An Exponential *Lévy* Model for Stocks Paying Discrete Dividends

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Abstract: By introducing the impact of discrete stochastic dividends on stock price dynamics, this article proposes an exponential *Lévy* model with dividend jump, and derives stock pricing formula in three cases respectively. One is that dividends are announced and paid at the same time; another is that dividends are announced in advance; the third is that dividend policy can be changed with regime.

Keywords: exponential lévy model; discrete dividends; stock pricing formula

1 Introduction

So far, stock pricing has been studied from different perspectives and methods by a lot of papers, we cite in particular [1-4]. The literatures have proposed a series of stock pricing models, such as Black-Scholes model, an exponential *Lévy* model, dividend discount Model and so on. In most cases, the stock will pay dividends in actual financial markets, so in recent years, studying the price dynamics of stock paying discrete stochastic dividends has become a hot, see[5,6]. However, with regard to exponential *Lévy* model mentioned in [1], as far as authors know, there is still no article to study the impact of stochastic dividends on stock pricing. This article considers the price dynamics of stock paying discrete stochastic dividends, proposes an exponential *Lévy* model with dividend jump, and deduces stock pricing formula in three cases respectively. One is that dividends are announced and paid at the same time; another is that dividends are announced in advance; the third is that dividend policy can be changed with regime.

2 The Exponential *Lévy* Model

Based on the existing works of the article [1], this section considers pricing of stock paying discrete stochastic dividends and proposes an exponential *Lévy* with dividend jump. Details are as follows:

(1) One riskless asset, such as short-term debt, its pricing formula is

$$B_t = \exp(rt).$$

Namely, risk-free interest rate is a constant, denoted by r.

(2) A stock price process paying discrete dividends can be denoted by $\{S(t), t \geq 0\}$, where $S(t)$ is the stock price at time t.

Prior to the first payment of dividend and between the adjacent two dividend-paying, the stock price process $\{S(t);\ t \geq 0\}$ follows the path of an exponential *Lévy* process.

Namely, $\forall\, t_j \leq t < t_{j+1}$, $j = 0, 1, 2, 3, \ldots\ldots$, we have

$$S(t) = S(t_j)\exp[X_{t-t_j}] = S(t_j)Y_{t-t_j}$$

where X_t is a *Lévy* process and $X(0) = 0$. Define a process Y_t as follows:

$$Y_t =: \exp[X_t],$$

then Y_t is an exponential *Lévy* process.

We assume there are times $t_k = kh;\ k = 1, 2, 3, \ldots.$, and at each time t_j the dividend paid is $D_j = a_j S(t_j^-)$; where $S(t_j^-)$ denotes the stock price just before the moment of the dividend payment. We assume that each a_j, j=1, 2,… is a positive constant number between 0 and 1. We set $t_0 = 0$, but it is not a Dividend-paying time.

3 Stock Pricing Formula

3.1 Stock Pricing Formula in Case of Dividends Announced and Paid at the Same Time

In this subsection, we consider the stock price process where dividends will be announced at time kh, k = 1, 2,⋯ and be paid at the same time. Specific ways about paying dividends can be seen in section 2. The stock price $S(tj\,)$ after dividend payment is the stock price just before the moment of dividend payment less corresponding dividend payment:

$$S(t_j) = S(t_j^-) - a_j S(t_j^-)$$

Lemma 3.1

$$Y_s \cdot Y_{t-s} = Y_t, \quad \forall_s \leq t \tag{1}$$

proof: By the definition of $Y(t)$, we have

$$\begin{aligned} Y_s \cdot Y_{t-s} &= \exp[X_s]\cdot\exp[X_{t-s}] \\ &= \exp[X_s + X_{t-s}] \end{aligned} \tag{2}$$

According to the property of stationary independent increments of *Lévy* process [7], we have

$$X_s + X_{t-s} = X_t \, .$$

Substituting above formula into (2), we get

$$Ys \cdot Yt{-}s = \exp[Xt] = Yt$$

Theorem 3.1 Given $a_0 = 0$, for $\forall t_k \le t < t_{k+1}$, $k = 0, 1, 2, \ldots$, the stock pricing formula is

$$S(t) = (1 - a_1)(1 - a_2)\ldots(1-a_k)\, S_0 \cdot Y_t, \tag{3}$$

which implies,

$$S(t_{k+1}^-) = (1-a_1)(1-a_2)\cdots(1-a_k)S_0 \cdot Y_{tk+1} \tag{4}$$

and

$$S(t_{k+1}) = (1-a_1)(1-a_2)\cdots(1-a_{k+1})S_0 Y_{t_{k+1}} \tag{5}$$

Proof: (1) When $k = 0$, for $0 \le t < t_1$,

$$S(t) = S_0 Y_t = S_0 \cdot \exp(X_t) \tag{6}$$

Because X is a continuous process, so $X(t_{j+1}^-) = X(t_{j+1})$. Substituting $t = t_1^-$ into (6), we have

$$S(t_1^-) = S_0 Y_{t_1^-} = S_0 \exp(X_{t1}) = S_0 Y_{t1} \tag{7}$$

In the case $t = t_1$, using (7), this leads to

$$\begin{aligned} D_1 &= a_1 S(t_1^-) \\ &= a_1 S_0 Y_{t1} \end{aligned}$$

then we have

$$S(t_1) = S(t_1^-) - D_1 = S_0 Y_{t_1} - a_1 S_0 Y_{t_1} = (1-a_1) S_0 Y_{t_1}$$

For $k = 0$, the theorem holds.

(2) Suppose the theorem holds for $k = n$, $n \ge 0$, we shall prove that the theorem holds for $k = n + 1$. Based on the assumption, for $t_n \le t < t_{n+1}$,

$$S(t) = (1-a_1)(1-a_2)\cdots(1-a_n)S_0 \cdot Y_t$$

$$S(t_{n+1}^-) = (1-a_1)(1-a_2)\cdots(1-a_n)S_0 \cdot Y_{t_{n+1}}$$

$$S(t_{n+1}) = (1-a_1)(1-a_2)\cdots(1-a_{n+1})S_0 Y_{t_{n+1}}$$

When $k = n + 1$, for $t_{n+1} \le t < t_{n+2}$, the stock pricing process follows the path of an exponential *Lévy* process with $S(t_{n+1})$ as initial value. By lemma 3.1, we have

$$\begin{aligned} S(t) &= S(t_{n+1}) \cdot Y_{t-tn+1} \\ &= (1-a_1)(1-a_2)\cdots(1-a_{n+1})S_0 Y_{t_{n+1}} y_{t-t_{n+1}} \\ &= (1-a_1)(1-a_2)\cdots(1-a_n)(1-a_{n+1})S_0 Y_t \end{aligned} \tag{8}$$

In the case $t = t_{n+2}^-$, using (8), this leads to

$$\begin{aligned} S(t_{n+2}^-) &= (1-a_1)(1-a_2)\cdots(1-a_n)(1-a_{n+1})S_0 Y_{t_{n+2}^-} \\ &= (1-a_1)(1-a_2)\cdots(1-a_n)(1-a_{n+1})S_0 \exp(X_{t_{n+2}^-}) \end{aligned}$$

X is a continuous process, so $X(t_{n+2}^-) = X(t_{n+2})$. Then we have

$$\begin{aligned} & S(t_{n+2}^-) \\ &= (1-a_1)(1-a_2)\cdots(1-a_n)(1-a_{n+1})S_0 \exp(X_{t_{n+2}}) \\ &= (1-a_1)(1-a_2)\cdots(1-a_n)(1-a_{n+1})S_0 Y_{t_{n+2}} \end{aligned}$$

In the case $t = t_{n+2}$

$$\begin{aligned} D_{n+2} &= a_{n+2} S(t_{n+2}^-) \\ &= a_{n+2}(1-a_1)(1-a_2)\cdots(1-a_n)(1-a_{n+1})S_0 Y_{t_{n+2}} \, , \end{aligned}$$

Then we have

$$\begin{aligned} S(t_{n+2}) &= S(t_{n+2}^-) - D_{n+2} \\ &= (1-a_1)(1-a_2)\cdots(1-a_{n+1})(1-a_{n+2})S_0 Y_{t_{n+2}} \end{aligned}$$

The proof of theorem is completed.

Remarks: In fact, about the development of stock price process, the intuitive understanding is very important. We need deriving stock pricing formula recursively at first and then prove it rigorously. By theorem 3.1, we know that the stock price process follows the path of an exponential *Lévy* process with jumps at dividend-paying time. For $t_k \le t < t_{k+1}$, $k = 0, 1, 2, \ldots$, k times dividends have been paid, so the parameters $a_1, a_2, \ldots, a_k$ are all known. We also know the value of exponential *Lévy* process Y_t, so the stock price at time t can be calculated immediately by (3).

3.2 Stock Pricing Formula in Case of Dividends Announced in Advance

In this subsection, we assumes that the dividends paid at time $t_k = kh$ are announced at time $(k-1)h + \varepsilon h$ and are equal to

$$D_k = a_k S((k-1+\varepsilon)h^-) \tag{9}$$

where ε is a given constant in (0, 1), a_k is a positive constant in (0,1) and $k = 1, 2, \ldots$. Let $b_k = e^{-r(1-\varepsilon)h} a_k$, the present value of future dividends at the announcement moment is

$$\begin{aligned} D_k^* &= e^{-r(1-\varepsilon)h} D_k \\ &= b_k S((k-1+\varepsilon)h^-) \end{aligned} \tag{10}$$

For $kh \le t < kh + \varepsilon h$, no dividend is announced or paid, then the stock pricing process follows the path of an exponential *Lévy* process with $S(kh)$ as initial value. For $kh + \varepsilon h \le t < (k+1)h$, namely from the announcement time until the moment of the dividend payment, the stock price $S(t)$ includes ex-dividend stock price process $S^{ex}(t)$ and the present value of the next known dividend payment at the time t,

$$\begin{aligned} S(t) &= S^{ex}(t) + D_{k+1} e^{-r((k+1)h-t)} \\ &= S^{ex}(t) + D_{k+1}^* e^{r(t-\varepsilon h)} \end{aligned} \tag{11}$$

Theorem 3.2 Set $b_0 = 0$, for $kh \le t < kh + \varepsilon h$, $k = 0, 1, 2, \ldots$ the stock pricing formula is

$$S(t) = (1-b_1)(1-b_2)\cdots(1-b_k)S_0Y_t \quad (12)$$

for $kh + \varepsilon h \le t < (k + 1)h$, the stock pricing formula is

$$S(t) = (1-b_1)(1-b_2)\cdots(1-b_k)S_0 \times [(1-b_{k+1})Y_t + b_{k+1}Y_{kh+\varepsilon h}e^{r(t-kh-\varepsilon h)}] \quad (13)$$

Proof: (1) When $k = 0$, for $0 \le t < \varepsilon h$, because X is a continuous process, from the definition of $Y(t)$, we have

$$S(t) = S_0Y_t, \quad S_{\varepsilon h-} = S_0Y_{\varepsilon h}, \quad D_1 = a_1S_{\varepsilon h-}.$$

For $\varepsilon h \le t < h$, let $S^{ex}(t)$ denote the ex-dividend stock price process, then $S^{ex}(t)$ follows the path of an exponential *Lévy* process with $S_{\varepsilon h-} - D_1^*$ as initial value. Using (9) and (10), we have

$$S^{ex}(t) = [S_0Y_{\varepsilon h} - D_1^*]Y_{t-\varepsilon h} = (1-b_1)S_0Y_t$$

Then by (10) and (11), we have

$$S(t) = Sex(t) + D_1^*e^{r(t-\varepsilon h)} = (1-b_1)S_0Y_t + b_1S_0Y_{\varepsilon h}e^{r(t-\varepsilon h)} \quad (14)$$

From (14), we have

$$S(h^-) = (1-b_1)S_0Y_{h-} + b_1S_0Y_{\varepsilon h}e^{r(h-\varepsilon h)}$$

According to (10), we have

$$b_1S_0Y_{\varepsilon h}e^{r(h-\varepsilon h)} = D_1^*e^{r(h-\varepsilon h)} = D_1$$

Because Y_t is a continuous process, we have

$$S(h) = S(h^-) - D_1 = (1-b_1)S_0Y_{h^-} = (1-b_1)S_0Y_h$$

Thus, when $k = 0$, the theorem holds.

(2) Suppose the theorem holds when $k = n$, namely, for $nh \le t < nh + \varepsilon h$,

$$S(t) = (1-b_1)(1-b_2)\cdots(1-b_n)S_0Y_t$$

for $nh + \varepsilon h \le t < (n + 1)h$,

$$S(t) = (1-b_1)(1-b_2)\cdots(1-b_n)S_0 \times [(1-b_{n+1})Y_t + b_{n+1}Y_{nh+\varepsilon h}e^{r(t-nh-\varepsilon h)}] \quad (15)$$

From (15), we have

$$S((n+1)h^-) = (1-b_1)(1-b_2)\cdots(1-b_n)S_0 \times [(1-b_{n+1})Y_{(n+1)h^-} + b_{n+1}Y_{nh+\varepsilon h}e^{r(h-\varepsilon h)}]$$

According to (10), we have

$$b_{n+1}Y_{nh+\varepsilon h}e^{r(h-\varepsilon h)} = D_{n+1}^*e^{r(h-\varepsilon h)} = D_{n+1}.$$

Because Y_t is a continuous process, we have

$$S((n+1)h) = S((n+1)h^-) - D_{n+1} = (1-b_1)(1-b_2)\cdots(1-b_{n+1})S_0Y_{(n+1)h}$$

When $k = n + 1$, for $(n + 1)h \le t < (n + 1)h + \varepsilon h$, the stock pricing process follows the path of an exponential Lévy process with $S((n + 1)h)$ as initial value, namely

$$S(t) = S((n+1)h)Y_{t-(n+1)h} = (1-b_1)(1-b_2)\cdots(1-b_n)(1-b_{n+1})S_0Y_t$$

for $(n+1)h + \varepsilon h \le t < (n+2)h$, using (10) and (11), this leads to

$$\begin{aligned} S(t) &= S^{ex}(t) + D_{n+2}^*e^{r(t-(n+1)h-\varepsilon h)} \\ &= [(1-b_1)(1-b_2)\cdots(1-b_n)(1-b_{n+1})S_0Y_{(n+1)h+\varepsilon h} \\ &\quad - D_{n+2}^*]Y_{t-(n+1)h-\varepsilon h} + D_{n+2}^*e^{r(t-(n+1)h-\varepsilon h)} \\ &= (1-b_1)(1-b_2)\cdots(1-b_{n+1})S_0[(1-b_{n+2})Y_t + b_{n+2}Y_{(n+1)h+\varepsilon h}e^{r(t-(n+1)h-\varepsilon h)}] \end{aligned}$$

The proof of theorem is completed.

Remarks: In case of dividends announced in advance, for $kh \le t < kh + \varepsilon h$, $k = 0, 1, 2, \ldots$, k times dividends have been paid, so parameters $a_1, a_2, \ldots, a_k$ are all known, and the values of $b_1, b_2, \ldots, b_k$ can be deduced. The value of exponential *Lévy* process Y_t at time t is also known, so the stock price at time t can be calculated immediately according to (12). For $kh + \varepsilon h \le t < (k + 1)h$, k previous dividends have been paid, while the $(k + 1)$-th dividends have already been announced in advance, so parameters $a(1), a(2), \ldots, a(k + 1)$ are all known. Then the values of $b_1, b_2, \ldots, b_{k+1}$ can be deduced. The value of exponential *Lévy* process Y_t at time t is also known, so the stock price at time t can be calculated immediately according to (13).

3.3 Stock Pricing Formula in Case of Changing Dividend Policy

Theorem 3.1 and Theorem 3.2 give the stock pricing formulas in the case of dynamic changes of the proportion of dividend payment. The results can be generalized to the more general case-dividend policy modulated by an external Markov chain.

We extend the model in section II to incorporate the dependence of dividend payment on the external economic environment, where the latter is modeled by a finite-state Markov chain $\{I(t)\}$. Suppose that the stock pays dividends Dj at discrete times $tj = jh$, the dividends have the form

$$D_j = a(I(t_j^-))S(t_j^-)$$

where $S(t_j^-)$ denotes the stock price just before the moment of the dividend payment. We assume that $a(I(t_j^-))$, $j = 1, 2, \ldots$ is a function depending on the state of $\{I(t)\}$ at time t. We set $t_0 = 0$, but it is not a dividend payment time. Suppose $\{I(t);\ t \geq 0\}$ is a homogeneous, irreducible and recurrent markov process with values in the set of $I = \{1, 2, 3, \ldots, m\}$ and its intensity matrix is $\Lambda = (\alpha_{ij})$, where $\alpha_{ii} = -\alpha_i$. The transition probability matrix of the corresponding Embedded chain of $\{I(t);\ t \geq 0\}$ is

$$P = (\rho_{ij}),\ \ \rho_{ij} = \begin{cases} 0, & i = j \\ \dfrac{\alpha_{ij}}{\alpha_i}, & i \neq j \end{cases}\ \ i, j \in J$$

Using theorem 3.1, we have

Theorem 3.3 Assume that the dividends are paid and announced at the same time, let $a_0 = 0$, for $t_k \leq t < t_{k+1}$, $k = 0, 1, 2, \ldots$, the stock pricing formula is

$$S(t) = (1 - a(I(t_1^-)))(1 - a(I(t_2^-)))\cdots(1 - a(I(t_k^-)))S_0 Y_t$$

proof: Completely similar to the proof of Theorem 3.1.

For Theorem 3.2, it has a similar extention. We will not repeat them here.

4 Conclusions

Based on the reality of stock paying discrete stochastic dividends in actual financial market, this article, considering the impact of discrete stochastic dividends on stock price dynamics, proposes an exponential *Lévy* model with dividend jump, and deduces the concrete stock pricing formulas in three cases. The method in this article has several advantages: there is no difficulty in coping with dividends; the stock price can be easily calculated at any time. In the future, it is deserved to study the formulas of European option and American option of exponential *Lévy* model with dividend jump.

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References

[1] Schoutens W., *Exotic options under levy models: An overview*, Journal of Computational and Applied Mathematics, 2006, 189: 526-538.

[2] Korn R., Rogers L. C. G., *Stock paying discrete dividends: Modeling and option pricing*, Journal of Derivatives, 2005, 13: 44-49.

[3] Black F., Scholes M., *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 1973, 81: 637-659.

[4] Chen Xu, Wan Jianping, *Option Pricing for Time-change Exponential levy Model Under Memm*, Acta Mathematicae Applicatae Sinica, 2007, 23(4): 651-664.

[5] Sakkas E., Le H., *A Markov-modulated model for stocks paying discrete dividends*, Insurance: Mathematics and Economics, 2009, 45: 19-24.

[6] Cassimon, D. Engelen, P. J. Thomassen, L. Van Wouwe, M., *Closed-form valuation of American call options on stocks paying multiple dividends*, Finance Research Letters, 2007, 4: 33-48.

[7] Steven E. Shreve, *Stochastic Calculus for finance* II *Continuous-Time Models*, Springer, 2004.

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