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New Probability Distributions in Astrophysics: VII. The Truncated Gamma-Pareto Distribution Applied to Cosmic Rays

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Abstract

Many astrophysical phenomena are modeled by an inverse power law distribution at high values of the random variable but often at low values of the random variable we have a departure from an inverse power law. In order to insert a continuous transition from low to high values of the random variable we analyse the truncated gamma-Pareto distribution in two versions by deriving the most important statistical parameters. The application of the results to the distribution in energy of cosmic rays allows deriving an analytical expression for the average energy, which is 2.6 GeV.

Keywords

Cosmic Rays

1. Introduction

The gamma-Pareto distribution was introduced by [1], where its most important statistical parameters were derived. The gamma-Pareto probability density function (PDF) models the long right tail characteristics as well the gamma-type left behaviour. The first application was to the monthly rainfall data from Jatiwangi station, Jakarta [2]. A comparison between the generalized linear model (GLM) and the gamma-Pareto was made by [3]. An application to the length of hospital stays was made by [4]. The inter-aircraft distances between two closest flying passenger aircraft were analysed by [5]. Two extensions are the Gamma-Pareto (IV) [6] and the weighted gamma-Pareto distribution (WGPD) [7]. In astrophysics, many phenomena are modeled by a Pareto or power law distribution in the absence of more accurate models. Two examples are the distribution in energy of the cosmic rays (CR) and the radio-flux versus frequency in extra-galactic radio-sources. The above astrophysical models require a distribution that is more flexible at low values of the random variable and therefore Sections 2 and 3 analyse the gamma-Pareto and the gamma-Pareto II distributions which add to the Pareto distribution the flexibility of the gamma distribution. The effect of right-truncation and bi-truncation on gamma-Pareto and gamma-Pareto II distributions are analysed in Sections 4, 5 and 6. Section 7 deals with the transition from a probability distribution to a function. Section 8 applies the obtained results to a sample of the diameters of the asteroids and to the distribution in energy of CR.

2. The Gamma-Pareto Distribution

The gamma-Pareto has PDF

$$f(x;\alpha,c,\theta)=\frac{\left(\frac{\theta}{x}\right)^{\frac{1}{c}}\ln\left(\frac{x}{\theta}\right)^{\alpha-1}}{x\Gamma(\alpha)c^{\alpha}}, \tag{1}$$

and is defined for $\alpha>0$, $c>0$, $\theta>0$ and $x>\theta$, see formula (2.1) in [1]. Its distribution function (DF) is

$$F(x;\alpha,c,\theta)=1+\frac{-\Gamma\left(\alpha+1,\frac{\ln(x)-\ln(\theta)}{c}\right)+x^{-\frac{1}{c}}\theta^{\frac{1}{c}}c^{-\alpha}\left(\ln(x)-\ln(\theta)\right)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{2}$$

Its average value, or mean, μ, is

$$\mu(\alpha,c,\theta)=\theta(1-c)^{-\alpha}, \tag{3}$$

its variance, σ^2, is

$$\sigma^2(\alpha,c,\theta)=\theta^2\left((1-2c)^{-\alpha}-(1-c)^{-2\alpha}\right), \tag{4}$$

its *r*th moment about the origin, μ_r', is

$$\mu_r'(\alpha,c,\theta)=(-cr+1)^{-\alpha}\theta^r, \tag{5}$$

its skewness, $\tilde{\mu}_3$, is

$$\tilde{\mu}_3(\alpha,c,\theta)=\frac{-3(-2c+1)^{-\alpha}(-c+1)^{-\alpha}+2(-c+1)^{-3\alpha}+(-3c+1)^{-\alpha}}{\left((-2c+1)^{-\alpha}-(-c+1)^{-2\alpha}\right)^{\frac{3}{2}}}, \tag{6}$$

its kurtosis, $\tilde{\mu}_4$, is

$$\tilde{\mu}_4(\alpha,c,\theta)=\frac{N}{D}, \tag{7}$$

with

$$N=-4(-c+1)^{3\alpha}(-3c+1)^{-\alpha}(-2c+1)^{2\alpha}+(-c+1)^{4\alpha}(-4c+1)^{-\alpha}(-2c+1)^{2\alpha}+6(-c+1)^{2\alpha}(-2c+1)^{\alpha}-3(-2c+1)^{2\alpha}, \tag{8}$$

$$D=\left((-c+1)^{2\alpha}-(-2c+1)^{\alpha}\right)^2, \tag{9}$$

and the mode, $Mode$, is at

$$Mode(\alpha, c, \theta) = e^{\frac{(\alpha-1)c}{c+1}} \theta. \tag{10}$$

Random generation of the variate X is obtained by solving the nonlinear equation

$$F(x; \alpha, c, \theta) = R, \tag{11}$$

where R is the unit rectangular variate and F is given by Equation (2). Once the elements, x_i, of the experimental sample with i varying between 1 and n are given, the parameter θ can be derived from the minimum of the sample minus a small quantity, see the discussion in [1] [8]. The two remaining parameters, α and c, can be derived by the numerical solution of the two following equations, which arise from the maximum likelihood estimator (MLE),

$$-n\alpha c - n\ln(\theta) + \left(\sum_{i=1}^{n} \ln(x_i)\right) = 0, \tag{12a}$$

$$-n\Psi(\alpha) - n\ln(c) + \left(\sum_{i=1}^{n} \ln\left(\ln(x_i) - \ln(\theta)\right)\right) = 0, \tag{12b}$$

where $\Psi(z)$ is the digamma or Psi function defined as

$$\Psi(z) = \Gamma'(z)/\Gamma(z), \tag{13}$$

where $\mathcal{R}z > 0$, see [9]. The Pareto PDF [10] as defined in [11] is

$$f(x; a, c) = a^c x^{-c-1} c, \tag{14}$$

with $c > 0$ and $a > 0$. **Figure 1** compares the PDFs of the Pareto and the gamma-Pareto distributions.

A second PDF for comparison is the lognormal, which, according to [11], is

$$f_{LN}(x; m, \sigma) = \frac{e^{-\frac{1}{2\sigma^2}\left(\ln\left(\frac{x}{m}\right)\right)^2}}{x\sigma\sqrt{2\pi}}, \tag{15}$$

where m is the median and σ a shape parameter. **Figure 2** compares the gamma-Pareto and the lognormal.

The third distribution which will be used for comparison is the double Pareto lognormal, see formula (22) in [12], which has PDF

$$\begin{aligned} &f(x; \alpha, \beta, \mu, \sigma) \\ &= \frac{1}{2}\alpha\beta\left(e^{\frac{1}{2}\alpha\left(\alpha\sigma^2 + 2\mu - 2\ln(x)\right)} erfc\left(\frac{1}{2}\frac{\left(\alpha\sigma^2 + \mu - \ln(x)\right)\sqrt{2}}{\sigma}\right)\right. \\ &\left. + e^{\frac{1}{2}\beta\left(\beta\sigma^2 - 2\mu + 2\ln(x)\right)} erfc\left(\frac{1}{2}\frac{\left(\beta\sigma^2 - \mu + \ln(x)\right)\sqrt{2}}{\sigma}\right)\right) x^{-1}(\alpha+\beta)^{-1}, \end{aligned} \tag{16}$$

where α and β are the Pareto coefficients for the upper and the lower tail, respectively, μ and σ are the lognormal body parameters, and *erfc* is the complementary error function. **Figure 3** compares the gamma-Pareto and the double Pareto lognormal.

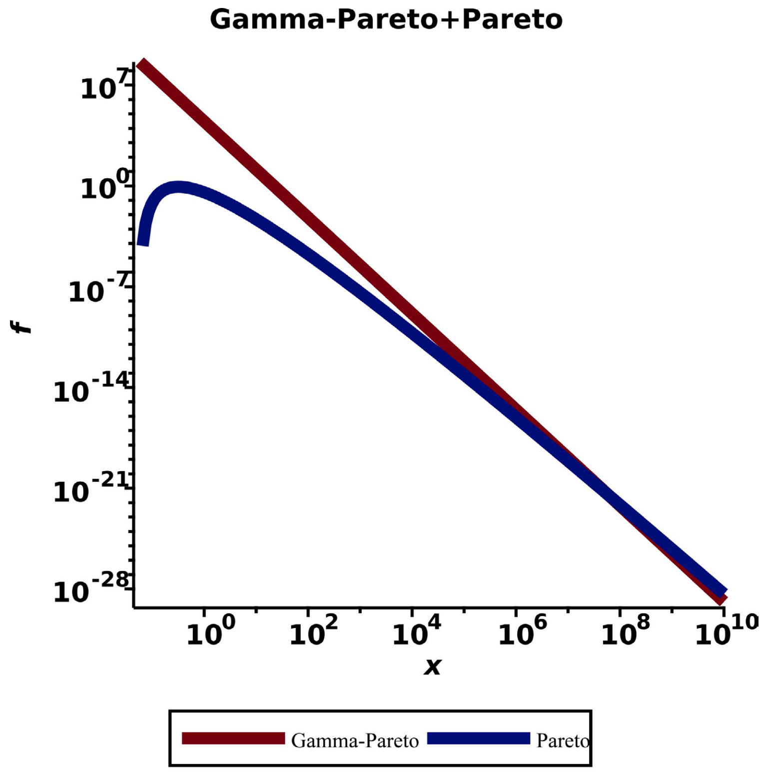

Figure 1. The PDF of the gamma-Pareto distribution with $\alpha = 6.716$, $c = 0.428$, and $\theta = 0.0574$ (blue line) and Pareto PDF with $a = 57.4$ and $c = 2.336$ (red line).

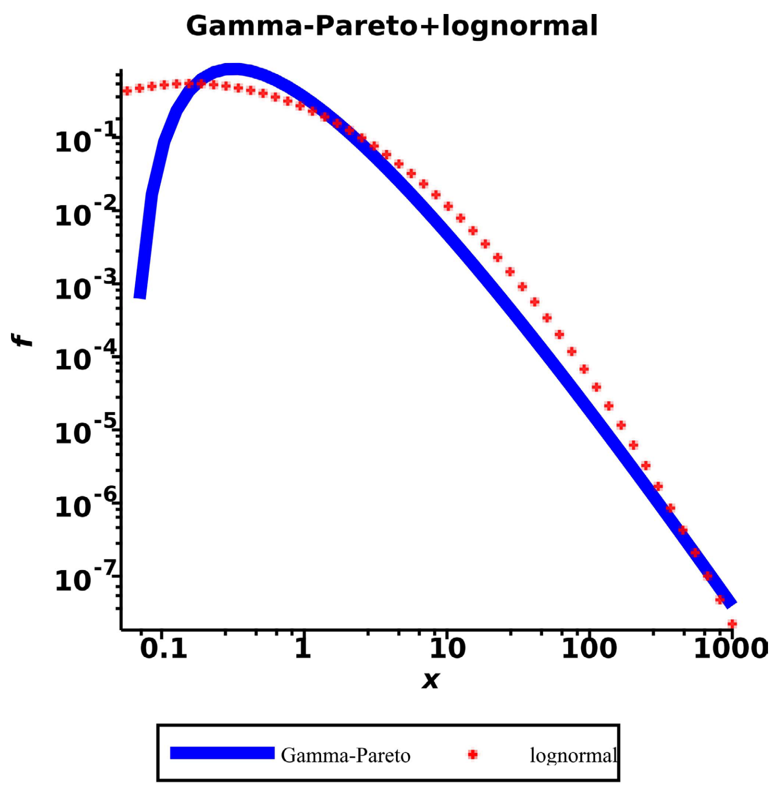

Figure 2. The PDF of the gamma-Pareto distribution with $\alpha = 6.716$, $c = 0.428$, and $\theta = 0.0574$ (blue line) and lognormal with $m = 1.5$, $\sigma = 1.5$ (red points).

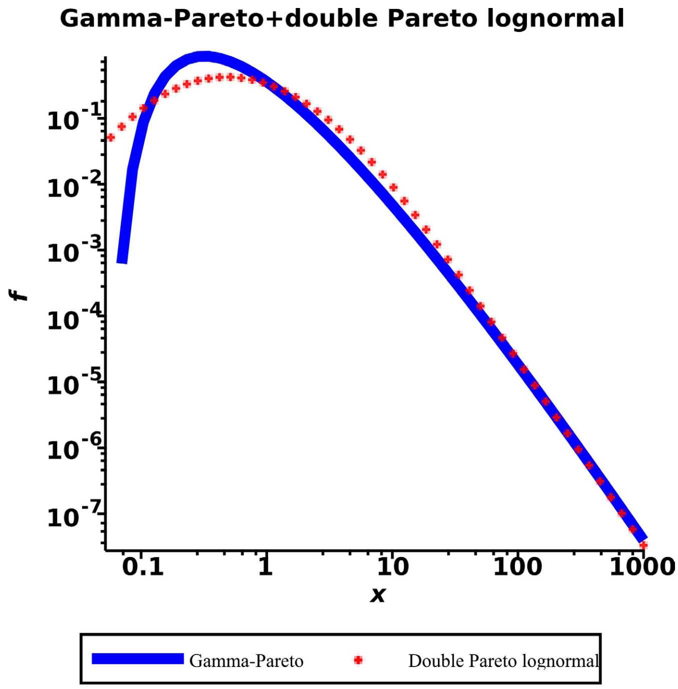

Figure 3. The PDF of the gamma-Pareto distribution with $\alpha = 6.716$, $c = 0.428$, and $\theta = 0.0574$ (blue line) and double Pareto lognormal with $\alpha = 1.8$, $\beta = 20$, $\mu = 0$, and $\sigma = 1$ (red points).

3. The Gamma-Pareto II Distribution

The translation $Y = X + \theta$ of a gamma-Pareto PDF, see Equation (1), in the random variable Y produces the gamma-Pareto II PDF

$$f(x;\alpha,c,\theta) = \frac{\theta^{\frac{1}{c}} (x+\theta)^{-1-\frac{1}{c}} \ln\left(1+\frac{x}{\theta}\right)^{\alpha-1}}{c^{\alpha}\Gamma(\alpha)}, \quad (17)$$

which is defined for $\alpha > 0$, $c > 0$, $\theta > 0$ and $x > 0$, see formula (2.4) in Table 1 of [1]. The DF is

$$F(x;\alpha,c,\theta) = 1 + \frac{-\Gamma\left(\alpha+1, \frac{\ln(x+\theta)-\ln(\theta)}{c}\right) + (x+\theta)^{-\frac{1}{c}} \theta^{\frac{1}{c}} c^{-\alpha} \left(\ln(x+\theta)-\ln(\theta)\right)^{\alpha}}{\alpha\Gamma(\alpha)}, \quad (18)$$

its average value is

$$\mu(\alpha,c,\theta) = \theta\left((1-c)^{-\alpha} - 1\right), \quad (19)$$

and the mode is at

$$Mode(\alpha,c,\theta) = \theta\left(\mathrm{e}^{\frac{c(\alpha-1)}{c+1}} - 1\right). \quad (20)$$

The three parameters α, c, θ are found in the framework of the MLE method by solving the three non-linear equations.

Table 1. Parameter estimates for different distributions applied to the diameters in NEOWISE.

Distribution	Equation	Parameters	Log Likelihood
gamma-Pareto	(1)	$c = 0.232$; $\theta = 0.081$; $\alpha = 19.43$	−36,070.22
gamma-Pareto R-truncated	(25)	$c = 0.222$; $\theta = 0.087$; $\alpha = 20.03$; $x_u = 450$	−36,093.94
bi-truncated gamma-Pareto	(29)	$c = 0.227$; $\theta = 0.087$; $\alpha = 19.56$; $x_l = 0.09$; $x_u = 450$	−36,093.07
gamma-Pareto II	(18)	$c = 0.35$; $\theta = 1.944$; $\alpha = 4.7$	−35,766.08
bi-truncated gamma-Pareto II	(33)	$c = 0.35$; $\theta = 1.944$; $\alpha = 4.7$; $x_l = 0.09$; $x_u = 450$	−35,762.30
lognormal	(16)	$m = 7.496$; $\sigma = 0.983$	−36,099.25
double Pareto lognormal	(17)	$\alpha = 2.981$; $\beta = 3$; $\mu = 2.025$; $\sigma = 0.979$	−36,166.91

$$\frac{-n\alpha c - n\ln(\theta) + \left(\sum_{i=1}^{n}\ln(x_i+\theta)\right)}{c^2} = 0, \tag{21a}$$

$$-n\ln(c) - n\Psi(\alpha) + \left(\sum_{i=1}^{n}\ln\left(\ln(x_i+\theta)-\ln(\theta)\right)\right) = 0, \tag{21b}$$

$$n - \left(\sum_{i=1}^{n}\frac{-\theta(c+1)\ln(x_i+\theta)+\theta(c+1)\ln(\theta)-x_i c(\alpha-1)}{\left(-\ln(x_i+\theta)+\ln(\theta)\right)(x_i+\theta)}\right) = 0. \tag{21c}$$

4. Right Truncation of the Gamma-Pareto Distribution

We now analyse a right truncated gamma-Pareto, see Equation (1), with PDF

$$f_T(x;\alpha,c,\theta,x_u) = \frac{\theta^{\frac{1}{c}} x^{\frac{-1-c}{c}} \ln\left(\frac{x}{\theta}\right)^{\alpha-1} c^{-\alpha}\alpha}{\alpha\Gamma(\alpha) - \Gamma\left(\alpha+1, \ln\left(\left(\frac{x_u}{\theta}\right)^{\frac{1}{c}}\right)\right) + c^{-\alpha}\ln\left(\frac{x_u}{\theta}\right)^{\alpha} x_u^{-\frac{1}{c}}\theta^{\frac{1}{c}}}, \tag{22}$$

which is defined for $\alpha > 0$, $c > 0$, $0 < \theta < x < x_u$ and the subscript T means truncation. Its DF is

$$F_T(x;\alpha,c,\theta,x_u) = \frac{x_u^{\frac{1}{c}}\left(-\Gamma\left(\alpha+1,\ln\left(\left(\frac{x}{\theta}\right)^{\frac{1}{c}}\right)\right)c^{\alpha} + c^{\alpha}\Gamma(\alpha+1) + \ln\left(\frac{x}{\theta}\right)^{\alpha}\left(\frac{x}{\theta}\right)^{-\frac{1}{c}}\right)}{c^{\alpha}x_u^{\frac{1}{c}}\Gamma(\alpha+1) - c^{\alpha}x_u^{\frac{1}{c}}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u}{\theta}\right)^{\frac{1}{c}}\right)\right) + \ln\left(\frac{x_u}{\theta}\right)^{\alpha}\theta^{\frac{1}{c}}}, \tag{23}$$

and its average value is

$$\mu_T(\alpha,c,\theta,x_u) = \frac{A}{c^{\alpha}x_u^{\frac{1}{c}}\Gamma(\alpha+1) - c^{\alpha}x_u^{\frac{1}{c}}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u}{\theta}\right)^{\frac{1}{c}}\right)\right) + \ln\left(\frac{x_u}{\theta}\right)^{\alpha}\theta^{\frac{1}{c}}}, \tag{24}$$

where

$$A = \ln\left(\frac{x_u}{\theta}\right)^{\alpha}\left(\Gamma(\alpha+1)c^{\alpha}x_u^{\frac{1}{c}}\ln\left(\frac{x_u\left(\frac{x_u}{\theta}\right)^{-c}}{\theta}\right)^{-\alpha}\theta - c^{\alpha}x_u^{\frac{1}{c}}\ln\left(\frac{x_u\left(\frac{x_u}{\theta}\right)^{-c}}{\theta}\right)^{-\alpha}\Gamma\left(\alpha+1,\ln\left(\frac{\theta\left(\frac{x_u}{\theta}\right)^{\frac{1}{c}}}{x_u}\right)\right)\theta + x_u\theta^{\frac{1}{c}}\right). \quad (25)$$

The two parameters θ and x_u are the minimum and the maximum of the sample. The two remaining parameters α and c are derived by MLE.

5. Right and Left Truncation of the Gamma-Pareto Distribution

The right- and left-truncated gamma-Pareto, see Equation (1), has PDF

$$f_{DT}(x;\alpha,c,\theta,x_l,x_u) = \frac{\theta^{\frac{1}{c}}x^{\frac{-1-c}{c}}\ln\left(\frac{x}{\theta}\right)^{\alpha-1}c^{-\alpha}}{\Gamma(\alpha)K}, \quad (26)$$

where

$$K = \frac{1}{\alpha\Gamma(\alpha)}\times\left(-\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u}{\theta}\right)^{\frac{1}{c}}\right)\right)+\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l}{\theta}\right)^{\frac{1}{c}}\right)\right) + c^{-\alpha}x_u^{-\frac{1}{c}}\theta^{\frac{1}{c}}\ln\left(\frac{x_u}{\theta}\right)^{\alpha} - c^{-\alpha}x_l^{-\frac{1}{c}}\theta^{\frac{1}{c}}\ln\left(\frac{x_l}{\theta}\right)^{\alpha}\right), \quad (27)$$

which is defined for $\alpha>0$, $c>0$, $0<\theta<x_l<x<x_u$ and the subscript DT means double truncation. The DF is

$$F_{DT}(\alpha,c,\theta,x_l,x_u) = \frac{1}{\alpha K\Gamma(\alpha)}\times\left(\ln\left(\frac{x}{\theta}\right)^{\alpha}c^{-\alpha}\left(\frac{x}{\theta}\right)^{-\frac{1}{c}} - c^{-\alpha}\ln\left(\frac{x_l}{\theta}\right)^{\alpha}\left(\frac{x_l}{\theta}\right)^{-\frac{1}{c}} + \Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l}{\theta}\right)^{\frac{1}{c}}\right)\right) - \Gamma\left(\alpha+1,\ln\left(\left(\frac{x}{\theta}\right)^{\frac{1}{c}}\right)\right)\right), \quad (28)$$

and its average value is

$$\mu_{DT}(\alpha,c,\theta,x_l,x_u) = \frac{1}{\alpha K\Gamma(\alpha)}\times\left(-\Gamma(\alpha)c^{-\alpha}\ln\left(\frac{x_l}{\theta}\right)^{\alpha}\ln\left(\left(\frac{x_l}{\theta}\right)^{-\frac{c-1}{c}}\right)^{-\alpha}\alpha\theta\right.$$

$$
\begin{aligned}
&+\Gamma(\alpha)\ln\left(\frac{x_u}{\theta}\right)^{\alpha}\ln\left(\frac{\left(\frac{x_u}{\theta}\right)^{-c}x_u}{\theta}\right)^{-\alpha}\alpha\theta \\
&+c^{-\alpha}\ln\left(\frac{x_l}{\theta}\right)^{\alpha}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l}{\theta}\right)^{-\frac{c-1}{c}}\right)\right)\ln\left(\left(\frac{x_l}{\theta}\right)^{-\frac{c-1}{c}}\right)^{-\alpha}\theta \\
&+c^{-\alpha}\theta^{\frac{1}{c}}\ln\left(\frac{x_u}{\theta}\right)^{\alpha}x_u^{\frac{c-1}{c}}-c^{-\alpha}x_l\left(\frac{x_l}{\theta}\right)^{-\frac{1}{c}}\ln\left(\frac{x_l}{\theta}\right)^{\alpha} \\
&\left.-\ln\left(\frac{x_u}{\theta}\right)^{\alpha}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u}{\theta}\right)^{-\frac{c-1}{c}}\right)\right)\ln\left(\frac{\left(\frac{x_u}{\theta}\right)^{-c}x_u}{\theta}\right)^{-\alpha}\theta\right).
\end{aligned}
\tag{29}
$$

6. The Truncated Gamma-Pareto II Distribution

The left and right truncated (bi-truncated) version of the gamma-Pareto II PDF, see Equation (18), is

$$
f_{DT}(x;\alpha,c,\theta,x_l,x_u)=\frac{\theta^{\frac{1}{c}}(x+\theta)^{-1-\frac{1}{c}}\ln\left(1+\frac{x}{\theta}\right)^{\alpha-1}}{c^{\alpha}\Gamma(\alpha)K'},
\tag{30}
$$

which is defined for $\alpha>0$, $c>0$, $x_l>0$, $x_u>0$, $\theta>0$, $x_l<x<x_u$ and

$$
\begin{aligned}
K'=&\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1}{c}}\right)\right)-\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1}{c}}\right)\right) \\
&+c^{-\alpha}\ln\left(\frac{x_u+\theta}{\theta}\right)^{\alpha}(x_u+\theta)^{-\frac{1}{c}}\theta^{\frac{1}{c}}-c^{-\alpha}\ln\left(\frac{x_l+\theta}{\theta}\right)^{\alpha}(x_l+\theta)^{-\frac{1}{c}}\theta^{\frac{1}{c}}.
\end{aligned}
\tag{31}
$$

Its DF is

$$
\begin{aligned}
&F_{DT}(\alpha,c,\theta,x_l,x_u) \\
&=\frac{1}{K'\alpha\Gamma(\alpha)}\times c^{-\alpha}\left(\ln(x+\theta)-\ln(\theta)\right)^{\alpha}(x+\theta)^{-\frac{1}{c}}\theta^{\frac{1}{c}} \\
&-c^{-\alpha}\left(\ln(x_l+\theta)-\ln(\theta)\right)^{\alpha}(x_l+\theta)^{-\frac{1}{c}}\theta^{\frac{1}{c}} \\
&+\Gamma\left(\alpha+1,\frac{\ln(x_l+\theta)-\ln(\theta)}{c}\right)-\Gamma\left(\alpha+1,\frac{\ln(x+\theta)-\ln(\theta)}{c}\right).
\end{aligned}
\tag{32}
$$

And its average value is

$$
\begin{aligned}
&\mu_{DT}(\alpha,c,\theta,x_l,x_u) \\
&=\frac{1}{K'\alpha\Gamma(\alpha)}\times c^{-\alpha}\left(-\ln\left(\frac{x_l+\theta}{\theta}\right)^{\alpha}\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1-c}{c}}\right)^{-\alpha}\Gamma(\alpha)\alpha\theta\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\ln\left(\frac{x_l+\theta}{\theta}\right)^{\alpha}\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1}{c}}\right)^{-\alpha}\Gamma(\alpha)\alpha\theta \\
&+\ln\left(\frac{x_u+\theta}{\theta}\right)^{\alpha}\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1-c}{c}}\right)^{-\alpha}\Gamma(\alpha)\alpha\theta \\
&-\ln\left(\frac{x_u+\theta}{\theta}\right)^{\alpha}\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1}{c}}\right)^{-\alpha}\Gamma(\alpha)\alpha\theta \\
&+\ln\left(\frac{x_l+\theta}{\theta}\right)^{\alpha}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1-c}{c}}\right)\right)\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1-c}{c}}\right)^{-\alpha}\theta \\
&-\ln\left(\frac{x_l+\theta}{\theta}\right)^{\alpha}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1}{c}}\right)\right)\ln\left(\left(\frac{x_l+\theta}{\theta}\right)^{\frac{1}{c}}\right)^{-\alpha}\theta \\
&-\ln\left(\frac{x_u+\theta}{\theta}\right)^{\alpha}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1-c}{c}}\right)\right)\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1-c}{c}}\right)^{-\alpha}\theta \qquad (33)\\
&+\ln\left(\frac{x_u+\theta}{\theta}\right)^{\alpha}\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1}{c}}\right)\right)\ln\left(\left(\frac{x_u+\theta}{\theta}\right)^{\frac{1}{c}}\right)^{-\alpha}\theta \\
&\left.-\ln\left(\frac{x_l+\theta}{\theta}\right)^{\alpha}\left(\frac{x_l+\theta}{\theta}\right)^{-\frac{1}{c}}x_l+\ln\left(\frac{x_u+\theta}{\theta}\right)^{\alpha}\left(\frac{x_u+\theta}{\theta}\right)^{-\frac{1}{c}}x_u\right).
\end{aligned}
$$

7. The Functions

When the data are presented as $y = f(x)$, an interpolating function is obtained by multiplying a PDF by a constant of normalization ϕ

$$y(x;\phi) = \phi \times PDF(x). \qquad (34)$$

The gamma-Pareto function, see Equation (1), is

$$\Psi(x;\alpha,c,\theta,\phi) = \phi\frac{\left(\frac{\theta}{x}\right)^{\frac{1}{c}}\ln\left(\frac{x}{\theta}\right)^{\alpha-1}}{x\Gamma(\alpha)c^{\alpha}}. \qquad (35)$$

The gamma-Pareto R-truncated function, see Equation (25), is

$$\Psi_T(x;\alpha,c,\theta,x_u,\phi) = \phi\frac{\theta^{\frac{1}{c}}x^{\frac{-1-c}{c}}\ln\left(\frac{x}{\theta}\right)^{\alpha-1}c^{-\alpha}\alpha}{\alpha\Gamma(\alpha)-\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u}{\theta}\right)^{\frac{1}{c}}\right)\right)+c^{-\alpha}\ln\left(\frac{x_u}{\theta}\right)^{\alpha}x_u^{-\frac{1}{c}}\theta^{\frac{1}{c}}}. \qquad (36)$$

The right and left truncated gamma-Pareto function, see Equation (29), is

$$\Psi_{DT}\left(x;\alpha,c,\theta,x_l,x_u,\phi\right)=\phi\frac{\theta^{\frac{1}{c}}x^{\frac{-1-c}{c}}\ln\left(\frac{x}{\theta}\right)^{\alpha-1}c^{-\alpha}}{\Gamma(\alpha)K}. \tag{37}$$

The gamma-Pareto II function, see Equation (18), is

$$\Psi_{II}\left(x;\alpha,c,\theta,\phi\right)=\phi\frac{\theta^{\frac{1}{c}}\left(x+\theta\right)^{-1-\frac{1}{c}}\ln\left(1+\frac{x}{\theta}\right)^{\alpha-1}}{c^{\alpha}\Gamma(\alpha)}. \tag{38}$$

The left and right truncated gamma-Pareto II function, see Equation (33), is

$$\Psi_{II,DT}\left(x;\alpha,c,\theta,x_l,x_u,\phi\right)=\phi\frac{\theta^{\frac{1}{c}}\left(x+\theta\right)^{-1-\frac{1}{c}}\ln\left(1+\frac{x}{\theta}\right)^{\alpha-1}}{c^{\alpha}\Gamma(\alpha)K'}. \tag{39}$$

The lognormal function, see Equation (16), is

$$\Psi_{LN}\left(x;m,\sigma,\phi\right)=\phi\frac{\mathrm{e}^{-\frac{1}{2\sigma^2}\left(\ln\left(\frac{x}{m}\right)\right)^2}}{x\sigma\sqrt{2\pi}}. \tag{40}$$

The double Pareto lognormal function, see Equation (17), is

$$\begin{aligned}&\Psi_{DP}\left(x;\alpha,\beta,\mu,\sigma,\phi\right)\\&=\phi\frac{1}{2}\alpha\beta\left(\mathrm{e}^{\frac{1}{2}\alpha\left(\alpha\sigma^2+2\mu-2\ln(x)\right)}erfc\left(\frac{1}{2}\frac{\left(\alpha\sigma^2+\mu-\ln(x)\right)\sqrt{2}}{\sigma}\right)\right.\\&\left.+\mathrm{e}^{\frac{1}{2}\beta\left(\beta\sigma^2-2\mu+2\ln(x)\right)}erfc\left(\frac{1}{2}\frac{\left(\beta\sigma^2-\mu+\ln(x)\right)\sqrt{2}}{\sigma}\right)\right)x^{-1}\left(\alpha+\beta\right)^{-1}.\end{aligned} \tag{41}$$

8. Astrophysical Applications

We present an initial application of the gamma-Pareto PDFs to a sample of asteroids and the second one to the distribution in energy of CR.

8.1. Application to the Asteroids

The Near-Earth Object Wide-Field Infrared Survey Explorer (NEOWISE) has measured the diameters in km of 10565 asteroids [13] and the corresponding catalog is available at VIZIER
http://vizier.u-strasbg.fr/viz-bin/VizieR-3?-source=J/AJ/152/63&-out.add=_r.
Table 1 presents the parameters of the gamma-Pareto PDFs here analysed and of two PDFs for comparison.

As an example, Figure 4 presents the graph of the gamma-Pareto R-truncated PDF and in Figure 5 the graph of the gamma-Pareto II PDF.

For reference, Figure 6 presents the graph of the lognormal PDF and Figure 7 presents the graph of the double Pareto lognormal PDF.

8.2. Application to Cosmic rays

The observed differential spectrum of CR according to [14] is presented in Figure 8 in the H case (I_H). The parameters of the distributions here analysed when applied to the above spectrum are presented in Table 2, where the parameter χ^2 represents the merit function [15]. Figure 9 presents the fit of the CR spectrum with the gamma-Pareto function and Figure 10 that with the gamma-Pareto II function.

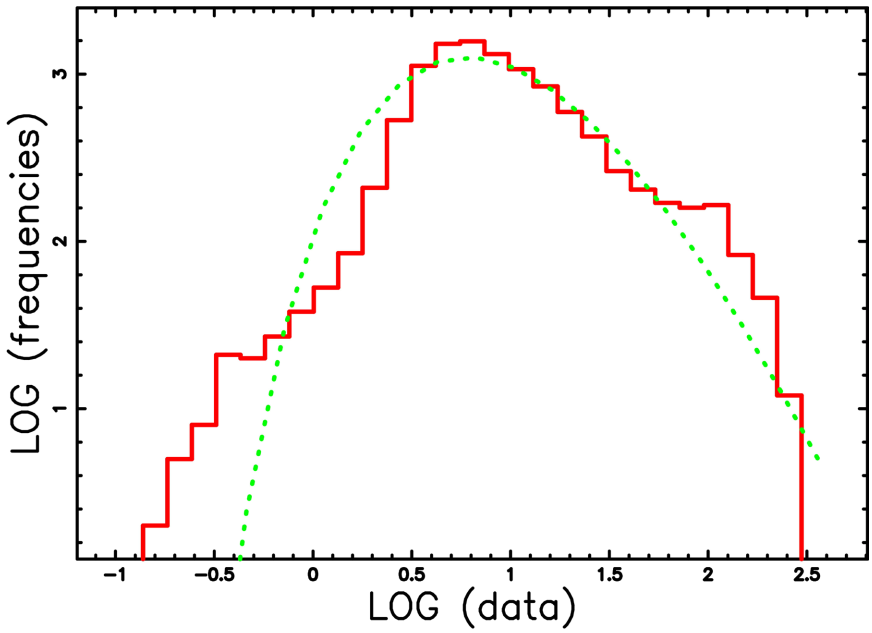

Figure 4. Empirical PDF of the distribution of the diameters of asteroids, from NEOWISE (red histogram), with a superposition of the gamma-Pareto R-truncated PDF (green dotted line) with parameters as in Table 1.

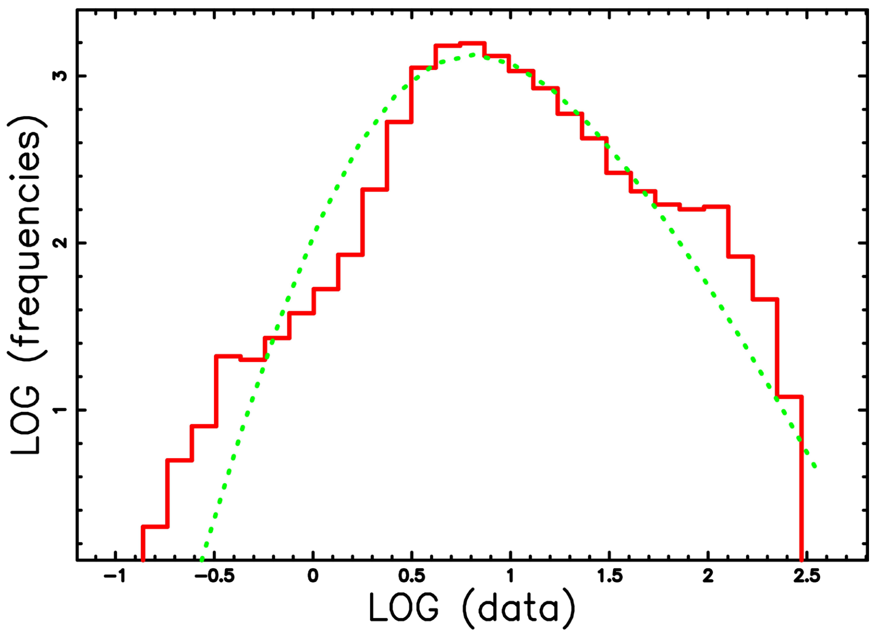

Figure 5. Empirical PDF of the distribution of the diameters of the asteroids in NEOWISE (red histogram) with a superposition of the gamma-Pareto II PDF (green dotted line) with parameters as in Table 1.

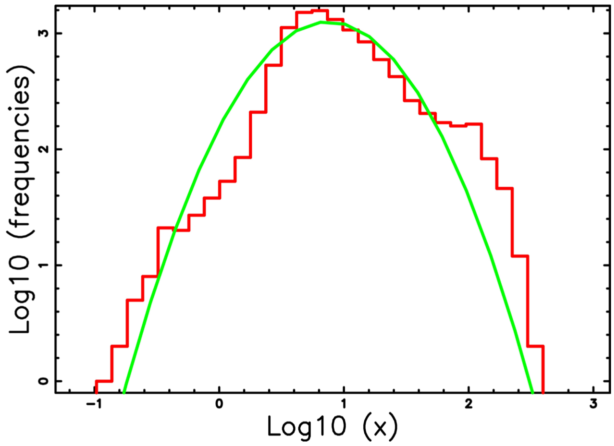

Figure 6. Empirical PDF of the distribution of the diameters of the asteroids in NEOWISE (red histogram) with a superposition of the lognormal PDF (green full line) with parameters as in Table 1.

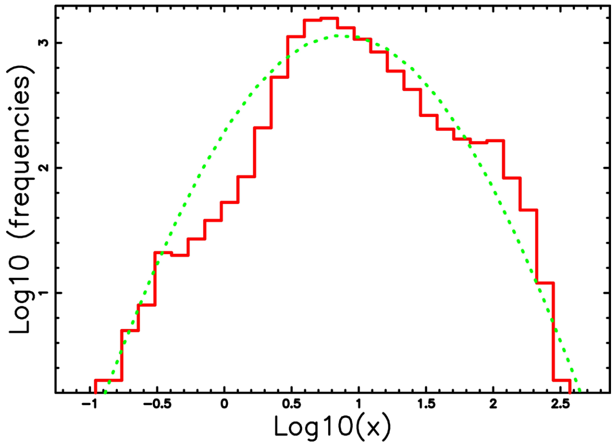

Figure 7. Empirical PDF of the distribution of the diameters of the asteroids in NEOWISE (red histogram) with a superposition of the double Pareto lognormal PDF (green dotted line) with parameters as in Table 1.

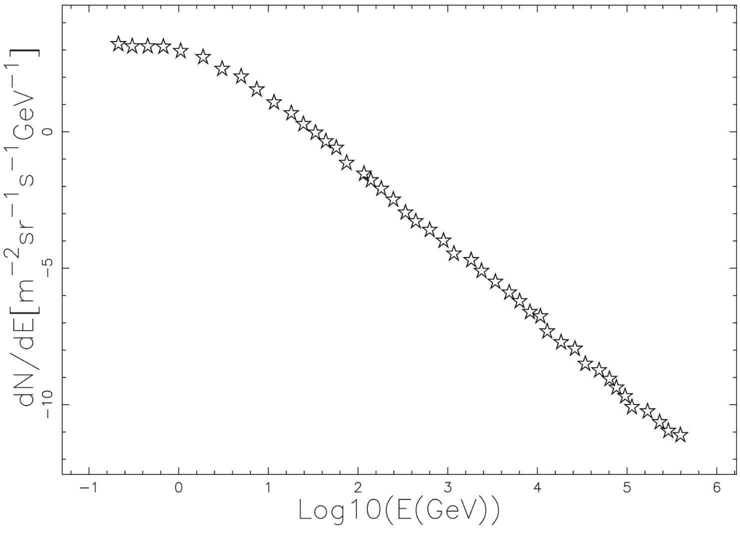

Figure 8. Flux of H versus energy per nucleus in GeV: experimental data (empty stars).

Table 2. Parameter estimates for different functions applied to cosmic rays.

Distribution	Equation	Parameters	$\langle E \rangle$ (GeV)	χ^2
gamma-Pareto	(38)	$\phi = 2110$; $c = 0.428$; $\theta = 0.057$; $\alpha = 6.716$	2.456	113.05
gamma-Pareto R-truncated	(39)	$\phi = 2106$; $c = 0.429$; $\theta = 0.057$; $\alpha = 6.687$; $x_u = 393587$	2.443	113.09
bi-truncated gamma-Pareto	(40)	$\phi = 2301$; $c = 0.455$; $\theta = 0.117$; $\alpha = 5.039$; $x_l = 0.212$; $x_u = 393587$	2.518	101.06
gamma-Pareto II	(41)	$\phi = 2696$; $c = 0.544$; $\theta = 1.395$; $\alpha = 1.34$	2.6	62.204
bi-truncated gamma-Pareto II	(42)	$\phi = 2373$; $c = 0.544$; $\theta = 1.4$; $\alpha = 1.33$; $x_l = 0.212$; $x_u = 2373$	2.956	62.205
lognormal	(43)	$\phi = 10442$; $m = 4.43 \times 10^{-3}$; $\sigma = 2.787$	0.215	284.55
double Pareto lognormal	(44)	$\phi = 4992$; $\alpha = 1.777$; $\beta = 0.147$; $\mu = 1.355$; $\sigma = 0.654$	1.413	72.94

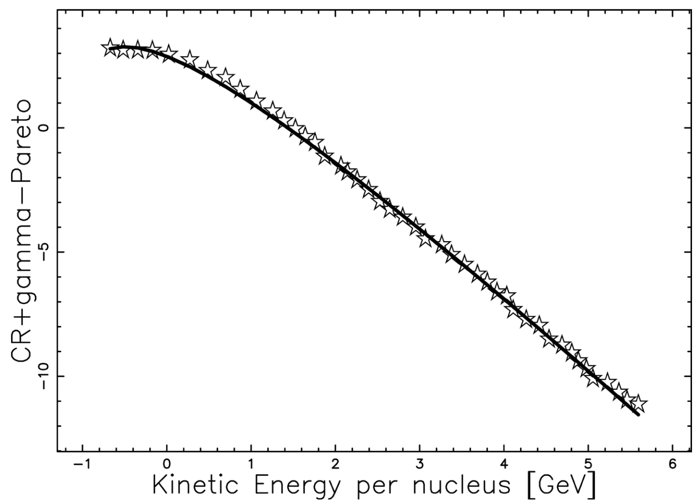

Figure 9. Flux of H versus energy per nucleus in GeV: experimental data (empty stars) and theoretical power law (full line) for the gamma-Pareto function, see Equation (38), with parameters as in Table 2.

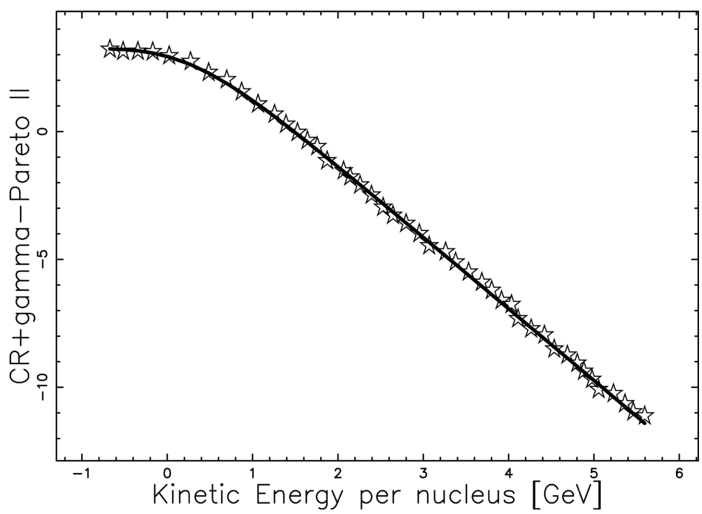

Figure 10. Flux of H versus energy per nucleus in GeV: experimental data (empty stars) and theoretical power law (full line) for the gamma-Pareto II function, see Equation (41), with parameters as in Table 2.

9. Conclusion

We have analysed the effect of truncation on the gamma-Pareto and the gamma-Pareto II distributions, deriving their distribution functions, average values, and variances. We made two applications to phenomena which present a long right tail often modeled with a power law behaviour, such as the Pareto PDF. In the case of the asteroids, the best results were obtained with the bi-truncated gamma-Pareto II, see Table 1, and in the case of cosmic rays (CR), with the bi-truncated gamma-Pareto, see Table 2. These models allow deriving some analytical formulae for the average value of the CR energy spectrum, which is 2.6 GeV for the bi-truncated gamma-Pareto II function.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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