + pu(\xi)u'(\xi) + qu^2(\xi)u'(\xi) + u'''(\xi) = 0. \tag{3.6}$$

Integrating Equation (3.6) once, it follows that

$$-cu(\xi) + \frac{1}{2}pu^2(\xi) + \frac{1}{3}qu^3(\xi) + u''(\xi) + c_1 = 0, \tag{3.7}$$

where c_1 is the integral constant. According to the elliptic equation method, the solutions of Equation (3.7) can be expanded as follows:

$$u(\xi)=\sum_{i=0}^{N}\beta_i\varphi^i(x,t)=\sum_{i=0}^{N}\beta_i\varphi^i(\xi), \tag{3.8}$$

where $\varphi(\xi)$ satisfy Equation (1.6) and Equation (3.3). Balancing the term $u^3(\xi)$ with term $u''(\xi)$ in Equation (3.7), we get $N=1$. Therefore Equation (3.7) has the following solution

$$u(\xi)=\beta_0+\beta_1\varphi(\xi), \tag{3.9}$$

where $\beta_1\neq 0$. Substituting Equation (3.3) and Equation (3.7) into Equation (3.9) and setting the coefficients of φ^i $(i=0,1,2,3)$ to zero, we obtain the following algebraic system

$$\begin{aligned}
&\varphi^3:\frac{1}{3}q\beta_1^3+2\beta_1A=0,\\
&\varphi^2:\frac{1}{2}p\beta_1^2+q\beta_0\beta_1^2-\frac{3}{2}\beta_1A\gamma_1=0,\\
&\varphi^1:q\beta_0^2\beta_1+A\beta_1\gamma_2+p\beta_0\beta_1-c\beta_1=0,\\
&\varphi^0:-c\beta_0+\frac{1}{2}p\beta_0^2+\frac{1}{3}q\beta_0^3-\frac{1}{2}\beta_1A\gamma_3+c_1=0.
\end{aligned} \tag{3.10}$$

With the help of Maple, we get

$$\begin{aligned}
&\beta_0=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}, \qquad c=-\frac{1}{2}q\beta_1\gamma_1\beta_0-\frac{1}{6}q\beta_1^2\gamma_2-q\beta_0^2,\\
&\beta_1=\pm\sqrt{\frac{-6A}{q}}, \qquad \gamma_3=-\frac{3q\beta_0^2\beta_1\gamma_1+2q\beta_0\beta_1^2\gamma_2+4q\beta_0^3+12c_1}{q\beta_1^3},
\end{aligned} \tag{3.11}$$

where $qA<0$, and $p,q,A,\gamma_1,\gamma_2,c_1$ are arbitrary constants.

Substituting (3.11) into Equation (3.9), we get the following travelling wave solutions of Equation (3.7)

$$u(\xi)=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}\pm\sqrt{\frac{-6A}{q}}\varphi(\xi), \tag{3.12}$$

where $\varphi(\xi)$ satisfies Equation (1.6). From Equation (3.12) and (2.1)-(2.5), we get the elliptic function solutions of Equation (3.7) as follows:

1) $A>0$, $q<0$, we obtain

$$u(\xi)=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}\pm\sqrt{\frac{-6A}{q}}\left[\frac{\alpha_2(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4}\right], \tag{3.13}$$

$$u(\xi)=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}\pm\sqrt{\frac{-6A}{q}}\left[\frac{\alpha_3(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_4(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3}\right], \tag{3.14}$$

and

$$u(\xi)=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}\pm\sqrt{\frac{-6A}{q}}\left[\frac{\alpha_1(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3}\right], \quad (3.15)$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_3-\alpha_2)(\alpha_4-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

Taking $A=6$, $\alpha_1=1$, $\alpha_2=2$, $\alpha_3=3$, $\alpha_4=4$, $p=1$, $q=-1$, $\gamma_1=1$, $\gamma_2=1$, $\gamma_3=\frac{35}{108}$, $c_1=1$, $\beta_1=6$, $\beta_0=-1$, $c=4$, the plane images of these solutions are shown in **Figures 1-6**, respective.

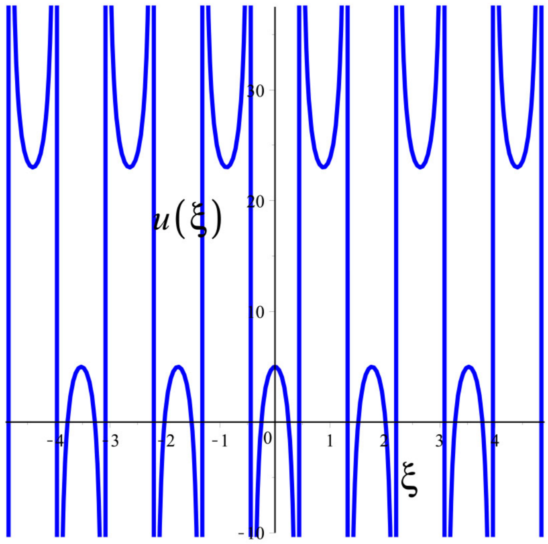

Figure 1. The 2D plot of (3.13).

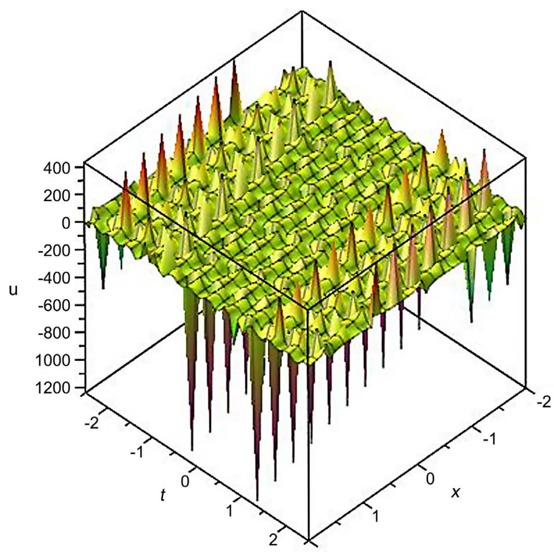

Figure 2. The 3D plot of (3.13).

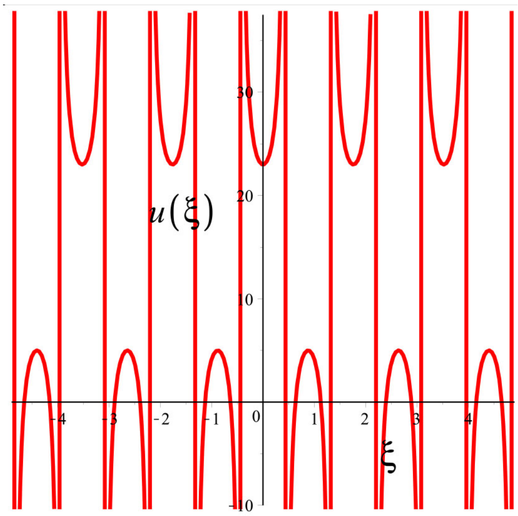

Figure 3. The 2D plot of (3.14).

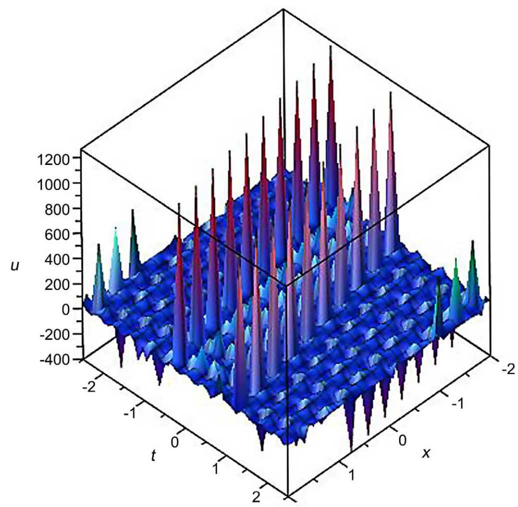

Figure 4. The 3D plot of (3.14).

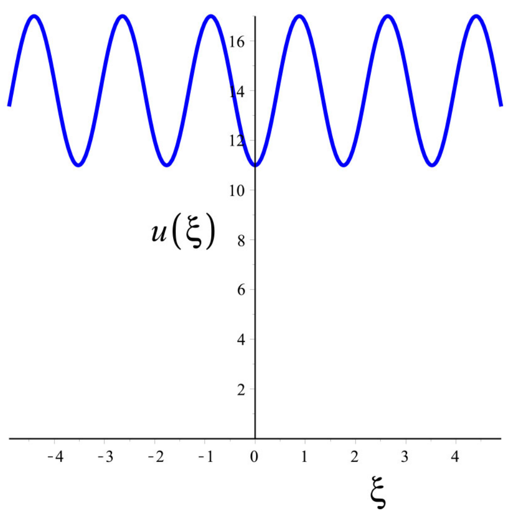

Figure 5. The 2D plot of (3.15).

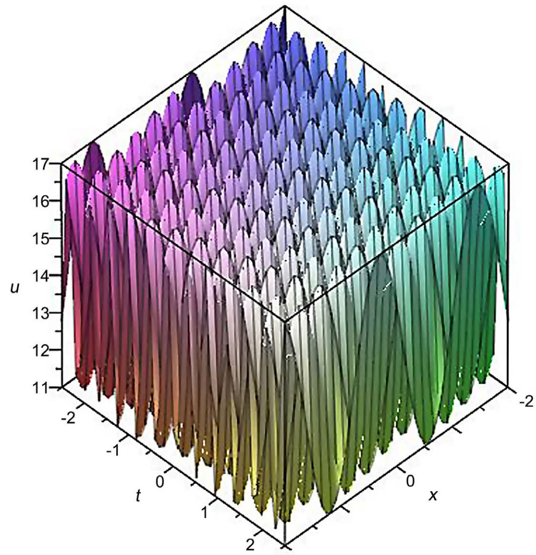

Figure 6. The 3D plot of (3.15).

2) $A<0$, $q>0$, we obtain

$$u(\xi)=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}\pm\sqrt{\frac{-6A}{q}}\left[\frac{\alpha_4(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_2-\alpha_4}\right], \quad (3.16)$$

and

$$u(\xi)=-\frac{2p\pm q\gamma_1\sqrt{\frac{-6A}{q}}}{4q}\pm\sqrt{\frac{-6A}{q}}\left[\frac{\alpha_2(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4}\right], \quad (3.17)$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_4-\alpha_3)(\alpha_2-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

Taking $A=-6$, $\alpha_1=1$, $\alpha_2=2$, $\alpha_3=3$, $\alpha_4=4$, $p=1$, $q=1$, $\gamma_1=1$, $\gamma_2=1$, $\gamma_3=\frac{23}{54}$, $c_1=1$, $\beta_1=6$, $\beta_0=-2$, $c=-4$, the plane images of these solutions are shown in **Figures 7-10**, respective.

3.3. A Fourth-Order Integrable Nonlinear Schrödinger Equation

Consider a fourth-order integrable nonlinear Schrödinger equation [28] [29]

$$\begin{aligned} &i\psi_t+\psi_{xx}+2|\psi|^2\psi+\gamma\left(\psi_{xxxx}+8|\psi|^2\psi_{xx}+6\bar{\psi}\psi_x^2\right.\\ &\left.+4\psi|\psi_x|^2+2\psi^2\bar{\psi}_{xx}+6|\psi|^4\psi\right)=0, \end{aligned} \quad (3.18)$$

where $\psi(x,t)$ is a function defined in complex, $\psi(x,t)$ and $\bar{\psi}(x,t)$ are complex conjugates of each other, γ is a non-zero real constant.

Firstly, we assume that

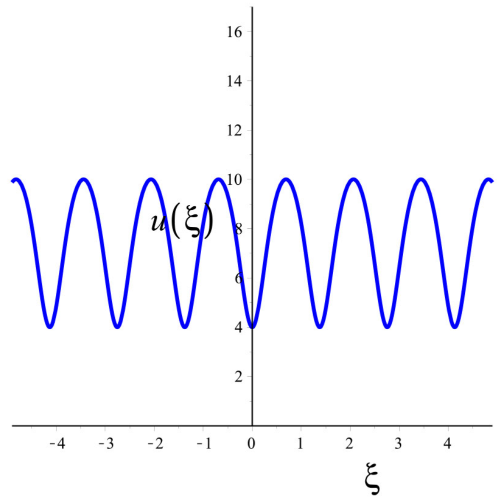

Figure 7. The 2D plot of (3.16).

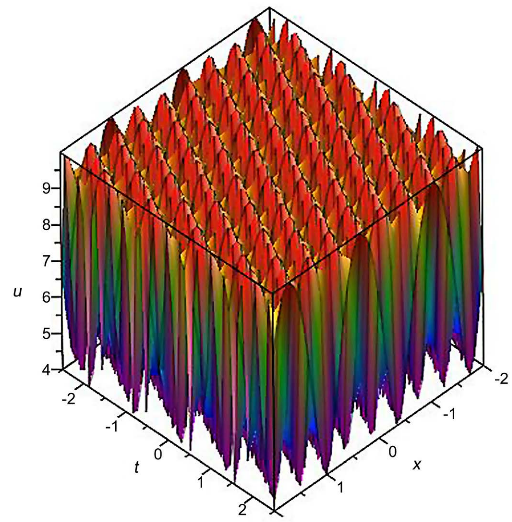

Figure 8. The 3D plot of (3.16).

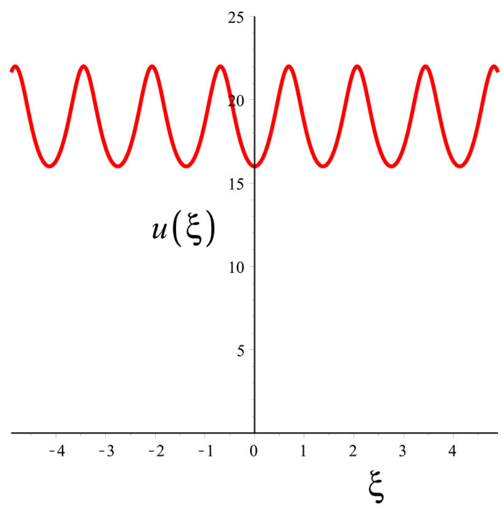

Figure 9. The 2D plot of (3.17).

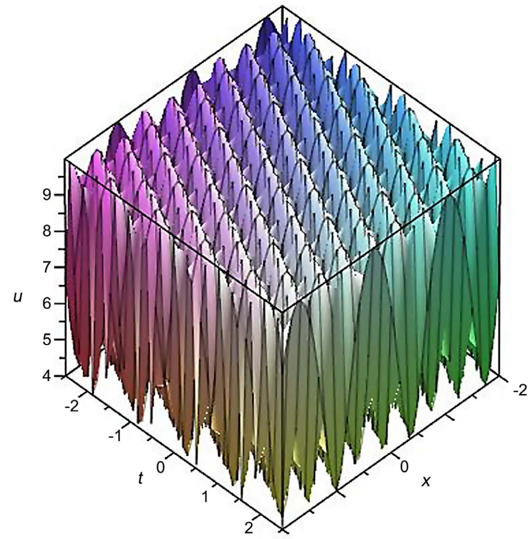

Figure 10. The 3D plot of (3.17).

$$\psi(x,t)=u(\xi)\mathrm{e}^{i\delta(x,t)},\quad \xi=x-ct,\quad \delta(x,t)=\kappa x-\eta t, \tag{3.19}$$

where c is the wave speed, $u(\xi)$ is a real function, κ is non-zero constant.

Substitute (3.19) to Equation (3.18), and letting the real part and imaginary part be zero, respectively. We obtain

$$\begin{aligned}&6\gamma u^5+\left(2-12\gamma\kappa^2\right)u^3+\left(\eta-\kappa^2+\gamma\kappa^4\right)u+10\gamma uu'^2\\&+10\gamma u^2u''+\left(1-6\gamma k^2\right)u''+\gamma u^{(4)}=0,\end{aligned} \tag{3.20}$$

and

$$4\gamma\kappa u^{(3)}+\left(2k-4\gamma\kappa^3-c\right)u'+24\gamma\kappa u^2u'=0, \tag{3.21}$$

where $u'=\frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi}$. Integrating Equation (3.21) once yields

$$4\gamma\kappa u''+\left(2k-4\gamma\kappa^3-c\right)u+8\gamma\kappa u^3=0. \tag{3.22}$$

Substitute Equation (3.22) to Equation (3.20), we get

$$\left(u^2(\xi)+\lambda_1\right)u''(\xi)-\frac{1}{2}u\left[u'(\xi)\right]^2+\frac{3}{2}u^5(\xi)+\lambda_2u(\xi)=0, \tag{3.23}$$

where

$$\lambda_1=\frac{4\gamma\kappa^3+c-2\kappa}{16\kappa\gamma},\quad \lambda_2=-\frac{5}{4}\kappa^4+\frac{3\kappa^2}{4\gamma}-\frac{3c\kappa}{8\gamma}+\frac{\eta}{4\gamma}-\frac{1}{8\gamma^2}+\frac{c}{16\gamma^2\kappa}.$$

From Equation (3.3), Equation (1.6) can be transformed into the following form

$$\left[\varphi'(\xi)\right]^2=A\left(\varphi^4(\xi)-\gamma_1\varphi^3(\xi)+\gamma_2\varphi^2(\xi)-\gamma_3\varphi(\xi)+\gamma_4\right), \tag{3.24}$$

where $\gamma_4=\alpha_1\alpha_2\alpha_3\alpha_4$. According to the elliptic equation method, we know that the solution of Equation (3.23) can be expressed as follows:

$$u(\xi)=\sum_{i=0}^{N}\beta_i\varphi^i(x,t)=\sum_{i=0}^{N}\beta_i\varphi^i(\xi), \tag{3.25}$$

where $\varphi(\xi)$ satisfy Equation (3.3) and Equation (3.24). Balancing the term $u^5(\xi)$ with term $u^2(\xi)u''(\xi)$ in Equation (3.23), we obtain $N=1$. Therefore Equation (3.23) has the following solution

$$u(\xi)=\beta_0+\beta_1\varphi(\xi), \tag{3.26}$$

where $\beta_1 \neq 0$. Substituting Equation (3.3), Equation (3.24) and Equation (3.26) into Equation (3.23) and setting the coefficients of $\varphi^i\,(i=0,1,2,3)$ to zero, we get the following algebraic system

$$\begin{aligned}
&\varphi^5: \beta_1^3 A+\beta_1^5=0,\\
&\varphi^4: \frac{7}{2}\beta_0\beta_1^2 A-\beta_1^3 A\gamma_1+\frac{15}{2}\beta_0\beta_1^4=0,\\
&\varphi^3: 2\left(\beta_0^2+\lambda_1\right)\beta_1 A-\frac{5}{2}\beta_0\beta_1^2 A\gamma_1+\frac{1}{2}\beta_1^3 A\gamma_2+15\beta_0^2\beta_1^3=0,\\
&\varphi^2: -\frac{3}{2}\left(\beta_0^2+\lambda_1\right)\beta_1 A\gamma_1+\frac{3}{2}\beta_0\beta_1^2 A\gamma_2+15\beta_0^3\beta_1^2=0,\\
&\varphi^1: \left(\beta_0^2+\lambda_1\right)\beta_1 A\gamma_2-\frac{1}{2}\beta_0\beta_1^2 A\gamma_3-\frac{1}{2}\beta_1^3 A\gamma_4+\frac{15}{2}\beta_0^4\beta_1+\lambda_2\beta_1=0,\\
&\varphi^0: -\frac{1}{2}\left(\beta_0^2+\lambda_1\right)\beta_1 A\gamma_3-\frac{1}{2}\beta_0\beta_1^2 A\gamma_4+\frac{3}{2}\beta_0^5+\lambda_2\beta_0=0.
\end{aligned} \tag{3.27}$$

With the help of Maple, we obtain

$$\begin{aligned}
&\beta_1=\pm\sqrt{-A},\quad \beta_0=\mp\frac{1}{4}\sqrt{-A}\gamma_1,\quad \lambda_1=\frac{1}{4}A\gamma_2-\frac{3A\gamma_1^2}{32},\\
&\gamma_4=-\frac{1}{2}\frac{1}{A^2}\left(A^2\gamma_2^2-\frac{5}{8}A^2\gamma_1^2\gamma_2-\frac{1}{2}A^2\gamma_1\gamma_3+\frac{15A^2\gamma_1^4}{128}+4\lambda_2\right),
\end{aligned} \tag{3.28}$$

where $A<0$, $\gamma_1,\gamma_2,\gamma_3,\lambda_2$ are arbitrary constants. Substituting (3.28) into Equation (3.26), we get the following travelling wave solutions of Equation (3.23)

$$u(\xi)=\mp\frac{1}{4}\sqrt{-A}\gamma_1\pm\sqrt{-A}\varphi(\xi), \tag{3.29}$$

where $\varphi(\xi)$ satisfy Equation (3.3) and Equation (3.24). From (2.4), (2.5) and Equation (3.29), we get the elliptic function solutions of Equation (3.23) as follows:

$$u(\xi)=\mp\frac{1}{4}\sqrt{-A}\gamma_1\pm\sqrt{-A}\left[\frac{\alpha_4(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_2-\alpha_4}\right], \tag{3.30}$$

and

$$u(\xi)=\mp\frac{1}{4}\sqrt{-A}\gamma_1\pm\sqrt{-A}\left[\frac{\alpha_2(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4}\right], \tag{3.31}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_4-\alpha_3)(\alpha_2-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

To the best of our knowledge, the solutions (3.30) and (3.31) have not been obtained in other papers.

3.4. Generalized Dullin-Gottwald-Holm Equation

We consider generalized Dullin-Gottwald-Holm equation [30] [31] [32] [33]

$$u_t - au_{xxt} + 2bu_x + cu^3u_x + du_{xxx} = a\left(2u_xu_{xx} + uu_{xxx}\right). \tag{3.32}$$

In [33], the authors studied the bifurcation of Equation (3.32) and obtained some travelling wave solutions by the modified simplest equation method. To our knowledge, there is no study on the exact solutions for Equation (3.32) other than [33]. Using (2.1)-(2.5), we will get some new traveling wave solutions to Equation (3.32).

Firstly, we introduce the traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - \mu t. \tag{3.33}$$

Substituting (3.33) into (3.32), integrating once and letting the integration constant be zero, we get

$$\lambda\left[u(\xi) - \beta\right]u''(\xi) + \frac{1}{2}\lambda u'^2(\xi) - \alpha u(\xi) - u^4(\xi) = 0, \tag{3.34}$$

where $\alpha = \frac{4(2b-\mu)}{c}$, $\beta = \frac{d}{a} - \mu$, $\lambda = \frac{4a}{c}$, $\lambda \neq 0$. We assume the solutions of Equation (3.34) as follows:

$$u(\xi) = \sum_{i=0}^{N} k_i\varphi^i(x,t) = \sum_{i=0}^{N} k_i\varphi^i(\xi). \tag{3.35}$$

Substituting (3.35) into Equation (3.34) and balancing the term $u^4(\xi)$ with term $u(\xi)u''(\xi)$ in Equation (3.34), we obtain $N = 1$. Therefore Equation (3.34) has the following solution

$$u(\xi) = k_0 + k_1\varphi(\xi), \tag{3.36}$$

where $k_1 \neq 0$. Substituting Equation (3.3), Equation (3.24) and Equation (3.36) into Equation (3.34), and taking the coefficients of $\varphi^i\left(i = 0,1,2,3,4\right)$ to zero, we get the following system of algebraic equations

$$\begin{aligned}
&\varphi^4 : \frac{5}{2}\lambda k_1^2 A - k_1^4 = 0,\\
&\varphi^3 : 2\lambda(-\beta + k_0)k_1 A - 2\lambda k_1^2 A\gamma_1 - 4k_0k_1^3 = 0,\\
&\varphi^2 : -\frac{3}{2}\lambda(-\beta + k_0)k_1 A\gamma_1 + \frac{3}{2}\lambda k_1^2 A\gamma_2 - 6k_0^2k_1^2 = 0,\\
&\varphi^1 : \lambda(-\beta + k_0)k_1 A\gamma_2 - \lambda k_1^2 A\gamma_3 - \alpha k_1 - 4k_0^3k_1 = 0,\\
&\varphi^0 : -\frac{1}{2}\lambda(-\beta + k_0)k_1 A\gamma_3 + \frac{1}{2}\lambda k_1^2 A\gamma_4 - \alpha k_0 - k_0^4 = 0.
\end{aligned} \tag{3.37}$$

With the help of Grobner basis elimination method and Maple, we obtain the following travelling wave solutions:

Case I: If $k_0 = 0$, γ_1, k_1, A and λ are non-zero constants, we get

$$\begin{aligned} &k_0=0,\quad \gamma_2=\gamma_1^2,\quad k_1=-\frac{\beta}{\gamma_1},\\ &A=\frac{2}{5}\frac{\beta^2}{\gamma_1^2\lambda},\quad \gamma_4=\gamma_1\gamma_3,\quad \alpha=-\frac{2}{5}\frac{\beta^3\left(\gamma_1^3-\gamma_3\right)}{\gamma_1^3}, \end{aligned} \tag{3.38}$$

where $\lambda,\beta,\gamma_1,\gamma_3$ are arbitrary constants. Substituting (3.38) into Equation (3.36), we get the following travelling wave solutions of Equation (3.34)

$$u(\xi)=-\frac{\beta}{\gamma_1}\varphi(\xi), \tag{3.39}$$

where $\varphi(\xi)$ satisfy Equation (3.3) and Equation (3.24). From (2.4), (2.5) and Equation (3.29), we get the elliptic function solutions of Equation (3.23) as follows:

1) $\lambda>0$, we obtain

$$u(\xi)=-\frac{\beta}{\gamma_1}\frac{\alpha_2(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4}, \tag{3.40}$$

$$u(\xi)=-\frac{\beta}{\gamma_1}\frac{\alpha_3(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_4(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3}, \tag{3.41}$$

and

$$u(\xi)=-\frac{\beta}{\gamma_1}\frac{\alpha_1(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3}, \tag{3.42}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_3-\alpha_2)(\alpha_4-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

2) $\lambda<0$, we get

$$u(\xi)=-\frac{\beta}{\gamma_1}\frac{\alpha_4(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_2-\alpha_4}, \tag{3.43}$$

and

$$u(\xi)=-\frac{\beta}{\gamma_1}\frac{\alpha_2(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4}, \tag{3.44}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_4-\alpha_3)(\alpha_2-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

Case II: If $\lambda A>0$, k_1 is a non-zero constant, we get

$$\begin{aligned}
&k_0=-\frac{1}{4}\left(\beta\pm\gamma_1\sqrt{\frac{5\lambda A}{2}}\right),\quad k_1=\pm\sqrt{\frac{5\lambda A}{2}},\quad \gamma_2=\frac{\beta^2+3\beta k_0+6k_0^2}{k_1^2},\\
&\alpha=-\frac{2}{5}k_1^3\gamma_3-\frac{2}{5}\beta^3-\frac{4}{5}\beta^2k_0-\frac{6}{5}\beta k_0^2-\frac{8}{5}k_0^3,\\
&\gamma_4=-\frac{1}{k_1^4}\left(\beta k_1^3\gamma_3+k_0k_1^3\gamma_3+2\beta^3k_0+4\beta^2k_0^2+6\beta k_0^3+3k_0^4\right),
\end{aligned}\tag{3.45}$$

where $A,\lambda,\beta,\gamma_1,\gamma_3$ are arbitrary constants. From Equation (3.36), Equation (3.38) and (2.1)-(2.5), we obtain

1) $A>0$, $\lambda>0$, we get

$$u(\xi)=-\frac{1}{4}\left(\beta\pm\gamma_1\sqrt{\frac{5\lambda A}{2}}\right)\pm\sqrt{\frac{5\lambda A}{2}}\frac{\alpha_2(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4},\tag{3.46}$$

$$u(\xi)=-\frac{1}{4}\left(\beta\pm\gamma_1\sqrt{\frac{5\lambda A}{2}}\right)\pm\sqrt{\frac{5\lambda A}{2}}\frac{\alpha_3(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_4(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_4)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3},\tag{3.47}$$

and

$$u(\xi)=-\frac{1}{4}\left(\beta\pm\gamma_1\sqrt{\frac{5\lambda A}{2}}\right)\pm\sqrt{\frac{5\lambda A}{2}}\frac{\alpha_1(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_2-\alpha_3)sn^2\left(\frac{\sqrt{A}}{2}\xi\beta,k\right)-\alpha_1+\alpha_3},\tag{3.48}$$

where $\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}$, $k=\sqrt{\frac{(\alpha_3-\alpha_2)(\alpha_4-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$.

2) $A>0$, $\lambda>0$, it follows that

$$u(\xi)=-\frac{1}{4}\left(\beta\pm\gamma_1\sqrt{\frac{5\lambda A}{2}}\right)\pm\sqrt{\frac{5\lambda A}{2}}\frac{\alpha_4(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)+\alpha_2-\alpha_4},\tag{3.49}$$

and

$$u(\xi)=-\frac{1}{4}\left(\beta\pm\gamma_1\sqrt{\frac{5\lambda A}{2}}\right)\pm\sqrt{\frac{5\lambda A}{2}}\frac{\alpha_2(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_3-\alpha_4)sn^2\left(\frac{\sqrt{-A}}{2}\xi\beta,k\right)-\alpha_2+\alpha_4},\tag{3.50}$$

where

$$\beta=\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)},\quad k=\sqrt{\frac{(\alpha_4-\alpha_3)(\alpha_2-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}.$$

The solutions (3.46)-(3.50) have not been obtained in [33], they are new ones.

4. Conclusion

In summary, we have obtained five new solutions for Equation (1.6), including three periodic solutions and two singular solutions. These solutions are not obtained in other papers. These new solutions are applied to three differential equations, including combined KdV-MKdV equation, a fourth-order integrable nonlinear Schrödinger equation and generalized Dullin-Gottwald-Holm equation, we have obtained new travelling wave solutions for these equations, respectively.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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