Journal of Software Engineering and Applications, 2022, 15, 19-58
https://www.scirp.org/journal/jsea
ISSN Online: 1945-3124
ISSN Print: 1945-3116

Some Considerations about Fuzzy Logic Based Decision Making by Autonomous Intelligent Actor

Alex Tserkovny

Applied AI Services, Brookline, USA
Email: atserkovny@yahoo.com

How to cite this paper: Tserkovny, A. (2022) Some Considerations about Fuzzy Logic Based Decision Making by Autonomous Intelligent Actor. *Journal of Software Engineering and Applications*, **15**, 19-58.
https://doi.org/10.4236/jsea.2022.152002

Received: January 7, 2022
Accepted: February 22, 2022
Published: February 25, 2022

Abstract

The article presents an approach toward an implementation of a fuzzy logic-based decision-making process by Autonomous Intelligent Actor (AIA) (© A. Tserkovny), when an input information is defined for its "strategic targeting" by a human operator in terms of a fuzzy incident geometry, whereas its "tactical" behavior (a navigation in space) is directed by fuzzy conditional inference rules. For implementing both elements of AIA decision-making a fuzzy logic [1] for formal geometric reasoning with extended objects is used. This fuzzy logic based fuzzification of axioms of an incidence geometry and a predicate apparatus [2] for AIA space orientation are also presented. The approach, offered in the article, extends predicates of a counter positioning of two objects and their mutual navigation into their fuzzy counterparts. The latter allows AIA to make certain "tactical" decisions.

Keywords

Fuzzy Logic, Implication, Conjunction, Disjunction, Fuzzy Predicate, Degree of Indiscernibility, Discernibility Measure, Extended Lines Sameness, AIA Orientation Principles

1. Introduction

The article introduces the notion of an *Autonomous Intelligent Actor* (*AIA*), which, in general terms, represents an *entity*, which would be able to make an independent decision about strategic and tactical behavior, given an ultimate goal, defined by a *human operator*. For this purpose, we have proposed to use both fuzzy incident geometry paradigm for AIA strategic planning and fuzzy conditional inference rules for its local orientation (tactical behavior). The article presents a *fuzzy incident geometry* (© A. Tserkovny), based on a *fuzzy logic* [1] (© A. Tserkovny). For a consistency's sake we use the same *fuzzy logic* for an AIA orientation, once the strategic positioning is achieved.

Axiomatic Geometry and Extended Objects for AIA Strategic Targeting

In this chapter we consider AIA Strategic Targeting Apparatus to be defined by not traditional primitives of Euclidean Geometry, such as points and lines, but by their extended in space counterparts. In other words, we are reinstating Euclidean geometry, including the concepts of crisp points and lines, by using different set of geometric primitives. One of our goals is to expand traditional axiomatization of Euclidean geometry by applying fuzzification technique.

Similarly, to [3] we will present a set of incident geometry axiom, which validates the conduct of points and lines in space.

*I*1) *Every two separate* ***points*** *p* and *q could be linked by at least one* ***line*** *l*, *which is incident both.*

*I*2) *Such a* ***line*** *is unique.*

*I*3) *Every* ***line*** *is incident with at least two points.*

*I*4) *At least three points exist that are not incident with the same* ***line****.*

Now we formulate some entities of geometry by applying so-called construction operators sequentially. An instance of a construction operator is

Connect: ***point*** × ***point*** → ***line****.*

It means these input two points are connected by the line through them. In accordance with axiom *I*2 *Connect* to is well-defined mathematical function, because resulting line is unique and always exists. Let us show that couple more examples of geometric construction operators of 2D incidence geometry are

Intersect: ***line*** × ***line*** → ***point***,

Parallel through point: ***line*** × ***point*** → ***line***

Note that the set of axioms of incidence geometry is just a subset of the axioms of Euclidean geometry.

2. Fuzzification of Incidence Geometry

2.1. Fuzzy Logic in Use

Now we present some basic operations in a fuzzy logic [1] we will use for all purposes of an article. We define the truth values of logical *antecedent* A and *consequent* B as $T(A)=a$ and $T(B)=b$ respectively. Then relevant set of proposed fuzzy logic operators is shown in **Table 1**. To get the truth values of these definitions we use well known logical properties such as $A \to b = \neg a \vee b$; $a \wedge b = \neg(\neg a \vee \neg b)$ and alike. In other words we are considering a many-valued system, characterized by the set of base *union* ($\bigcup$) and *intersection* ($\bigcap$) operations with relevant *complement*, defined as $T(\neg A)=1-T(A)$. In addition, the operators $\downarrow$ and $\uparrow$ are expressed as negations of the $\bigcup$ and $\bigcap$ correspondingly.

From practical point of view and for illustration purposes only the real interval $\Re = [0,1]$ would be presented by the set of 11 values, *i.e.*, we are considering the following set $V_{11} = \{0, 0.1, 0.2, \cdots, 0.9, 1\}$, which we shall use *as a universe of discourse* in all our future exercises. Table 2 shows the operation *implication*

Table 1. The logical operations of a fuzzy logic in use.

Name	Designation	Value
Tautology	A^{I}	1
Controversy	A^{O}	0
Negation	$\neg A$	$1-A$
Disjunction	$A \vee B$	$\begin{cases} a \cdot b, a+b<1, \\ 1, a+b \geq 1 \end{cases}$
Conjunction	$A \wedge B$	$\begin{cases} a \cdot b, a+b>1, \\ 0, a+b \leq 1 \end{cases}$
Implication	$A \rightarrow B$	$\begin{cases} (1-a) \cdot b, a>b, \\ 1, a \leq b \end{cases}$
Equivalence	$A \leftrightarrow B$	$\begin{cases} (1-a) \cdot b, a<b, \\ 1, a=b \\ (1-b) \cdot a, b<a, \end{cases}$
Pierce Arrow	$A \downarrow B$	$\begin{cases} (1-a) \cdot (1-b), a+b<1, \\ 0, a+b \geq 1 \end{cases}$
Shaffer Stroke	$A \uparrow B$	$\begin{cases} (1-a) \cdot (1-b), a+b>1, \\ 1, a+b \leq 1 \end{cases}$

Table 2. The operation implication $I(a, b)$ in fuzzy logic in use.

$a \rightarrow b$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	1	1	1	1	1	1	1	1	1	1	1
0.1	0	1	1	1	1	1	1	1	1	1	1
0.2	0	0.08	1	1	1	1	1	1	1	1	1
0.3	0	0.07	0.14	1	1	1	1	1	1	1	1
0.4	0	0.06	0.12	0.18	1	1	1	1	1	1	1
0.5	0	0.05	0.1	0.015	0.2	1	1	1	1	1	1
0.6	0	0.04	0.08	0.12	0.16	0.2	1	1	1	1	1
0.7	0	0.03	0.06	0.09	0.12	0.15	0.18	1	1	1	1
0.8	0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	1	1	1
0.9	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	1	1
1	0	0	0	0	0	0	0	0	0	0	1

in fuzzy logic in use, which unique feature is the following

$$F_1(a,b) = (1-a)\cdot b, a > b \tag{2.1}$$

From where we have

$$I(a,b) = \begin{cases} F_1(a,b), a > b; \\ 1, a \le b, \end{cases} = \begin{cases} (1-a)\cdot b, a > b, \\ 1, a \le b \end{cases} \tag{2.2}$$

2.2. Geometric Primitives as Fuzzy Predicates

Let us denote some incidence geometry predicates as *p*(*a*) ("*a* is a point"), *l*(*a*) ("*a* is a line"), and *inc*(*a*, *b*) ("*a* and *b* are incident"). Traditionally predicates are interpreted by crisp relations. The predicate expressing equality can be denotes by *eq*(*a*, *b*) ("*a* and *b* are equal"). For example, $eq: N \times N \to \{0,1\}$ is a switch function between every pair of *equal* to every pair of *distinct* objects from the set *N*. Both predicates *p*(.) and *l*(.), with only one symbol input argument are *unary*, whereas *binary* predicates, like *inc*(.,.) and *eq*(.,.), accept pairs of symbols as an input. In this work by applying fuzzy predicate logic, we are re-interpreting all types of crisp relations predicates by their fuzzy counterparts. For instance, a binary fuzzy relation *eq* is defined as function $eq: N \times N \to [0,1]$, assigning a real number $\lambda \in [0,1]$ to *every* pair of objects from *N*. In other words, every two objects of *N* are equal to some degree. The degree of equality of two objects *a* and *b* may be 1 or 0 as in the crisp case, but can as well be 0.9, expressing that *a* and *b* are *almost* equal.

Note that point-predicate *p*(.) for Cartesian point does not change when the point is rotated, *i.e.* rotation-invariance could be a main characteristic of "point likeness" with respect to geometric operations. In other words "point likeness" should be kept in a relevant fuzzy predicate expressing the extended subsets of $R^{1.}$ Let us define

$$\begin{aligned} \theta_{\min}(A) &= \min_t \left| ch(A) \cap \left\{ c(A) + t \cdot R_\alpha \cdot (0,1)^T \mid t \in \Re \right\} \right| \\ \theta_{\max}(A) &= \max_t \left| ch(A) \cap \left\{ c(A) + t \cdot R_\alpha \cdot (0,1)^T \mid t \in \Re \right\} \right| \end{aligned} \tag{2.3}$$

as the minimal and maximal diameter of the convex hull *ch*(*A*) of $A \subseteq Dom$, respectively. The convex hull in certain way normalizes (eliminates irregularities) the sets A and B. *c* (*A*) denotes the centroid of *ch*(*A*), and *Rα* denotes the rotation matrix by angle *α* (**Figure 1(a)**) [3].

Let's describe the fuzzy point-predicate *p*(.), given the fact that *A* set is bounded and both *ch*(*A*) and *c* (*A*) exist, by

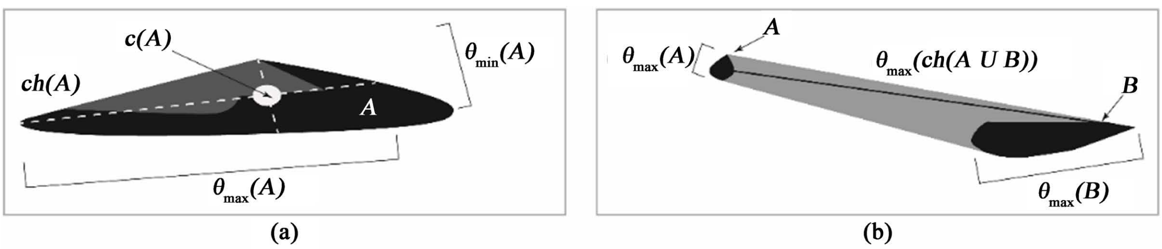

Figure 1. (a) Cartesian point A minimal and maximal diameters. Note: (b) A and B point's distinctness.

$$p(A) = \theta_{\min}(A)/\theta_{\max}(A). \tag{2.4}$$

If we express the degree to which the convex hull of a Cartesian point set *A* is rotation-invariant as $A \subseteq Dom$ *p*(.) and if *p*(*A*) = 1, then *ch*(*A*) is perfectly rotation invariant and it is a disc. And since *A* is assumed to be two-dimensional, then inequality $\theta_{\max}(A) \neq 0$ always holds. In addition to *p*(.), the fuzzy line-predicate is defined as a compliment to fuzzy-point one

$$l(A) = 1 - p(A) \tag{2.5}$$

To define the degree to which a Cartesian point set $A \subseteq Dom$ is sensitive to rotation and since we only regard convex hulls, then a fuzzy version of the incidence-predicate *inc*(.,.) would be a binary fuzzy relation between Cartesian point sets *A*, $B \subseteq Dom$:

$$inc(A,B) = \max\left(\left|ch(A) \cap ch(B)\right| / \left|ch(A)\right|, \left|ch(A) \cap ch(B)\right| / \left|ch(B)\right|\right) \tag{2.6}$$

Here in (2.6) we select the greater one of two convex hulls of *A* and *B* and this fuzzy relation measures the relative overlaps of them. Here *A* and *B* are considered as *an incident to degree one*, if |*ch*(*A*)| denotes the area occupied by *ch*(*A*). The greater *inc*(*A*, *B*), "the more incident" are *A* and *B*: If $A \subseteq B$ or $B \subseteq A$, then *inc*(*A*, *B*) = 1.

Contrariwise to *inc*(.,.), we define a graduated equality predicate *eq*(.,.) between the bounded Cartesian point sets *A*, $B \subseteq Dom$ as follows:

$$eq(A,B) = \min\left(\left|ch(A) \cap ch(B)\right| / \left|ch(A)\right|, \left|ch(A) \cap ch(B)\right| / \left|ch(B)\right|\right) \tag{2.7}$$

where *eq*(*A*, *B*) measures the minimal relative overlap of *A* and *B*, whereas the negation $\neg eq(A,B) = 1 - eq(A,B)$ measures the degrees to which the two-point sets do not overlap: if *eq*(*A*, *B*) ≈ 0, then *A* and *B* are "almost disjoint".

Then we can define the following measure of "distinctness of points" *dp*(.) of two extended objects from **Figure 1(b)** as

$$dp(A,B) = \max\left(0, 1 - \max\left(\theta_{\max}(A), \theta_{\max}(B)\right) / \theta_{\max}\left(ch(A \cup B)\right)\right) \tag{2.8}$$

It is apparent, that the greater *dp*(*A*, *B*), the more *A* and *B* behave like distinct Cartesian points with respect to connection. Indeed, for Cartesian points *a* and *b*, we would have *dp*(*A*, *B*) =1. If the distance between the Cartesian point sets *A* and *B* is infinitely big, then *dp*(*A*, *B*) = 1 as well. If $\max\left(\theta_{\max}(A), \theta_{\max}(B)\right) > \theta_{\max}\left(ch(A \cup B)\right)$ then *dp*(*A*, *B*) = 0.

2.3. Formalization of Fuzzy Predicates

To formalize fuzzy predicates, defined in subchapter 2.2 both *implication* → and *conjunction* operators are defined as in **Table 1**:

$$A \wedge B = \begin{cases} a \cdot b, a + b > 1, \\ 0, a + b \le 1 \end{cases} \tag{2.9}$$

$$A \to B = \begin{cases} (1-a) \cdot b, a > b, \\ 1, a \le b \end{cases} \tag{2.10}$$

In our further discussions we will also use the *disjunction* operator from the same table.

$$A \vee B = \begin{cases} a \cdot b, a+b<1, \\ 1, a+b \geq 1 \end{cases} \tag{2.11}$$

Now let us re-define the set of fuzzy predicates (2.6)-(2.8), using proposed fuzzy logic's operators.

PROPOSITION 1.

If fuzzy predicate *inc*(…) is defined as in (2.6) and *conjunction* operator is defined as in (2.9), then

$$inc(A,B) = \begin{cases} \max(a,b), a+b>1, \\ 0, a+b \leq 1 \end{cases} \tag{2.12}$$

Proof: Let's present (2.6) as follows:

$$\frac{|ch(A) \cap ch(B)|}{|ch(A)|} = \frac{A \cdot B}{A} = B \text{ and } \frac{|ch(A) \cap ch(B)|}{|ch(B)|} = \frac{A \cdot B}{B} = A \tag{2.13}$$

Therefore from (2.6) and (2.13) we are getting (2.12). (Q.E.D.).

PROPOSITION 1.

If fuzzy predicate *eq*(…) is defined as in (2.7) and *disjunction* operator is defined as in (2.11), then

$$eq(A,B) = \begin{cases} \min(a,b), a+b<1, \\ 1, a+b \geq 1 \end{cases} \tag{2.14}$$

Proof:

From (2.7) and (2.13) we are getting (2.14). (Q.E.D.).

COROLLARY 1.

If fuzzy predicate *eq*(*A*, *B*) is defined as (2.14), then the following type of *transitivity* is taking place

$$eq(a,c) \to eq(a,b) \wedge eq(b,c) \tag{2.15}$$

where $A, B, C \subseteq Dom$, and *Dom* is *partially ordered* space, *i.e.*, either $A \subseteq B \subseteq C$ or wise versa. (Note: both *conjunction* and *implication* operations are defined in Table 1).

Proof:

From (2.14) we have

$$\begin{aligned} eq(A,C) &= \begin{cases} \min(a,c), a+c<1, \\ 1, a+c \geq 1 \end{cases} \\ eq(A,B) &= \begin{cases} \min(a,b), a+b<1, \\ 1, a+b \geq 1 \end{cases} \\ eq(B,C) &= \begin{cases} \min(b,c), b+c<1, \\ 1, b+c \geq 1 \end{cases} \end{aligned} \tag{2.16}$$

Let $a<b<c \leq 1$ and $a+c<1$, $a+b<1$, $b+c<1$ from (2.16) we have

$$\begin{aligned} eq(a,c) &= \min(a,c), \\ eq(a,b) &= \min(a,b), \\ eq(b,c) &= \min(b,c). \end{aligned} \tag{2.17}$$

From (2.17), given $a < b < c \le 1$, we are getting

$$\begin{aligned} eq(a,c) &= a, \\ eq(a,b) &= a, \\ eq(b,c) &= b. \end{aligned} \tag{2.18}$$

Given (2.9) and (2.18) we have $a < a \cdot b$ and therefore

$$eq(a,c) \to eq(a,b) \wedge eq(b,c)$$

From (2.17), given $a > b > c \le 1$, we are getting

$$\begin{aligned} eq(a,c) &= c, \\ eq(a,b) &= b, \\ eq(b,c) &= c. \end{aligned} \tag{2.19}$$

Given (2.9) and (2.19) we have $c < b \cdot c$ and therefore

$$eq(a,c) \to eq(a,b) \wedge eq(b,c). \text{ (Q.E.D.).}$$

PROPOSITION 2.

If fuzzy predicate *dp*(…) is defined as in (2.8) and *disjunction* operator is defined as in (2.11), then

$$dp(A,B) = \begin{cases} 1-a, a+b \ge 1 \,\&\, a \ge b, \\ 1-b, a+b \ge 1 \,\&\, a < b, \\ 0, a+b < 1 \end{cases} \tag{2.20}$$

Proof:

From (2.8) we get the following:

$$dp(A,B) = \max\left\{0, 1 - \frac{\max(A,B)}{A \cup B}\right\}$$

For the case, when $a + b < 1$, and given (2.11)

$$dp(A,B) = \max\left\{0, 1 - \frac{\max(a,b)}{a \cdot b}\right\} = \begin{cases} \max\left\{0, 1 - \dfrac{1}{b}\right\}, a > b, \\ \max\left\{0, 1 - \dfrac{1}{a}\right\}, a < b. \end{cases} \tag{2.21}$$

Since $1 - \frac{1}{b} < 0$ and $1 - \frac{1}{a} < 0$, given $a, b \in [0,1]$ and from (2.21) we are getting

$$dp(A,B) \equiv 0. \tag{2.22}$$

For the case, when $a + b \ge 1$, and given (2.11)

$$dp(A,B) = \max\left\{0, 1 - \frac{\max(a,b)}{a \cdot b}\right\} = \begin{cases} \max\{0, 1-a\}, a > b, \\ \max\{0, 1-b\}, a < b. \end{cases} = \begin{cases} 1-a, a > b, \\ 1-b, a < b. \end{cases} \tag{2.23}$$

From (2.22) and (2.23) we are getting (2.20) (Q.E.D.).

2.4. Fuzzy Axiomatization of Incidence Geometry

By using the fuzzy predicates formalized in subchapter 2.3, we propose the set of axioms as fuzzy version of incidence geometry in the language of a fuzzy logic [1] as follows:

I1': $dp(a,b) \to \sup_c \left[l(c) \wedge inc(a,c) \wedge inc(b,c) \right]$

I2': $dp(a,b) \to \left[l(c) \to \left[inc(a,c) \to \left[inc(b,c) \to l(c') \to \left[inc(a,c') \to \left[inc(b,c') \to eq(c,c') \right] \right] \right] \right] \right]$

I3': $l(c) \to \sup_{a,b} \left\{ p(a) \wedge p(b) \wedge \neg eq(a,b) \wedge inc(a,c) \wedge inc(b,c) \right\}$

I4': $\sup_{a,b,c,d} \left[p(a) \wedge p(b) \wedge p(c) \wedge l(d) \to \neg\left(inc(a,d) \wedge inc(b,d) \wedge inc(c,d) \right) \right]$

In axioms I1'-I4' we also use a set of operations (2.9)-(2.11).

PROPOSITION 3.

If fuzzy predicates *dp*(…) and *inc*(…) are defined like (2.20) and (2.12) respectively, then axiom *I*1 ′is fulfilled for the set of logical operators from a fuzzy logic [1]. (For *every two distinct point a and b, at least one line l exists that is incident with a and b.*)

Proof:

From (2.15)

$$
\begin{aligned}
inc(A,C) &= \begin{cases} \max(a,c), a+c > 1, \\ 0, a+c \le 1 \end{cases} \\
inc(B,C) &= \begin{cases} \max(b,c), b+c > 1, \\ 0, b+c \le 1 \end{cases}
\end{aligned} \tag{2.24}
$$

Given (2.9) we are getting

$$
inc(A,C) \wedge inc(B,C) = \begin{cases} inc(A,C) \cdot inc(B,C), inc(A,C) + inc(B,C) > 1, \\ 0, inc(A,C) + inc(B,C) \le 1 \end{cases} \tag{2.25}
$$

Given (2.24) and (2.25)

$$
inc(A,C) \wedge inc(B,C) = \begin{cases} \max(A,C) \cdot \max(B,C), \max(A,C) + \max(B,C) > 1, \\ 0, \max(A,C) + \max(B,C) \le 1 \end{cases} \tag{2.26}
$$

But

$$
\begin{aligned}
&\sup_c \left[l(c) \wedge inc(a,c) \wedge inc(b,c) \right] \\
&= \sup_c \left[l(c) \wedge \begin{cases} C^2, 2 \cdot C > 1, \\ 0, 2 \cdot C \le 1 \end{cases} \right] = \sup_c \left[l(c) \wedge \begin{cases} C^2, C > 0.5, \\ 0, C \le 0.5 \end{cases} \right]
\end{aligned} \tag{2.27}
$$

In (2.27) we have the following

$$
\begin{cases} C^2, C > 0.5, \\ 0, C \le 0.5 \end{cases} \in (0.25, 1] \tag{2.28}
$$

From (2.28) we are getting

$$\sup_c \left[l(c) \wedge \begin{cases} C^2, C > 0.5, \\ 0, C \le 0.5 \end{cases} \right] \equiv 1 \tag{2.29}$$

From (2.20) we always have $dp(A,B) < 1$, therefore

$$dp(a,b) \to \sup_c \left[l(c) \wedge inc(a,c) \wedge inc(b,c) \right] \text{ (Q.E.D.).}$$

PROPOSITION 4.

If fuzzy predicates *dp*(…), *eq*(…) and *inc*(…) are defined like (2.35), (2.16) and (2.15) respectively, then axiom I2' is fulfilled for the set of logical operators from a fuzzy logic [1]. (For every two distinct point *a* and *b*, at least one line *l* exists that is incident with *a* and *b* and such a line is unique)

Proof:

Let's take a look at the following implication:

$$inc(b,c') \to eq(c,c') \tag{2.30}$$

But from (2.14) we have

$$eq(C,C') = \begin{cases} \min(c,c'), c + c' < 1, \\ 1, c + c' \ge 1 \end{cases} \tag{2.31}$$

And from (2.15)

$$inc(B,C') = \begin{cases} \max(b,c'), b + c' > 1, \\ 0, b + c' \le 1 \end{cases} \tag{2.32}$$

From (2.31) and (2.32) we see, that $inc(B,C') \le eq(C,C')$, which means that

$$inc(b,c') \to eq(c,c') \equiv 1,$$

Therefore, the following is also true

$$inc(a,c') \to \left[inc(b,c') \to eq(c,c') \right] \equiv 1 \tag{2.33}$$

Now let's look at the following implication $inc(b,c) \to l(c')$. Since $inc(b,c) \ge l(c')$, we are getting $inc(b,c) \to l(c') \equiv 0$. Considering (2.33) we have the following

$$inc(b,c) \to l(c') \to \left[inc(a,c') \to \left[inc(b,c') \to eq(c,c') \right] \right] \equiv 1 \tag{2.34}$$

Since from (2.12) $inc(a,c) \le 1$, then with taking into account (2.34) we've gotten the following:

$$inc(a,c) \to \left[inc(b,c) \to l(c') \to \left[inc(a,c') \to \left[inc(b,c') \to eq(c,c') \right] \right] \right] \equiv 1 \tag{2.35}$$

Since $l(c) \le 1$, from (2.35) we are getting:

$$l(c) \to \left[inc(a,c) \to \left[inc(b,c) \to l(c') \to \left[inc(a,c') \right.\right.\right.$$
$$\left.\left.\left.\to \left[inc(b,c') \to eq(c,c') \right] \right] \right] \right] \equiv 1$$

Finally, because $dp(a,b) \le 1$ we have

$$dp(a,b) \le \left\{ l(c) \to \left[inc(a,c) \to \left[inc(b,c) \to l(c') \to \left[inc(a,c') \right.\right.\right.\right.$$
$$\left.\left.\left.\left.\to \left[inc(b,c') \to eq(c,c') \right] \right] \right] \right] \right\}$$

(Q.E.D.).

PROPOSITION 6.

If fuzzy predicates *eq*(…) and *inc*(…) are defined like (2.14) and (2.12) respectively, then axiom I3' is fulfilled for the set of logical operators from a fuzzy logic [1]. (*Every line is incident with at least two points.*)

Proof:

It was already shown in (2.28) that $inc(a,c) \wedge inc(b,c) \in (0.25,1]$

And from (2.14) we have

$$eq(A,B) = \begin{cases} \min(a,b), a+b<1, \\ 1, a+b \geq 1 \end{cases}$$

The negation $\neg eq(A,B)$ will be

$$\neg eq(A,B) = \begin{cases} \max(a,b), a+b<1, \\ 0, a+b \geq 1 \end{cases} \tag{2.36}$$

Given (2.36) and (2.9) we get

$$\neg eq(A,B) \wedge 1 = \begin{cases} \max(a,b), \max(a,b)+1>0, \\ 0, \text{otherwise} \end{cases} = \max(a,b) \tag{2.37}$$

But

$$\sup_{a,b}\{p(a) \wedge p(b) \wedge \neg eq(a,b) \wedge inc(a,c) \wedge inc(b,c)\} = 1 \wedge \max(a,b) \wedge a \cdot b = 1.$$

And given, that $l(c)<1$ we are getting

$$l(c) \to \sup_{a,b}\{p(a) \wedge p(b) \wedge \neg eq(a,b) \wedge inc(a,c) \wedge inc(b,c)\} \equiv 1 \text{ (Q.E.D.).}$$

PROPOSITION 7.

If fuzzy predicate *inc*(…) is defined like (2.15), then axiom *I4'* is fulfilled for the set of *logical operators from* a fuzzy logic [1]. (*At least three points exist that are not incident with the same line.*)

Proof:

From (2.12) we have

$$inc(A,D) = \begin{cases} \max(a,d), a+d>1, \\ 0, a+d \leq 1 \end{cases}$$

$$inc(B,D) = \begin{cases} \max(b,d), b+d>1, \\ 0, b+d \leq 1 \end{cases}$$

$$inc(C,D) = \begin{cases} \max(c,d), c+d>1, \\ 0, c+d \leq 1 \end{cases}$$

But from (2.36) which we have

$$\left(inc(a,d) \wedge inc(b,d) \wedge inc(c,d)\right) = \left(\max(a,d) \wedge \max(b,d) \wedge \max(c,d)\right) \tag{2.38}$$

From (2.38)

$$\neg\left(inc(a,d) \wedge inc(b,d) \wedge inc(c,d)\right) = \neg\left(\max(a,d) \wedge \max(b,d) \wedge \max(c,d)\right) \tag{2.39}$$

From (2.39)
$\neg(\max(a,d)\wedge\max(b,d)\wedge\max(c,d))=(\min(a,d)\vee\min(b,d)\vee\min(c,d))$ or $(inc(a,d)\wedge inc(b,d)\wedge inc(c,d))=1\wedge inc(c,d)\equiv 1$, from where we have $\neg(inc(a,d)\wedge inc(b,d)\wedge inc(c,d))\equiv 0$. Since $l(d)\equiv 0\,|\,d=1$ we are getting $l(d)=\neg(inc(a,d)\wedge inc(b,d)\wedge inc(c,d))$, which could be interpreted like $l(d)\to\neg(inc(a,d)\wedge inc(b,d)\wedge inc(c,d))=1$, from which we finally get the following $\sup_{a,b,c,d}[p(a)\wedge p(b)\wedge p(c)\wedge 1]\equiv 1$ (Q.E.D.).

2.5. Equality of Extended Lines Is Graduated

In [3] it was shown that the location of the extended points creates a constraint on the location of an incident extended line. It was also mentioned, that in traditional geometry this location constraint fixes the position of the line uniquely. And therefore, in case points and lines are allowed to have extension this is not the case. Consequently, Euclid's First postulate does not apply: **Figure 2** shows that if two distinct extended points P and Q are incident (*i.e.*, overlap) with two extended lines L and M, then L and M are not necessarily equal.

Yet, in most cases, L and M are "closer together", *i.e.*, "more equal" than arbitrary extended lines that have only one or no extended point in common. The further P and Q move apart from each other, the more similar L and M become. One way to model this fact is to allow *degrees of equality* for extended lines. In other words, the equality relation *is* graduated: It allows not only for Boolean values, but for values in the whole interval [0, 1].

2.6. Incidence of Extended Points and Lines

As it was demonstrated in [3], there is a reasonable assumption to classify an extended point and an extended line as incident, if their extended representations in the underlying metric space overlap. We do this by modelling incidence by the subset relation:

Definition 1: For an extended point $\boldsymbol{A}$, and an extended line $\boldsymbol{L}$ we define the *incidence* relation by

$$R_{inc}(A,L):=(A\subseteq L)\in\{0,1\}, \tag{2.40}$$

where the subset relation $\subseteq$ refers to A and L as subsets of the underlying metric space. The extended incidence relation (2.40) is a Boolean relation, assuming either the truth value 1 (*true*) or the truth value 0 (*false*). It is well known that since a Boolean relation is a special case of a graduated relation, *i.e.*, since $\{0,1\}\subset[0,1]$, we will be able to use relation (2.40) as part of fuzzified Euclid's

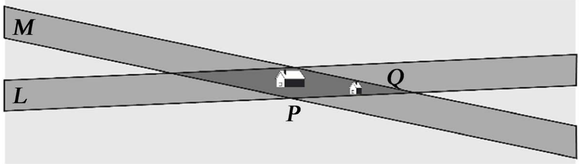

Figure 2. Two extended points do not uniquely determine the location of an incident extended line.

first postulate later on.

2.7. Equality of Extended Points and Lines

As stated in previous chapters, equality of extended points, and equality of extended lines is a matter of degree. Geometric reasoning with extended points and extended lines relies heavily on the metric structure of the underlying coordinate space. Consequently, it is reasonable to model graduated equality as inverse to distance.

2.7.1. Metric Distance

In [3] was mentioned that a pseudo metric distance, or pseudo metric, is a map $d: M^2 \to \Re^+$ from domain M into the positive real numbers (including zero), which is minimal, symmetric, and satisfies the triangle inequality:

$$\forall a,b \in [0,1] \Rightarrow \begin{cases} d(a,a)=0 \\ d(a,b)=d(b,a) \\ d(a,b)+d(b,c) \geq d(a,c). \end{cases} \tag{2.41}$$

d is called a metric, if additionally holds:

$$d(a,b)=0 \Leftrightarrow a=b, \tag{2.42}$$

Well known examples of metric distances are the Euclidean distance, or the Manhattan distance. Another example is the elliptic metric for the projective plane defined in (2.42) [3]. The "upside-down-version" of a pseudo metric distance is a fuzzy equivalence relation w.r.t. in proposed t-norm fuzzy logic. We will use this particular fuzzy logic to formalize Euclid's first postulate for extended primitives in chapter 4. The reason for choosing a proposed fuzzy logic is its strong connection to metric distance.

2.7.2. Fuzzy Equivalence Relations

As mentioned above, the "upside-down-version" of a pseudo metric distance is a fuzzy equivalence relation w.r.t. the proposed t-norm ^. A fuzzy equivalence relation is a fuzzy relation $e: M^2 \to [0,1]$ on a domain M, which is reflexive, symmetric and ^ -transitive:

$$\forall a,b \in [0,1] \Rightarrow \begin{cases} e(a,b)=1 \\ e(a,b)=e(b,a) \\ e(a,b) \wedge e(b,c) \leq e(a,c). \end{cases} \tag{2.43}$$

PROPOSITION 9.

If Fuzzy Equivalence Relation is defined (**Table 1**) as the following

$$e(a,b) = A \leftrightarrow B = \begin{cases} (1-a)\cdot b, a<b, \\ 1, a=b \\ (1-b)\cdot a, b<a, \end{cases} \tag{2.44}$$

then conditions (2.43) are satisfied.

Proof:

1) Reflexivity: $e(a,a)=1$ comes from (2.44) because $a \equiv a$.

2) Symmetricity: $e(a,b)=e(b,a)$.

$$e(a,b)=\begin{cases}(1-a)\cdot b, a<b,\\ 1, a=b\\ (1-b)\cdot a, b<a,\end{cases}, \text{ but } e(b,a)=\begin{cases}(1-b)\cdot a, b<a,\\ 1, a=b\\ (1-a)\cdot b, a<b,\end{cases}, \text{ therefore}$$

$$e(a,b)\equiv e(b,a) \quad \text{(Q.E.D.).}$$

3) Transitivity: $e(a,b)\wedge e(b,c)\le e(a,c) \mid \forall a,b,c\in L[0,1]$-lattice.

$$\text{From (2.52) let } F_1(a,c)=e(a,c)=\begin{cases}(1-a)\cdot c, a<c,\\ 1, a=c\\ (1-c)\cdot a, c<a,\end{cases} \tag{2.45}$$

$$\text{and } e(b,c)=\begin{cases}(1-b)\cdot c, b<c,\\ 1, b=c\\ (1-c)\cdot b, c<b,\end{cases}, \text{ then}$$

$$F_2(a,c)=e(a,b)\wedge e(b,c)=\begin{cases}e(a,b)\cdot e(b,c), e(a,b)+e(b,c)>1\\ 0, e(a,b)+e(b,c)\le 1\end{cases} \tag{2.46}$$

But

$$e(a,b)\cdot e(b,c)=\begin{cases}(1-a)\cdot b\cdot(1-b)\cdot c, a>b>c,\\ 1, a=b=c,\\ a\cdot b\cdot(1-b)\cdot(1-c), a<b<c\end{cases} \tag{2.47}$$

But since in (2.47) $\forall b\in[0,1]\Rightarrow f(b)=b\cdot(1-b)\in[0,0.25]$ and therefore from (2.45) and (2.46) the following is taking place $F_2(a,c)=f(b)\cdot F_1(a,c)<e(a,c) \mid a\ne b\ne c$ and $F_2(a,c)=e(a,c) \mid a=b=c$. In other words, we are getting the proof of the fact that $F_2(a,c)\le F_1(a,c)\Leftrightarrow e(a,b)\wedge e(b,c)\le e(a,c) \mid \forall a,b,c\in L[0,1]$ (Q.E.D.). Note that relation $e(a,b)$ is called a *fuzzy equality relation*, if additionally, separability holds, *i.e.*, $e(a,b)=1\Leftrightarrow a=b$, then let us define a *pseudo metric distance* $d(a,b)$ for domain *M*, normalized to 1, as

$$e(a,b)=1-d(a,b) \tag{2.48}$$

From (2.44) we are getting

$$d(a,b)=\begin{cases}(1-a)\cdot b, a>b,\\ 0, a=b\\ (1-b)\cdot a, b>a,\end{cases} \tag{2.49}$$

2.7.3. Approximate Fuzzy Equivalence Relations

In [3] it was mentioned, that *graduated equality* of *extended lines* compels *graduated equality* of *extended points*. **Figure 3(a)** sketches a situation where two extended lines *L* and *M* intersect in an extended point *P*. If a third extended line *L'* is very similar to *L*, its intersection with *M* yields an extended point *P'* which is very similar to *P*. It is desirable to model this fact. To do so, it is necessary

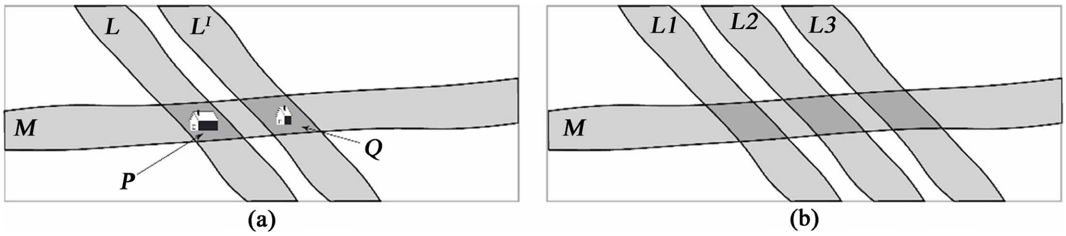

Figure 3. (a) Graduated equality of extended lines compels graduated equality of extended points. (b) Equality of extended lines is not transitive.

to allow graduated equality of extended points.

Figure 3(b) illustrates that an equality relation between extended objects need not be transitive. This phenomenon is commonly referred to as the Poincare paradox. The Poincare paradox is named after the famous French mathematician and theoretical physicist Henri Poincare, who repeatedly pointed this fact out, e.g., in [3], referring to indiscernibility in sensations and measurements. Note that this phenomenon is usually insignificant, if positional uncertainty is caused by stochastic variability. In measurements, the stochastic variability caused by measurement inaccuracy is usually much greater than the indiscernibility caused by limited resolution. For extended objects, this relation is reversed: The extension of an object can be interpreted as indiscernibility of its contributing points. In the present paper we assume that the extension of an object is being compared with the indeterminacy of its boundary. Then in [3] we also shown, that for modelling the Poincare paradox we can replace a *graduated context transitivity* by a weaker form:

$$e(a,b) \wedge e(b,c) \wedge dis(b) \le e(a,c) \tag{2.50}$$

Here $dis: M \to [0,1]$ is a lower-bound measure (*discernibility measure*) for the degree of transitivity that is permitted by *q*. A pair (*e*, *dis*) that is reflexive, symmetric and weakly transitive (2.50) is called an *approximate fuzzy* $\wedge$ *-equivalence relation*. Let us rewrite (2.50) as follows

$$F_2(a,c) \wedge dis(b) \le F_1(a,c) \tag{2.51}$$

where $F_2(a,c), F_1(b,c)$ are defined in (2.46) and (2.45) correspondingly and given (2.47) we found that

$$dis(b) \equiv b \cdot (1-b) \tag{2.52}$$

Since $\forall a \in [0,1] \Rightarrow dis(a) \in [0, 0.25]$, therefore (2.51) holds.

In [3] we also mentioned that an *approximate fuzzy* $\wedge$ *-equivalence relation* is the upside-down version of a so-called *pointless pseudo metric space* (δ, s):

$$\begin{aligned} &\delta(a,a) = 0 \\ &\delta(a,b) = \delta(b,a) \\ &\delta(a,b) \vee \delta(b,c) \vee s(b) \ge \delta(a,c) \end{aligned} \tag{2.53}$$

Here, $\delta: M \to \Re^+$ is a (not necessarily metric) distance between extended regions, and $s: M \to \Re^+$ is a *size measure* and we are using an *operation dis-*

junction (2.11) also shown in **Table 1**. Inequality $\delta(b,c) \vee s(b) \geq \delta(a,c)$ is a weak form of the triangle inequality. It corresponds to the weak transitivity (2.50) of the *approximate fuzzy* $\wedge$*-equivalence relation e*. In case the size of the domain *M* is normalized to 1, *e* and *dis* can be represented by [3]

$$e(a,b) = 1 - \delta(a,b), \quad dis(b) = 1 - s(b) \tag{2.54}$$

Note, that $\forall a \in [0,1] \Rightarrow s(a) \in [0.75,1]$

PROPOSITION 10.

If a distance between extended regions $\delta(a,b)$ from (2.53) and *pseudo metric distance* $d(a,b)$ for domain *M*, normalized to 1 be the same, *i.e.* $\delta(a,b) = d(a,b)$, then inequality $\delta(a,b) \vee \delta(b,c) \vee s(b) \geq \delta(a,c)$ holds.

Proof:

From (2.50), applying De Morgan's rule we have:

$$\neg(\neg e(a,b) \vee \neg e(b,c) \vee \neg dis(b)) \leq e(a,c) \tag{2.55}$$

And from (2.55) we are getting

$$\neg(\delta(a,b) \vee \neg\delta(b,c) \vee s(b)) \leq e(a,c), \tag{2.56}$$

$$\text{or} \quad (\delta(a,b) \vee \delta(b,c) \vee s(b)) \geq \neg e(a,c) \tag{2.57}$$

Therefore, we've gotten

$$\delta(a,b) \vee \delta(b,c) \vee s(b) \geq \delta(a,c) \quad \text{(Q.E.D.)}.$$

But as it was mentioned in [3], given a *pointless pseudo metric space* (δ, s) for extended regions on a normalized domain, equations (2.57) define an *approximate fuzzy* $\wedge$*-equivalence relation* (e, dis) by simple logical negation. The so defined equivalence relation on the one hand complies with the Poincare paradox, and on the other hand retains enough information to link two extended points (or lines) via a third. For used fuzzy logic an example of a *pointless pseudo metric space* is the set of extended points with the following measures:

$$\delta(A,B) := \inf\{d(a,b) \mid a \in A, b \in B\}, \tag{2.58}$$

$$s(A) := \sup\{d(a,b) \mid a,b \in A\}, \tag{2.59}$$

It is easy to show that (2.58) and (2.59) are satisfied, because from (2.49) $d(a,b) \in [0,1] \mid \forall c,a,b \in [0,1]$. A *pointless metric distance* of extended lines can be defined in the dual space [3]:

$$\delta(L,M) := \inf\{d(l',m') \mid l \in L, m \in M\}, \tag{2.60}$$

$$s(L) := \sup\{d(l',m') \mid l,m \in L\}, \tag{2.61}$$

2.7.4. Boundary Conditions for Granularity

As it was mentioned in [3], in exact coordinate geometry, points and lines do not have size. Therefore, distance of points does not matter in the formulation of Euclid's first postulate. If points and lines are allowed to have extension, both, size and distance matter. **Figure 4** depicts the location constraint on an extended line L that is incident with the extended points A and B.

The location constraint can be interpreted as *tolerance in the position of* L. In Figure 4(a) the distance of A and B is *large* with respect to the sizes of A and B, and with respect to the width of L. The resulting positional tolerance for L is *small*. In Figure 4(b), the distance of A and B is *smaller* than it is in Figure 4(a). As a consequence, the positional tolerance for L becomes *larger*. In Figure 4(b), A and B have the same distance than in Figure 4(a), but their sizes are increased. Again, positional tolerance of L increases. Therefore, a formalization of Euclid's first postulate for extended primitives must take all three parameters into account: the distance of the extended points, their size, and the size of the incident line.

Figure 5 illustrates this case: Despite the fact that A and B are distinct extended points that are both incident with L, they do not specify any *directional constraint* for L. Consequently, the directional parameter of the extended lines L and L' in Figure 5 may assume its maximum (at 90˚). If we measure similarity (*i.e.*, graduated equality) as inverse to distance, and if we establish a distance measure between extended lines that depends on all parameters of the line's parameter space, then L and L' in Figure 5 must have *maximum* distance. In other words, their degree of equality is zero, even though they are distinct and incident with A and B. The above observation can be interpreted as *granularity*: If we interpret the extended line L in Figure 5 as a *sensor*, then the extended points P and Q are indiscernible for L. Note that in this context grain size is not constant but depends on the line that serves as a *sensor*. Based on above mentioned a granularity enters Euclid's first postulate, if points and lines have extension: If

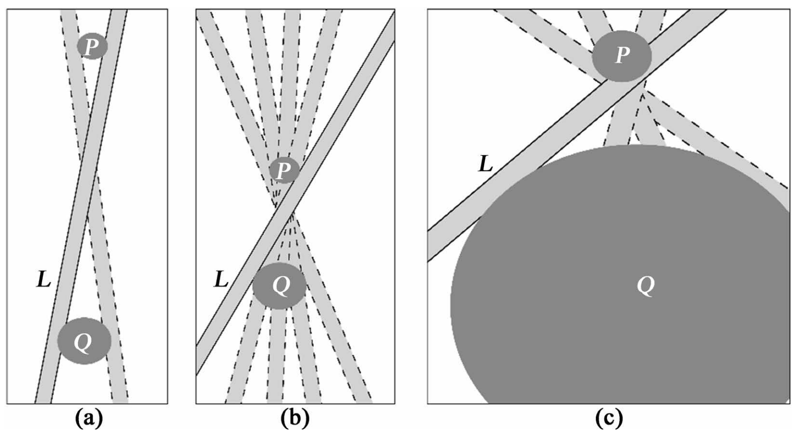

Figure 4. Size and distance matter.

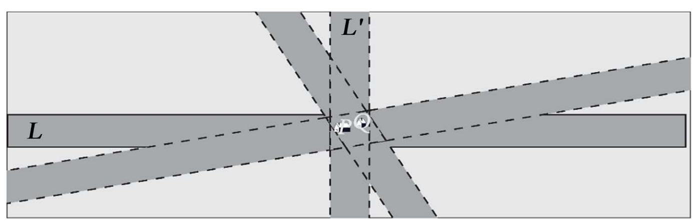

Figure 5. A and B are indiscernible for L.

two extended points P and Q are *too close* and the extended line L is *too broad*, then P and Q are *indiscernible* for L. Since this relation of *indiscernibility* (equality) depends not only on P and Q, but also on the extended line L, which acts as a *sensor*, we denote it by $e(P,Q)[L]$, where L serves as an additional parameter for the equality of P and Q. In [3] the following three boundary conditions to specify a reasonable behavior of $e(P,Q)[L]$ were proposed:

1) If $s(L) \geq \delta(P,Q) + s(P) + s(Q)$, then P and Q impose no direction constraint on L (cf. **Figure 5**), *i.e.*, P and Q are *indiscernible* for L to degree 1: $e(P,Q)[L] = 1$.

2) If $s(L) < \delta(P,Q) + s(P) + s(Q)$, then P and Q impose some direction constraint on L, but in general do not fix its location unambiguously. Accordingly, the degree of indiscernibility of P and Q lies between zero and one: $0 < e(P,Q)[L] <= 1$.

3) If $s(L) < \delta(P \setminus P,Q) + s(P) + s(Q)$ and $P = p$, $Q = q$ and $L = l$ is crisp, then $s(L) = s(P) = s(Q) = 0$. Consequently, p and q determine the direction of l unambiguously, and all positional tolerance disappears. For this case we demand $e(P,Q)[L] = 0$.

In this paper we are proposing an alternative approach to one from [4] to model granulated equality.

PROPOSITION 11.

If *Fuzzy Equivalence Relation* $e(A, B)$ is defined in (2.52) and the width $s(L)$ of *extended line* L is defined in (2.61), then $e(A,B)[L]$, the degree of *indiscernibility* of A and B could be calculated as follows:

$$e(A,B)[L] \equiv e(A,B) \wedge s(L), \tag{2.62}$$

And would satisfy a reasonable behavior, defined in 1 - 2. Here $\wedge$ is *conjunction* operator from **Table 1**.

Proof:

From (2.9), (2.62) and (2.44) we have:

$$\begin{aligned} e(A,B)[L] &\equiv e(A,B) \wedge s(L) \\ &= \begin{cases} e(A,B) \cdot s(L), e(A,B) + s(L) > 1, \\ 0, e(A,B) + s(L) \leq 1 \end{cases} \end{aligned} \tag{2.63}$$

but from (2.44)

$$e(A,B) = \begin{cases} (1-a) \cdot b, a < b, \\ 1, a = b \\ (1-b) \cdot a, b < a, \end{cases}, \tag{2.64}$$

therefore, we have the following:

1) If A and B impose no direction constraint on L which means that $s(L) = 1$ and $\delta(A,b) = 0 \Rightarrow e(A,B) = 1$, then $e(A,B)[L] = 1$ (proof of point 1).

2) If A and B impose some direction constraint on L, but in general do not fix its location unambiguously, then from (2.62)-(2.64) we are getting

$$e(A,B)[L]=\begin{cases}(1-a)\cdot b,\cdot s(L),(1-a)\cdot b+s(L)>1,a<b,\\0,(1-a)\cdot b+s(L)\le 1,\\(1-b)\cdot a\cdot s(L),(1-a)\cdot b+s(L)>1,b<a,\end{cases}\in(0,1)$$ (proof of point 2).

3) If $A = a$, $B = b$ and $L = l$ are crisp, which means that values of a and b are either 0 or 1 and since $s(L) = 0$, then $e(A,B)[L]=0$ (proof of point 3).

3. Fuzzification of Euclid's First Postulate

3.1. A Euclid's First Postulate Formalization

In previous chapter we identified and formalized a number of new qualities that enter into Euclid's first postulate, if extended geometric primitives are assumed. We are now in the position of formulating a fuzzified version of Euclid's first postulate. To do this, we first split the postulate

"Two distinct points determine a line uniquely." (3.1)

into two sub sentences:

"*Given two distinct points, there exists at least one line that passes through them.*" (3.2)

"If more than one line passes through them, then they are equal." (3.3)

These sub sentences can be formalized in Boolean predicate logic given R_{inc} from (2.40) as follows

$$\forall a,b,\exists l,\left[R_{inc}(a,l)\wedge R_{inc}(b,l)\right] \tag{3.4}$$

$$\begin{aligned}&\forall a,b,l,m\left[\neg(a=b)\right]\wedge\left[R_{inc}(a,l)\wedge R_{inc}(b,l)\right]\\&\wedge\left[R_{inc}(a,m)\wedge R_{inc}(b,m)\right]\to(l=m)\end{aligned} \tag{3.5}$$

A verbatim translation of (3.4) and (3.5) into the syntax of a fuzzy logic we use yields

$$\inf_{A,B}\sup_{L}\left[R_{inc}(A,L)\wedge R_{inc}(B,L)\right] \tag{3.6}$$

$$\begin{aligned}&\inf_{A,B,L,M}\left\{\left[\neg e(A,B)\right]\wedge\left[R_{inc}(A,L)\wedge R_{inc}(B,L)\right]\right.\\&\left.\wedge\left[R_{inc}(A,M)\wedge R_{inc}(B,M)\right]\to e(L,M)\right\}\end{aligned} \tag{3.7}$$

where A, B denote extended points, L, M denote extended lines. The translated existence property (3.6) can be adopted as it is, but the translated uniqueness property (3.7) must be adapted to include *granulated equality* of extended points. In contrast to the Boolean case, the degree of equality of two given extended points is not constant but depends on the extended line that acts as a *sensor*. Consequently, the term $\neg e(A,B)$ on the left-hand side of (3.7) must be replaced by two terms, $\neg e(A,B)[L]$ and $\neg e(A,B)[M]$, one for each line, L and M, respectively:

$$\begin{aligned}&\inf_{A,B,L,M}\left\{\left[\neg e(A,B)[L]\wedge\neg e(A,B)[M]\right]\wedge\left[R_{inc}(A,L)\wedge R_{inc}(B,L)\right]\right.\\&\left.\wedge\left[R_{inc}(A,M)\wedge R_{inc}(B,M)\right]\to e(L,M)\right\}\end{aligned} \tag{3.8}$$

We have to use weak transitivity of graduated equality. For this reason, the *discernibility measure* of extended connection $\overline{A}\overline{B}$ between extended points A and B must be added into (3.8)

$$\inf_{A,B,L,M}\left\{\left[\neg e(A,B)[L]\wedge\neg e(A,B)[M]\wedge dis\left(\overline{A}\overline{B}\right)\right]\wedge\left[R_{inc}(A,L)\wedge R_{inc}(B,L)\right]\right. \\ \left.\wedge\left[R_{inc}(A,M)\wedge R_{inc}(B,M)\right]\rightarrow e(L,M)\right) \tag{3.9}$$

But from (2.44) we get

$$\neg e(A,B)[L]=\neg e(A,B)\wedge s(L)=\begin{cases}\neg e(A,B)\cdot s(L),\neg e(A,B)+s(L)>1,\\ 0,\neg e(A,B)+s(L)\le 1\end{cases} \tag{3.10}$$

and

$$\neg e(A,B)[M]=\neg e(A,B)\wedge s(M)=\begin{cases}\neg e(A,B)\cdot s(M),\neg e(A,B)+s(M)>1,\\ 0,\neg e(A,B)+s(M)\le 1\end{cases} \tag{3.11}$$

By using (3.10) and (3.11) in (3.9) we get

$$\neg e(A,B)[L]\wedge\neg e(A,B)[M] \\ =\begin{cases}\neg e(A,B)\cdot s(L)\cdot\neg e(A,B)\cdot s(M),2\cdot\neg e(A,B)+s(L)+s(M)>1,\\ 0,2\cdot\neg e(A,B)+s(L)+s(M)\le 1\end{cases} \tag{3.12}$$

Since from (2.40) we have $\left[R_{inc}(A,L)\wedge R_{inc}(B,L)\right]\wedge\left[R_{inc}(A,M)\wedge R_{inc}(B,M)\right]\equiv 1$, then (3.9) could be rewritten as follows

$$\inf_{A,B,L,M}\left\{\left[\neg e(A,B)[L]\wedge\neg e(A,B)[M]\wedge dis\left(\overline{A}\overline{B}\right)\right]\wedge 1\rightarrow e(L,M)\right\} \tag{3.13}$$

It means that the "sameness" of extended lines $e(L, M)$ depends on $\left[\neg e(A,B)[L]\wedge\neg e(A,B)[M]\wedge dis\left(\overline{A}\overline{B}\right)\right]$ only and could be calculated by (3.12) and (2.52) respectively.

3.2. Fuzzy Logical Inference for Euclid's First Postulate

Similarly, to an approach from [3], we suggest to use the same fuzzy logic (**Table 1**) and correspondent logical inference [5] [6] [7] and [8] to determine the value of $e(L, M)$. For this purpose, let us represent a value of following $E(a,b,l,m)=\neg e(A,B)[L]\wedge\neg e(A,B)[M]$ from (3.12) and $D(a,b)=dis\left(\overline{A},\overline{B}\right)$ from (2.52) functions. Note, that values from both $E(a,b,l,m)\in\left[E_{\min},E_{\max}\right]$ and $D(a,b)\in\left[D_{\min},D_{\max}\right]$. In our case $E(a,b,l,m)\in[0,1]$ $D(a,b)\in[0,0.25]$. We represent E as a *fuzzy set* forming linguistic variable, described by a triplet of the form $E=\left\{\left\langle E_i,X,\tilde{E}\right\rangle\right\}$, $E_i\in T(x)$, $\forall i\in[0,CardX]$, where $T_i(x)$ is extended term set of the linguistic variable "*degree of indiscernibility*" from **Table 3**, $\tilde{E}$ is normal fuzzy set represented by membership function $\mu_E:X\rightarrow[0,1]$, where $X=\{0,1,2,\cdots,10\}$ universe set and $CardX$ is power set of the set U. We will use the following mapping $\alpha:\tilde{E}\rightarrow X\mid x_i=Ent\left[(CardX-1)\times E_i\right]\mid\forall i\in[0,CardX]$, where

$$\tilde{E} = \int_X \mu_E(x)/x \tag{3.14}$$

To determine the estimates of the membership function in terms of singletons from (3.14) in the form $\mu_E(x_i)/x_i \mid \forall i \in [0, CardX]$ we propose the following procedure.

$$\begin{gathered}\forall i \in [0, CardX], \forall E_i \in [0,1], \\ \mu(x_i) = 1 - \frac{1}{CardX - 1} \times \left| x_i - Ent\left[(CardX - 1) \times E_i\right] \right|,\end{gathered} \tag{3.15}$$

We also represent D as a *fuzzy set* forming linguistic variable, described by a triplet of the form $D = \{\langle D_j, Y, \tilde{D} \rangle\}$, $D_j \in T(y)$, $\forall j \in [0, CardY]$, where $T_j(y)$ is extended term set of the linguistic variable "*discernibility measure* "from **Table 3**, $\tilde{D}$ is normal fuzzy set represented by membership function $\mu_D : Y \to [0,1]$

We will use the following mapping

$\beta : \tilde{D} \to Y \mid y_j = Ent\left[(CardY - 1) \times D_j\right] \mid \forall j \in [0, CardY]$, where

$$\tilde{D} = \int_Y \mu_D(y)/y \tag{3.16}$$

On the other hand, to determine the estimates of the membership function in terms of singletons from (3.16) in the form $\mu_D(y_j)/y_j \mid \forall j \in [0, CardY]$ we propose the following procedure.

$$\begin{gathered}\forall j \in [0, CardY], \forall D_j \in [0, 0.25], \\ \mu(y_j) = 1 - \frac{1}{CardY - 1} \times \left| y_j - Ent\left[(CardY - 1) \times D_j / 0.25\right] \right|,\end{gathered} \tag{3.17}$$

Let us represent *e* (*L*, *M*) as a *fuzzy set* $\tilde{S}$, forming linguistic variable, described by a triplet of the form $S = \{\langle S_k, Z, \tilde{S} \rangle\}$, $S_k \in T(z)$, $\forall k \in [0, CardZ]$, where $T_k(z)$ is extended set term set of the linguistic variable "*extended lines sameness"* from **Table 3**. $\tilde{S}$ is normal fuzzy set represented by membership function $\mu_S : Z \to [0,1]$, where $Z = \{0, 1, 2, \cdots, 10\}$ universe set and $CardZ$ is power set of the set *Z*. We will use the following mapping

$\gamma : \tilde{S} \to Z \mid z_k = Ent\left[(CardZ - 1) \times S_k\right] \mid \forall k \in [0, CardZ]$, were

$$\tilde{S} = \int_Z \mu_s(z)/z \tag{3.18}$$

Again, to determine the estimates of the membership function in terms of singletons from (3.18) in the form $\mu_S(z_k)/z_k \mid \forall k \in [0, CardZ]$ we propose the following procedure.

$$\begin{gathered}\forall k \in [0, CardZ], \forall S_k \in [0,1], \\ \mu(z_k) = 1 - \frac{1}{CardZ - 1} \times \left| z_k - Ent\left[(CardZ - 1) \times S_k\right] \right|,\end{gathered} \tag{3.19}$$

To get an estimates of values of *e*(*L*,*M*) or "*extended lines sameness"*, represented by fuzzy set $\tilde{S}$ from (3.18) given the values of $E(a, b, l, m)$ or "*degree of indiscernibility*" and $D(a, b)$ "*discernibility measure*", represented by fuzzy sets $\tilde{E}$ from (3.14) and $\tilde{D}$ from (3.16) respectively, we will use a *Fuzzy Conditional*

Table 3. Linguistic variables in fuzzy logical inference for Euclid's first postulate.

Value of variable			$x_i \in X, y_j \in Y, z_k \in Z$
"*Degree of indiscernibility*"	"*Discernibility measure*"	"*Extended lines sameness*"	$\forall i, j, k \in [0,10]$
Lowest	highest	nothing in common	0
very low	almost highest	very far	1
Low	high	far	2
bit higher than low	pretty high	bit closer than far	3
almost average	bit higher than average	almost average distance	4
Average	average	average	5
bit higher than average	almost average	bit closer than average	6
pretty high	bit higher than low	pretty close	7
High	low	close	8
almost highest	very low	almost the same	9
Highest	lowest	the same	10

Inference Rule, formulated by means of *"common sense"* as a following conditional clause:

$$P = \text{"IF (}\tilde{E}\text{ is }X\text{) AND (}\tilde{D}\text{ is }Y\text{), THEN (}\tilde{S}\text{ is }Z\text{)"} \quad (3.20)$$

In other words, we use fuzzy conditional inference of the following type [7]:

$$\begin{array}{l} \text{Ant 1: If } e \text{ is } E \text{ and } d \text{ is } D \text{ then } s \text{ is } S \\ \text{Ant2: } e \text{ is } E' \text{ and } d \text{ is } D' \\ \text{----------------------------------}, \\ \text{Cons: } s \text{ is } S'. \end{array} \quad (3.21)$$

where $E, E' \subseteq X$, $D, D' \subseteq Y$ and $S, S' \subseteq Z$.

Now for fuzzy sets (3.14), (3.16) and (3.18) a *binary relationship* for the fuzzy conditional proposition of the type (3.20) and (3.21) for fuzzy logic we use so far is defined as

$$\begin{aligned} &R(A_1(e,d), A_2(s)) = [E \times X \cap D \times Y] \\ &\to Z \times S = \int_{X \times Y \times Z} ([\mu_E(x)/x \wedge \mu_D(y)/y] \to \mu_S(z))/(x,y,z) \end{aligned} \quad (3.22)$$

Given (2.10) and since we consider that $CardX = CardY = CardZ$, then expression (3.22) looks like

$$\begin{aligned} &[\mu_E(x) \wedge \mu_D(y)] \\ &\to \mu_S(z) = \begin{cases} (1 - [\mu_E(x) \wedge \mu_D(y)]) \cdot \mu_S(z), [\mu_E(x) \wedge \mu_D(y)] > \mu_S(z), \\ 1, [\mu_E(x) \wedge \mu_D(y)] \le \mu_S(z). \end{cases} \end{aligned} \quad (3.23)$$

where $[\mu_E(x) \wedge \mu_D(y)]$ is $\min[\mu_E(x), \mu_D(y)]$. It is well known that given a *unary relationship* $R(A_1(e,d)) = E' \cap D'$ one can obtain the consequence $R(A_2(e))$ by applying compositional rule of inference (CRI) to $R(A_1(e,d))$

and $R(A_1(e,d), A_2(e))$ of type (3.22):

$$R(A_2(s)) = E' \cap D' \circ R(A_1(e,d), A_2(s))$$
$$= \int_{X\times Y} [\mu_{E'}(x) \wedge \mu_{D'}(y)]/(x,y) \circ \int_{X\times Y\times Z} [\mu_E(x) \wedge \mu_D(y)] \to \mu_S(z)/(x,y,z) \quad (3.24)$$
$$= \int_Z \bigcup_{x\in X, y\in Y} \{[\mu_{E'}(x) \wedge \mu_{D'}(y)] \wedge ([\mu_E(x) \wedge \mu_D(y)] \to \mu_S(z))\}/z$$

But for practical purposes we will use another *Fuzzy Conditional Rule* (*FCR*)

$$R(A_1(e,d), A_2(s)) = (P \times U \to V \times S) \cap (\neg P \times U \to V \times \neg S)$$
$$= \int_{U\times V} (\mu_P(u) \to \mu_S(v)) \wedge ((1-\mu_P(u)) \to (1-\mu_S(v)))/(u,v) \quad (3.25)$$

where $P = E \cap D$ and $U = X = Y$, therefore from (3.25) we are getting

$$R(A_1(e,d), A_2(s)) = (\mu_P(u) \to \mu_S(v)) \wedge ((1-\mu_P(u)) \to (1-\mu_S(v)))$$
$$= \begin{cases} (1-\mu_P(u)) \cdot \mu_S(v), \mu_P(u) > \mu_S(v), \\ 1, \mu_P(u) = \mu_S(v), \\ (1-\mu_s(v)) \cdot \mu_P(u), \mu_P(u) < \mu_S(v). \end{cases} \quad (3.26)$$

The *FCR* from (3.26) gives more reliable results.

3.3. Example

To build a binary relationship matrix of type (3.25) we us use a conditional clause of type (3.20):

P = "IF (S is "lowest") AND (D is "highest"), THEN (E is "nothing in common") (3.27)

To build membership functions for fuzzy sets S, D and E we use (3.15), (3.17) and (3.19) respectively.

In (3.27) the membership functions for fuzzy set S (for instance) would look like:

$$\mu_s(\text{"lowest"}) = 1/0 + 0.9/1 + 0.8/2 + 0.7/3 + 0.6/4 + 0.5/5 + 0.4/6 + 0.3/7 + 0.2/8 + 0.1/9 + 0/10 \quad (3.28)$$

Same membership functions we use for fuzzy sets D and E.

From (3.26) we have $R(A_1(s,d), A_2(e))$ from **Table 4**.

Suppose from (3.12) a current estimate of $E(a,b,l,m) = 0.6$ and from (2.62) $D(a,b) = 0.25$. By using (3.15) and (3.17) respectively we got (see **Table 2**)

$$\mu_E(\text{"bit higher than average"}) = 0.4/0 + 0.5/1 + 0.6/2 + 0.7/3 + 0.8/4 + 0.9/5 + 1/6 + 0.9/7 + 0.8/8 + 0.7/9 + 0.6/10$$

$$\mu_D(\text{"pretty high"}) = 0.7/0 + 0.8/1 + 0.9/2 + 1/3 + 0.9/4 + 0.8/5 + 0.7/6 + 0.6/7 + 0.5/8 + 0.4/9 + 0.3/10$$

It is apparent that:

$$R(A_1(s',d')) = \mu_E(u) \wedge \mu_D(u) = 0.4/0 + 0.5/1 + 0.6/2 + 0.7/3 + 0.8/4 + 0.8/5 + 0.7/6 + 0.6/7 + 0.5/8 + 0.4/9 + 0.3/10$$

Table 4. Binary relationship matrix of a current example.

$p \to s$	**1**	**0.9**	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**	**0**
1	1	0	0	0	0	0	0	0	0	0	0
0.9	0	1	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0
0.8	0	0.08	1	0.14	0.12	0.1	0.08	0.06	0.04	0.02	0
0.7	0	0.07	0.14	1	0.18	0.15	0.12	0.09	0.06	0.03	0
0.6	0	0.06	0.12	0.18	1	0.2	0.16	0.12	0.08	0.04	0
0.5	0	0.05	0.1	0.15	0.2	1	0.2	0.15	0.1	0.05	0
0.4	0	0.04	0.08	0.12	0.16	0.2	1	0.18	0.12	0.06	0
0.3	0	0.03	0.06	0.09	0.12	0.15	0.18	1	0.14	0.07	0
0.2	0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	1	0.08	0
0.1	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	1	0
0	0	0	0	0	0	0	0	0	0	0	1

By applying compositional rule of inference (CRI) to $R\left(A_1\left(s',d'\right)\right)$ and $R\left(A_1\left(s,d\right),A_2\left(e\right)\right)$ from Table 4.

$R\left(A_2\left(e'\right)\right)=R\left(A_1\left(s',d'\right)\right)\circ R\left(A_1\left(s,d\right),A_2\left(e\right)\right)$ we got the following:

$$R\left(A_2\left(e'\right)\right)=\mu_E\left(u\right)\wedge\mu_D\left(u\right)$$
$$=0.4/0+0.5/1+0.6/2+0.7/3+0.8/4+0.8/5$$
$$+0.7/6+0.6/7+0.5/8+0.4/9+0.3/10$$

It is obvious that the value of fuzzy set S is laying between terms *"almost average distance"* and *"average distance"* (see Table 2), which means that approximate values for $e(L.M)$ are $e\left(L,M\right)\in\left[0.5,0.6\right]$.

4. Logical Principles of AIA Orientation

4.1. Preliminary Considerations

Let consider that both Target and Object, a subject of mutual navigation, to be presented as octagons, depicted on Figure 6. We use octagons for simplification's sake only. Given the fact that we are studying a *projection-based model*, both targets and objects could be presented as follows:

$T=\left\{t_j\right\};j=\overline{1,n}$. Where j is number of heights of a Target, whereas $O=\left\{o_i\right\};i=\overline{1,m}$ and i is number of heights of an Object. Both a target and an object could be presented in three-dimensional space as follows:

$$t_j\in T=\left\{x_j^t,y_j^t,z_j^t\right\};\ j=\overline{1,n},\ o_j\in O=\left\{x_i^o,y_i^o,z_i^o\right\};i=\overline{1,m}. \tag{4.1}$$

On the other hand, from Figure 6 each value of both a Target and an Object coordinate could be presented as a pair of minimal and maximal (per 3D coordinate) values of them. For targets, in particular

$$\forall j=\overline{1,n}\mid x_{\min}^T=\min_j\{x_j^t\},x_{\max}^T=\max_j\left\{x_j^t\right\},\ y_{\min}^T=\min_j\left\{y_j^t\right\},y_{\max}^T=\max_j\left\{y_j^t\right\},$$
$$z_{\min}^T=\min_j\left\{z_j^t\right\},z_{\max}^T=\max_j\left\{z_j^t\right\}. \tag{4.2}$$

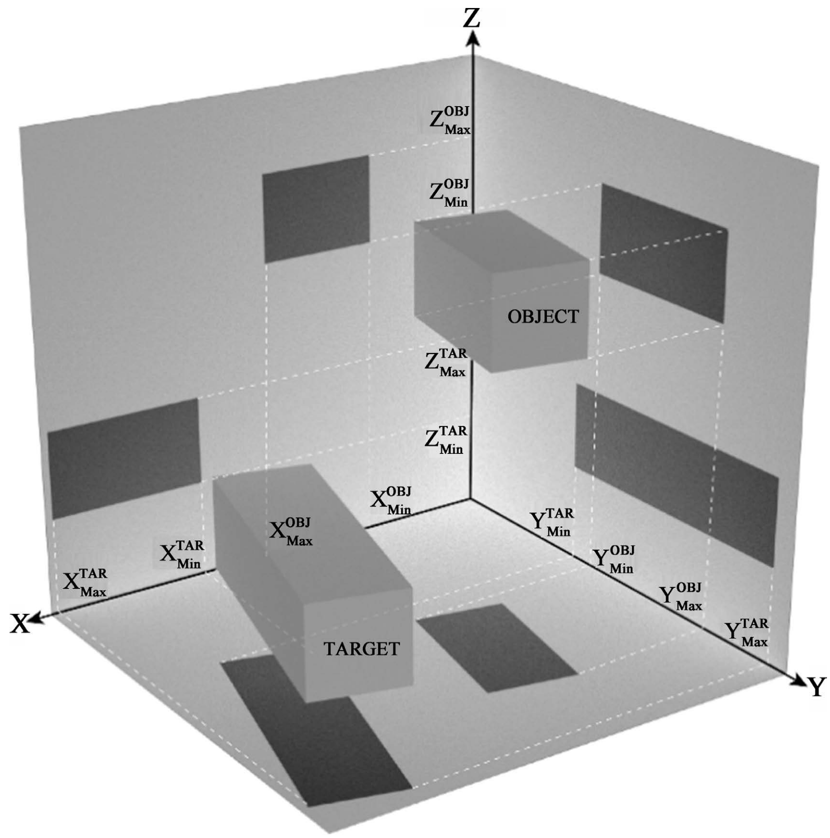

Figure 6. Object and target space representation.

By analogy, for objects we are getting:

$$\forall i=\overline{1,m} \mid x^O_{\min}=\min_i\{x^o_i\}, x^O_{\max}=\max_i\{x^o_i\},\ y^O_{\min}=\min_i\{y^o_i\}, y^O_{\max}=\max_i\{y^o_i\}, \\ z^O_{\min}=\min_i\{z^o_i\}, z^O_{\max}=\max_i\{z^o_i\}. \tag{4.3}$$

It is import to consider the following features of both target and object from **Figure 6**, *i.e.*, width (w), depth (d) and height (h) of them:

$$\begin{aligned} w^T &= \left|x^T_{\max}-x^T_{\min}\right|\cdot\tan\left(\theta(x,y)\right); \\ d^T &= \left|y^T_{\max}-y^T_{\min}\right|\cdot\tan\left(\theta(x,z)\right); \\ h^T &= \left|z^T_{\max}-z^T_{\min}\right|\cdot\tan\left(\theta(z,y)\right), \end{aligned} \tag{4.4}$$

$$\begin{aligned} w^O &= \left|x^O_{\max}-x^O_{\min}\right|\cdot\tan\left(\theta(x,y)\right); \\ d^O &= \left|y^O_{\max}-y^O_{\min}\right|\cdot\tan\left(\theta(x,z)\right); \\ h^O &= \left|z^O_{\max}-z^O_{\min}\right|\cdot\tan\left(\theta(z,y)\right). \end{aligned} \tag{4.5}$$

where $\theta(x,y)$, $\theta(x,z)$ and $\theta(z,y)$ will be defined in (4.19).

4.2. Predicates of Two Entities Mutual Relations

Considering (4.2)-(4.5) we can formulate some logical predicates, which would describe mutual positioning of two players in the paradigm of a *projection–based model*. Let us define predicates as relation symbols, describing a variety of positions of two entities in a space in a connection to each other.

4.2.1. Size Comparison Predicates

We define size comparison of Target and Object as the following relation symbols, given (4.4) and (4.5)

$$\begin{aligned} &LARGER(T,O) \Rightarrow w^T > w^O \& | d^T > d^O \& h^T > h^O, \\ &SMALLER(T,O) \Rightarrow w^T < w^O \& | d^T < d^O \& h^T < h^O \end{aligned} \tag{4.6}$$

$$\begin{aligned} &LARGER(O,T) \Rightarrow w^O > w^T \& | d^O > d^T \& h^O > h^T, \\ &SMALLER(O,T) \Rightarrow w^O < w^T \& | d^O < d^T \& h^O < h^T \end{aligned} \tag{4.7}$$

4.2.2. Mutual Positioning Predicates

We also define mutual positioning of a Target and an Object from **Figure 6** as the following relation symbols, given (4.2) and (4.3)

$$HIGHER(T,O) \Rightarrow z_{\min}^T \geq z_{\max}^O, LOWER(T,O) \Rightarrow z_{\min}^T \leq z_{\max}^O \tag{4.8}$$

$$HIGHER(O,T) \Rightarrow z_{\min}^O \geq z_{\max}^T, LOWER(O,T) \Rightarrow z_{\min}^O \leq z_{\max}^T \tag{4.9}$$

$$ATLEFT(T,O) \Rightarrow x_{\max}^T \leq x_{\min}^O, ATRIGHT(T,O) \Rightarrow x_{\max}^T \geq x_{\min}^O \tag{4.10}$$

$$ATLEFT(O,T) \Rightarrow x_{\max}^O \leq x_{\min}^T, ATRIGHT(O,T) \Rightarrow x_{\max}^O \geq x_{\min}^T \tag{4.11}$$

$$BEHIND(T,O) \Rightarrow y_{\max}^T \leq y_{\min}^O, INFRONT(T,O) \Rightarrow y_{\max}^T \geq y_{\min}^O \tag{4.12}$$

$$BEHIND(O,T) \Rightarrow y_{\max}^O \leq y_{\min}^T, INFRONT(O,T) \Rightarrow y_{\max}^O \geq y_{\min}^T \tag{4.13}$$

$$ONTOP(T,O) \Rightarrow z_{\min}^T = z_{\max}^O, ONBOTTOM(T,O) \Rightarrow z_{\max}^T = z_{\min}^O \tag{4.14}$$

$$ONTOP(O,T) \Rightarrow z_{\min}^O = z_{\max}^T, ONBOTTOM(O,T) \Rightarrow z_{\max}^O = z_{\min}^T \tag{4.15}$$

4.2.3. Preconditions for Actions and Entity Shape Estimation

Before formulation of a possible actions, which could be performed by certain entities, and given (4.2) and (4.3) we have to consider for each entity the following points in 3-dimentional space $T_{center} = \{x_{center}^T, y_{center}^T, z_{center}^T\}$ for a Target and $O_{center} = \{x_{center}^O, y_{center}^O, z_{center}^O\}$ for an Object correspondingly, These points could define some conditional center of a gravity for each of them (median points in space)

$$x_{center}^T = \frac{x_{\max}^T + x_{\min}^T}{2}, x_{center}^O = \frac{x_{\max}^O + x_{\min}^O}{2} \tag{4.16}$$

$$y_{center}^T = \frac{y_{\max}^T + y_{\min}^T}{2}, y_{center}^O = \frac{y_{\max}^O + y_{\min}^O}{2} \tag{4.17}$$

$$z_{center}^T = \frac{z_{\max}^T + z_{\min}^T}{2}, z_{center}^O = \frac{z_{\max}^O + z_{\min}^O}{2} \tag{4.18}$$

We use the following 3-coordinate mapping, depicted in **Figure 7** to define an angel in 3D space between a Target and an Object, *i.e.*, between points T_{center} and O_{center}.

Where each dimensional angel (mapping) $\theta(x,y), \theta(x,z)$ and $\theta(z,y)$ is defined as follows

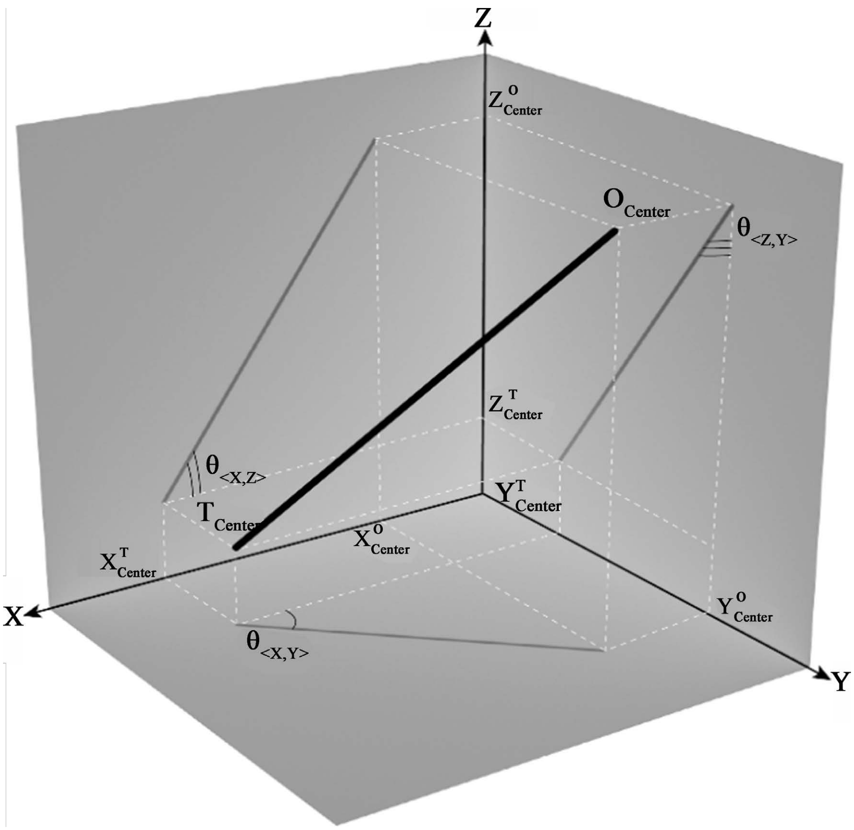

Figure 7. Angels mapping for object and target.

$$\theta(x,y)=\begin{cases}\dfrac{\pi}{2}-\arctan\left(\dfrac{y_{center}^{T}-y_{center}^{O}}{x_{center}^{T}-x_{center}^{O}}\right), x_{center}^{T}>x_{center}^{O},\\ \dfrac{3\pi}{2}-\arctan\left(\dfrac{y_{center}^{T}-y_{center}^{O}}{x_{center}^{T}-x_{center}^{O}}\right), x_{center}^{T}<x_{center}^{O};\\ 0, x_{center}^{T}=x_{center}^{O}, y_{center}^{T}>y_{center}^{O},\\ 1, x_{center}^{T}=x_{center}^{O}, y_{center}^{T}<y_{center}^{O},\end{cases}$$

$$\theta(x,z)=\begin{cases}\dfrac{\pi}{2}-\arctan\left(\dfrac{z_{center}^{T}-z_{center}^{O}}{x_{center}^{T}-x_{center}^{O}}\right), x_{center}^{T}>x_{center}^{O},\\ \dfrac{3\pi}{2}-\arctan\left(\dfrac{z_{center}^{T}-z_{center}^{O}}{x_{center}^{T}-x_{center}^{O}}\right), x_{center}^{T}<x_{center}^{O};\\ 0, x_{center}^{T}=x_{center}^{O}, z_{center}^{T}>y_{center}^{O},\\ 1, x_{center}^{T}=x_{center}^{O}, z_{center}^{T}<z_{center}^{O},\end{cases} \tag{4.19}$$

$$\theta(z,y)=\begin{cases}\dfrac{\pi}{2}-\arctan\left(\dfrac{y_{center}^{T}-y_{center}^{O}}{z_{center}^{T}-z_{center}^{O}}\right), z_{center}^{T}>z_{center}^{O},\\ \dfrac{3\pi}{2}-\arctan\left(\dfrac{y_{center}^{T}-y_{center}^{O}}{z_{center}^{T}-z_{center}^{O}}\right), z_{center}^{T}<z_{center}^{O},\\ 0, z_{center}^{T}=z_{center}^{O}, y_{center}^{T}>y_{center}^{O},\\ 1, z_{center}^{T}=z_{center}^{O}, y_{center}^{T}<y_{center}^{O},\end{cases}$$

Let us also introduce the following values, which define entities coordinate derivation in space.

1) Vertical (Z-dimension) derivation would be presented as

$$\forall x, y \mid x \in [x_{\min}, x_{\max}], y \in [y_{\min}, y_{\max}] \Rightarrow \Delta z(x, y) = z^{\max}(x, y) - z^{\min}(x, y) \quad (4.20)$$

2) Horizontal (X-dimension) derivation would be presented as

$$\forall z, y \mid z \in [z_{\min}, z_{\max}], y \in [y_{\min}, y_{\max}] \Rightarrow \Delta x(z, y) = x^{\max}(z, y) - x^{\min}(z, y) \quad (4.21)$$

3) Y-dimension derivation would be presented as

$$\forall x, z \mid x \in [x_{\min}, x_{\max}], z \in [z_{\min}, z_{\max}] \Rightarrow \Delta y(x, z) = y^{\max}(x, z) - y^{\min}(x, z) \quad (4.22)$$

4.2.4. Entity Shape Estimation Predicates

We define the following predicates by using (4.20)-(4.22)

1) Entity has *right geometric form* (**RGF**)

$$\begin{aligned} &RGF(E) \Rightarrow \Delta z^E(x^E, y^E) \equiv const \,\&\, \Delta x^E(z^E, y^E) \equiv const \\ &\&\, \Delta y^E(x^E, y^E) \equiv const \end{aligned} \quad (4.23)$$

2) Entity has *flat left* and *right surfaces* (**FLRS**)

$$FLRS(E) \Rightarrow \Delta x^E(z^E, y^E) \equiv const, \forall z^E, y^E \mid z^E \in [z^E_{\min}, z^E_{\max}], y^E \in [y^E_{\min}, y^E_{\max}] \quad (4.24)$$

3) Entity has *flat top* and *bottom* (**FTB**)

$$FTB(E) \Rightarrow \Delta z^E(x^E, y^E) \equiv const, \forall x^E, y^E \mid x^E \in [x^E_{\min}, x^E_{\max}], y^E \in [y^E_{\min}, y^E_{\max}] \quad (4.25)$$

4) Entity has *flat front* and *back surfaces* (**FFBS**)

$$FFBS(E) \Rightarrow \Delta y^E(x^E, z^E) \equiv const, \forall x^E, z^E \mid x^E \in [x^E_{\min}, x^E_{\max}], z^E \in [z^E_{\min}, z^E_{\max}] \quad (4.26)$$

4.2.5. Docking Positioning Predicates

We define the following predicates by using (4.16)-(4.18)

1) Object *docks in front* of a Target (**DIF**)

$$DIF(O,T) \Rightarrow x^T_{center} = x^O_{center} \,\&\, z^T_{center} = z^O_{center} \,\&\, y^T_{\max} = y^O_{\min} \quad (4.27)$$

2) Object *docks at back* of a Target (**DAB**)

$$DAB(O,T) \Rightarrow x^T_{center} = x^O_{center} \,\&\, z^T_{center} = z^O_{center} \,\&\, y^T_{\min} = y^O_{\max} \quad (4.28)$$

3) Object *docks at left* of a Target (**DAL**)

$$DAL(O,T) \Rightarrow y^T_{center} = y^O_{center} \,\&\, z^T_{center} = z^O_{center} \,\&\, x^T_{\min} = x^O_{\max} \quad (4.29)$$

4) Object *docks at right* of a Target (**DAR**)

$$DAR(O,T) \Rightarrow y^T_{center} = y^O_{center} \,\&\, z^T_{center} = z^O_{center} \,\&\, x^T_{\max} = x^O_{\min} \quad (4.30)$$

5) Object *docks on top* of a Target (**DOT**)

$$DOT(O,T) \Rightarrow x^T_{center} = x^O_{center} \,\&\, y^T_{center} = y^O_{center} \,\&\, z^T_{\max} = z^O_{\min} \quad (4.31)$$

6) Object *docks under* (*at bottom*) of a Target (**DUN**)

$$DUN(O,T) \Rightarrow x^T_{center} = x^O_{center} \,\&\, y^T_{center} = y^O_{center} \,\&\, z^T_{\min} = z^O_{\max} \quad (4.32)$$

It is well known fact, that distance $\boldsymbol{d}$ between two points $a,b \in \Re^3$ represents the Euclidean distance in $\Re^3$. Let us define Euclidean distance for a Target and an Object (between points T_{center} and O_{center}) from **Figure 7** the following way

$$d(T,O)=\sqrt{\left(x_{center}^{T}-x_{center}^{O}\right)^{2}+\left(y_{center}^{T}-y_{center}^{O}\right)^{2}+\left(z_{center}^{T}-z_{center}^{O}\right)^{2}}, \quad (4.33)$$

We have to take into account the fact that (4.33) presents idealistic case for two points in space, whereas for Target and Object from **Figure 6** we have to use their real size values. For this purpose, given (4.4) and (4.5) we introduce the following.

$$\dim^{T}=\max\left\{w^{T},d^{T},h^{T}\right\};\ \dim^{O}=\max\left\{w^{O},d^{O},h^{O}\right\}. \quad (4.34)$$

Therefore, the real distance between Target and Object from **Figure 6**, given (4.33), (4.34) could be defined like that

$$d_{Real}(T,O)=d(T,O)-dim^{T}/2-dim^{O}/2 \quad (4.35)$$

We define the degree $N_{\left(dim^{T},dim^{O}\right)}(T,O)$, to which a Target and an Object, *i.e.*, two points $T_{center},O_{center}\in\Re^3$, given (4.34) and (4.25) are *near each other*, by using an idea from [9] as

$$N_{\left(dim^{T},dim^{O}\right)}(T,O)=\begin{cases}1, d_{Real}(T,O)\le dim^{T},\\ 0, d_{Real}(T,O)>dim^{T}+dim^{O},\\ \dfrac{dim^{T}+dim^{O}-d_{Real}(T,O)}{dim^{O}}, \text{othervise.}\end{cases} \quad (4.36)$$

Note how $N_{\left(dim^{T},dim^{O}\right)}(T,O)$ can be regarded as the degree to which the distance between a Target and an Object is at most about dim^{T}. The value of dim^{O} defines how flexible "about dim^{T}" is interpreted. On the other hand, the degree $F_{\left(dim^{T},dim^{O}\right)}(T,O)$ to which a Target and an Object, *i.e.*, two points $T_{center},O_{center}\in\Re^3$, given (4.34) and (4.25) are *far from each other* is defined like $F_{\left(dim^{T},dim^{O}\right)}(T,O)=1-N_{\left(dim^{T},dim^{O}\right)}(T,O)$, *i.e.*

$$F_{\left(dim^{T},dim^{O}\right)}(T,O)=\begin{cases}0, d_{Real}(T,O)\le dim^{T},\\ 1, d_{Real}(T,O)>dim^{T}+dim^{O},\\ \dfrac{d_{Real}(T,O)-dim^{T}}{dim^{O}}, \text{othervise.}\end{cases} \quad (4.37)$$

The relationship between $N_{\left(dim^{T},dim^{O}\right)}(T,O)$ and $F_{\left(dim^{T},dim^{O}\right)}(T,O)$ on one hand, and $d_{Real}(T,O)$ on the other is depicted in **Figure 8**.

We conclude this work with the last important fuzzy feature, related to a human like perception of a world by a set of terms like *"North-East-South-West"*, when each of sub terms could be represented by an angle $\theta(x,y)$ from (4.19). Using a technique from [3] we model a vague cardinal direction by three parameters: $\theta, dev\theta$ and $\Delta\theta$, where θ is the most prototypical angle for each cardinal direction, *i.e.* "North", "East", "South" or "West" and the allowed deviation

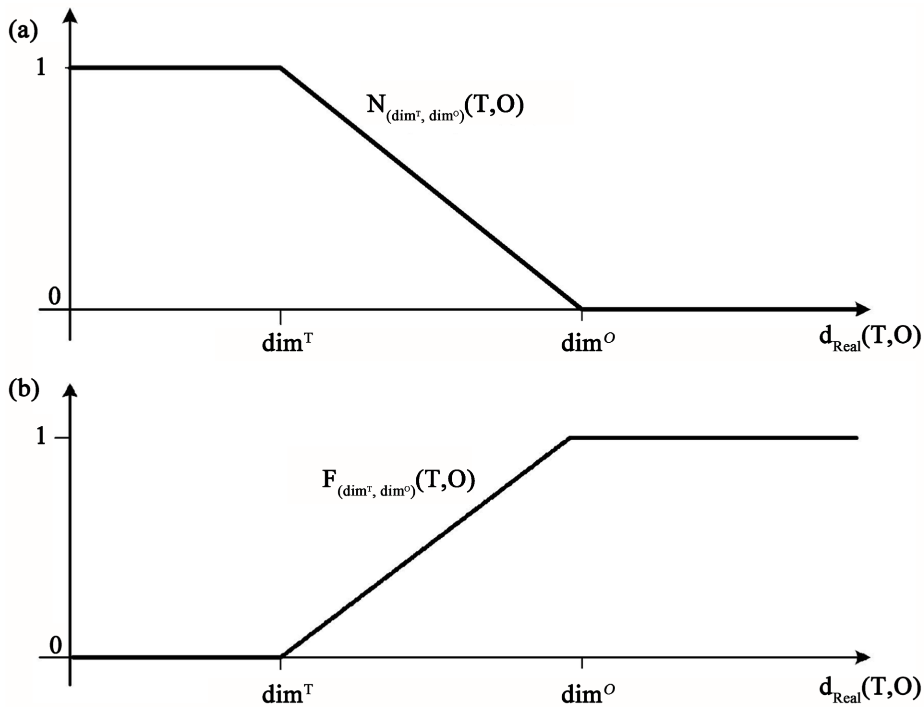

Figure 8. Relationship between the degree to which points $T_{center}, O_{center} \in \Re^3$ are near (a) or far (b) from each other, and their distance $d_{Real}(T,O)$.

from θ is about $dev\theta$. Again $\Delta\theta$ models how flexible "about $dev\theta$" is interpreted. We define $D_{(\theta(x,y),dev\theta,\Delta\theta)}(T,O)$ as

$$D_{(\theta(x,y),dev\theta,\Delta\theta)}(T,O)=\begin{cases}1, & ad(\theta(x,y),\theta)\le dev\theta,\\ 0, & ad(\theta(x,y),\theta)>dev\theta+\Delta\theta,\\ \dfrac{dev\theta+\Delta\theta-ad(\theta(x,y),\theta)}{\Delta\theta}, & \text{othervise.}\end{cases}$$

where $ad(\ldots)$ represents the unsigned angular difference [3], *i.e.*, for θ_1 and θ_2 in R, we define

$$ad(\theta_1,\theta_2)=\min\left(norm(\theta_2-\theta_1),2\pi-norm(\theta_2-\theta_1)\right),$$

where $norm(\theta)=\theta+2k\pi$ and k is the unique integer satisfying $\theta+2k\pi\in[0,2\pi]$. By analogy, we define the positive angular difference pad for θ_1 and θ_2 in $\Re$ by

$$pad(\theta_1,\theta_2)=norm(\theta_2-\theta_1).$$

For example, "East" could be modelled by the fuzzy relation $D_{\left(\frac{\pi}{2},\frac{\pi}{8},\frac{\pi}{4}\right)}$. The fuzzy set of points, which are "East" of a reference point T_{center}, using this interpretation, is displayed in **Figure 9**. In this figure, membership degrees $D_{\left(\frac{\pi}{2},\frac{\pi}{8},\frac{\pi}{4}\right)}(T,O)$ for various points O_{center} are depicted using grayscale colors, black being membership degree 1 and white being 0.

Now we are going to put together a set of an elements of Object/Target mutual

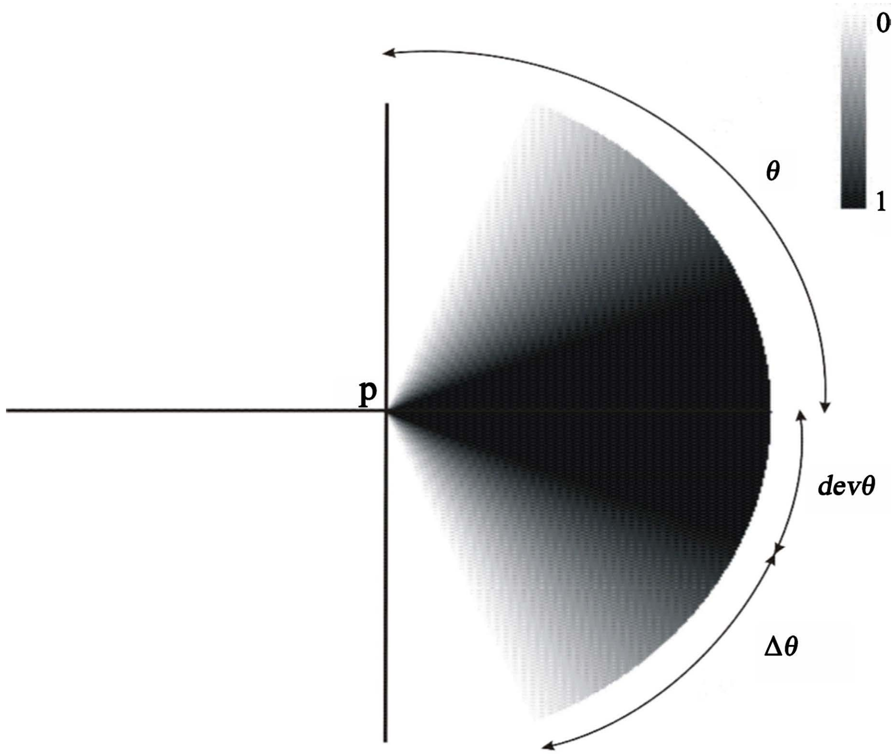

Figure 9. Fuzzy set $D_{\left(\frac{\pi}{2},\frac{\pi}{8},\frac{\pi}{4}\right)}(p,.)$ of points which are east of *p*.

behavior. Let us formulate the "Tactical Level" of an Object decision making mechanism, the goal of which is to figure out if there is a way to achieve the ultimate outcome of its potential ACTIONs.

4.3. Feasibility of a Goal Setting

Let us define a notion of a Pre-conditions and GOALs of an Object and a Target mutual interaction. From the position of common sense, we consider a set of *Size Comparison Predicates* from (4.6), (4.7), *Mutual Positioning Predicates* from (4.8)-(4.15) and *Entity Shape Estimation Predicates* from (4.23)-(4.26) as a set of Pre-conditions to find out if any particular GOAL of an Object/Target setting is achievable. We also consider the set of *Docking Positioning Predicates* from (4.27)-(4.32) as an Object action GOALs (for the sake of current discussion).

We have to note that by the term *Docking* we presume unfriendly (no specific docking mechanism in place) approaching of a Target by an Object. We also presume that the size of an Object is *SMALLER*, than a Target one. It means that both (4.38) and (4.39) as well, as all sentences from an APPENDIX, should also include $T\left(SMALLER\left(O,Tar\right)\right)$ term.

A feasibility of achieving a Target by an Object could be described by the set of predicates based logical sentences of the following structure.

$$T\left(RGF\left(O\right)\right)\wedge T\left(RGF\left(Tar\right)\right)\wedge T\left(HIGHER\left(O,Tar\right)\right)\Rightarrow T\left(DOT\left(O,Tar\right)\right) \quad (4.38)$$

This (4.38) logical sentence formulates a specific rule, which could be presented by natural language as follows: "IF both a Target and an Object have *right geometric form* AND an Object is positioned ***HIGHER***, than a Target AND Object's size is *SMALLER*, THEN the GOAL of an Object being docked on *TOP* of a Target is **achievable**.

Similarly, the following sentence formulates an opposite logical outcome.

$$T\left(RGF\left(O\right)\right)\wedge T\left(RGF\left(Tar\right)\right)\wedge T\left(LOWER\left(O,Tar\right)\right)\Rightarrow F\left(DOT\left(O,Tar\right)\right) \quad (4.39)$$

This (4.39) logical sentence formulates a specific rule, which could be presented in natural language as follows: "IF both a Target and an Object have *right geometric form* AND an Object is positioned ***LOWER***, than a Target AND Object's size is *SMALLER*, than Target's one, THEN the GOAL of an Object being docked on *TOP* of a Target is **NOT achievable**.

Let us also consider a couple cases from an APPENDIX, which describe another logical presumption.

$$T\left(RGF\left(O\right)\right)\wedge T\left(RGF\left(Tar\right)\right)\wedge T\left(ATLEFT\left(O,Tar\right)\right)\Rightarrow T\left(DAL\left(O,Tar\right)\right) \quad (4.40)$$

This (4.40) logical sentence formulates a specific rule, which could be presented by natural language as follows: "IF both a Target and an Object have *right geometric form* AND an Object is positioned ***ATLEFT*** of a Target AND Object's size is *SMALLER*, THEN the GOAL of an Object being docked at *LEFT* of a Target is **achievable**.

$$T\left(RGF\left(O\right)\right)\wedge T\left(RGF\left(Tar\right)\right)\wedge T\left(ATRIGHT\left(O,Tar\right)\right)\Rightarrow F\left(DAL\left(O,Tar\right)\right) \quad (4.41)$$

This (4.41) logical sentence formulates a specific rule, which could be presented by natural language as follows: "IF both a Target and an Object have *right geometric form* AND an Object is positioned ***ATRIGHT*** of a Target AND Object's size is *SMALLER*, THEN the GOAL of an Object being docked at *LEFT* of a Target is **NOT achievable**.

We have to point, that some sentences from an APPENDIX have a form of a prohibition. For instance, when the Entity has *flat left* and *right surfaces* (**FLRS**), then the following is taking place

$$T\left(FLRS\left(O\right)\right)\wedge T\left(FLRS\left(Tar\right)\right)\Rightarrow F\left(DOT\left(O,Tar\right)\right) \quad (4.42)$$

This (4.42) logical sentence formulates a specific rule, which could be presented by natural language as follows: "IF both a Target and an Object have *flat left* and *right surfaces*, THEN the GOAL of an Object being docked on *TOP* of a Target is **NOT achievable**. The same logic (prohibition) is applied for an OBJECT for being at the *BOTTOM*, *FRONT* or *BACK* of a Target.

Let's underscore the fact, that (4.38)-(4.42) represent *feasibility of a goal settings* in its Boolean form and, of course, could not be always applied to a real-world situation. Therefore, we shall try to introduce its fuzzy counterpart.

4.4. Fuzzification of Feasibility of a Goal Setting

We are not pretending to generalize the way to re-interpret *Size Comparison Predicates* from (4.6), (4.7), *Mutual Positioning Predicates* from (4.8)-(4.15) and

Entity Shape Estimation Predicates from (4.23)-(4.26), as well as *Docking Positioning Predicates* from (4.27)-(4.32) in fuzzy way, but still try to apply a certain uniform approach, extensively used in [5] and [6]. For this purpose, we are normalizing

1) Vertical (Z-dimension) derivation from (4.20)

$$\forall x, y \mid x \in [x_{\min}, x_{\max}], y \in [y_{\min}, y_{\max}], \forall i \in [1, N], \forall j \in [1, M]$$
$$\Rightarrow \Delta z(x_i, y_j)^{norm} = \frac{z(x_i, y_j) - z^{\min}(x, y)}{z^{\max}(x, y) - z^{\min}(x, y)}. \tag{4.43}$$

Note, that from (4.43) we can calculate the following

$$\Delta z(x, y)^{norm} = \frac{\sum_{N \times M} \Delta z(x_i, y_j)^{norm}}{N \times M}, \tag{4.44}$$

$$\text{where } \Delta z(x, y)^{norm} \in [0,1]. \tag{4.45}$$

To define if an object/target has *Right Geometrical Form* on Vertical (Z-dimension) we propose the following

$$\Delta z(x, y) = \frac{\left|\Delta z(x, y)^{norm} - 0.5\right|}{0.5}, \tag{4.46}$$

In (4.46) the value of 0.5 means that there are a lot of different "tops" and "bottoms" of $\Delta z(x, y)^{norm}$ on a vertical surface of an entity, which effectively resulted in $\Delta z(x, y) = 0$. Otherwise, when $\Delta z(x, y)^{norm} \approx 0$ or $\Delta z(x, y)^{norm} \approx 0$, we could consider a vertical surface as geometrically perfect one (flat).

We represent $\Delta z(x, y)$ from (4.46) as a *fuzzy set*, forming linguistic variable, described by a triplet of the form $\Delta Z = \left\{\left\langle \Delta Z_i, U_z, \Delta\tilde{Z} \right\rangle\right\}$, $\Delta Z_i \in T(u_z)$, $\forall i \in [0, CardU_z]$, where $T_i(u)$ is extended term set of the linguistic variable "*Vertical derivation* "from **Table 5**, $\tilde{Z}$ is normal fuzzy set with correspondent membership function $\mu_{\Delta z} : U_z \to [0,1]$.

We will use the following mapping

$\alpha : \widetilde{\Delta\tilde{Z}} \to U_z \mid u_i = Ent\left[(CardU_z - 1) \times \Delta Z_i\right] \mid \forall i \in [0, CardU_z]$, were

$$\widetilde{\Delta Z} = \int_{U_z} \mu_{\Delta z}(u_z)/u_z \tag{4.47}$$

On the other hand, similarly to the previous cases, to determine the estimates of the membership function in terms of singletons from (4.46) in the form $\mu_{\Delta z_i}(\Delta z_i)/\Delta z_i \mid \forall i \in [0, CardU_z]$ we propose the following procedure.

$$\forall i \in [0, CardU_z], \mu_{\Delta z_i}(\Delta z_i) = 1 - \frac{1}{CardU_z - 1} \times \left|i - Ent\left[(CardU_z - 1) \times \Delta Z_i\right]\right| \tag{4.48}$$

2) Horizontal (X-dimension) derivation from (4.21)

$$\forall z, y \mid z \in [z_{\min}, z_{\max}], y \in [y_{\min}, y_{\max}], \forall k \in [1, L], \forall j \in [1, M]$$
$$\Rightarrow \Delta x(z_k, y_j)^{norm} = \frac{x(z_k, y_j) - x^{\min}(z, y)}{x^{\max}(z, y) - x^{\min}(z, y)}. \tag{4.49}$$

Note, that from (4.49) we can calculate the following

$$\Delta x(z,y)^{norm}=\frac{\sum_{L\times M}\Delta x(z_k,y_j)^{norm}}{L\times M},\tag{4.50}$$

$$\text{where }\ \Delta x(z,y)^{norm}\in[0,1].\tag{4.51}$$

To define if an object/target has *Right Geometrical Form* on Horizontal (X-dimension) we propose the following

$$\Delta x(z,y)=\frac{\left|\Delta x(z,y)^{norm}-0.5\right|}{0.5},\tag{4.52}$$

Once again, we represent $\Delta x(z,y)$ from (4.52) as a *fuzzy set*, forming linguistic variable, described by a triplet of the form $\Delta X=\left\{\left\langle \Delta X_i,U_X,\Delta\tilde{X}\right\rangle\right\}$, $\Delta X_i\in T(u_x)$, $\forall i\in[0,CardU_x]$, where $T_i(u)$ is extended term set of the linguistic variable "*Horizontal derivation* "from Table 5, $\tilde{X}$ is normal fuzzy set with correspondent membership function $\mu_{\Delta x}:U_x\to[0,1]$.

We will use the following mapping

$\beta:\widetilde{\Delta\tilde{X}}\to U_x\mid u_i=Ent\left[(CardU_x-1)\times\Delta X_i\right]\mid\forall i\in[0,CardU_x]$, were

$$\widetilde{\Delta X}=\int_{U_x}\mu_{\Delta x}(u_x)/u_x\ .\tag{4.53}$$

On the other hand, similarly to the previous cases, to determine the estimates of the membership function in terms of singletons from (4.51) in the form $\mu_{\Delta x_i}(\Delta x_i)/\Delta x_i\mid\forall i\in[0,CardU_x]$ we propose the following procedure.

$$\forall i\in[0,CardU_x],\mu_{\Delta x_i}(\Delta x_i)=1-\frac{1}{CardU_x-1}\times\left|i-Ent\left[(CardU_x-1)\times\Delta X_i\right]\right|\tag{4.54}$$

3) Y-dimension derivation from (4.22)

$$\begin{gathered}\forall x,z\mid x\in[x_{\min},x_{\max}],z\in[z_{\min},z_{\max}],\forall i\in[1,N],\forall k\in[1,L]\\ \Rightarrow\Delta y(x_i,z_k)^{norm}=\frac{y(x_i,z_k)-y^{\min}(x,z)}{y^{\max}(x,z)-y^{\min}(x,z)}.\end{gathered}\tag{4.55}$$

Note, that from (4.49) we can calculate the following

$$\Delta y(x,z)^{norm}=\frac{\sum_{N\times L}\Delta y(x_i,z_k)^{norm}}{N\times L},\tag{4.56}$$

$$\text{where }\ \Delta y(x,z)^{norm}\in[0,1].\tag{4.57}$$

To define if an object/target has *Right Geometrical Form* on Y-dimension we propose the following

$$\Delta y(x,z)=\frac{\left|\Delta y(x,z)^{norm}-0.5\right|}{0.5}\tag{4.58}$$

Once again, we represent $\Delta y(x,z)$ from (4.58) as a *fuzzy set*, forming linguistic variable, described by a triplet of the form $\Delta Y=\left\{\left\langle \Delta Y_i,U_Y,\Delta\tilde{Y}\right\rangle\right\}$, $\Delta Y_i\in T(u_y)$, $\forall i\in[0,CardU_y]$, where $T_i(u)$ is extended term set of the linguistic variable "*Y-dimension derivation* "from Table 5, $\tilde{X}$ is normal fuzzy set with correspondent membership function $\mu_{\Delta y}:U_y\to[0,1]$.

Table 5. Linguistic variables for dimensions derivation and geometric form.

Value of variable		$\Delta z_i \in U_z, \Delta x_i \in U_x, \Delta y_i \in U_y$, $rgf_i \in U_{RGF} \quad \forall i \in [0,4]$
" Vertical/Horizontal/ Y-dimension derivation "	*Right geometric form*	
ideal surface	perfect	0
very low	acceptable	1
low	in doubt	2
medium	far from right	3
high	not right at all	4

We will use the following mapping
$\Omega : \widetilde{\Delta \tilde{Y}} \to U_y \mid u_i = Ent\left[\left(CardU_y - 1\right) \times \Delta Y_i\right] \mid \forall i \in \left[0, CardU_y\right]$, were

$$\widetilde{\Delta Y} = \int_{U_y} \mu_{\Delta y}\left(u_y\right) / u_y \ . \tag{4.59}$$

On the other hand, similarly to the previous cases, to determine the estimates of the membership function in terms of singletons from (4.5) in the form $\mu_{\Delta y_i}\left(\Delta y_i\right) / \Delta y_i \mid \forall i \in \left[0, CardU_y\right]$ we propose the following procedure.

$$\forall i \in \left[0, CardU_y\right], \mu_{\Delta y_i}\left(\Delta y_i\right) = 1 - \frac{1}{CardU_y - 1} \times \left| i - Ent\left[\left(CardU_y - 1\right) \times \Delta Y_i\right] \right| \tag{4.60}$$

Note that from (4.23)-(4.26) we have the following predicate based logical statement

$$T\left(FLRS\left(E\right)\right) \wedge T\left(FTB\left(E\right)\right) \wedge T\left(FFBS\left(E\right)\right) \Rightarrow T\left(RGF\left(E\right)\right), \tag{4.61}$$

Which means that the *right geometric form* of an entity depends on *Vertical/Horizontal/Y-dimension derivations*, therefore we represent *RGF* from (4.61) as a *fuzzy set*, forming linguistic variable, described by a triplet of the form $RGF = \left\{\left\langle RGF_i, U_{RGF}, \widetilde{RGF}\right\rangle\right\}$, $RGF_i \in T\left(u_{rgf}\right)$, $\forall i \in \left[0, CardU_{RGF}\right]$, where $T_i\left(u\right)$ is extended term set of the linguistic variable "*right geometric form"* from **Table 5**, $\widetilde{RGF}$ is normal fuzzy set with correspondent membership function $\mu_{rgf} : U_{RDF} \to [0,1]$. We will use the following mapping
$\alpha : \widetilde{RGF} \to U_{RGF} \mid u_i = Ent\left[\left(CardU_{RGF} - 1\right) \times RGF_i\right] \mid \forall i \in \left[0, CardU_{RGF}\right]$, *i.e.*

$$\widetilde{RGF} = \int_{U_{RGF}} \mu_{RGF}\left(u_{rgf}\right) / u_{rgf} \ . \tag{4.62}$$

On the other hand, similarly to the previous cases, to determine the estimates of the membership function in terms of singletons from (4.5) in the form $\mu_{rgf_i}\left(rgf_i\right) / rgf_i \mid \forall i \in \left[0, CardU_{RGF}\right]$ we propose the following procedure.

$$\forall i \in \left[0, CardU_{RGF}\right],$$
$$\mu_{rgf_i}\left(rgf_i\right) = 1 - \frac{1}{CardU_{RGF} - 1} \times \left| i - Ent\left[\left(CardU_{RGF} - 1\right) \times RGF_i\right] \right| \tag{4.63}$$

To convert (4.61) into fuzzy logic-based statement and terms from **Table 5** *we use* a *Fuzzy Conditional Inference Rule*, formulated by means of *"common* *sense*" as a following conditional clause:

P = "IF ($\widetilde{\Delta Z}$ is Z) AND ($\widetilde{\Delta X}$ is X) AND ($\widetilde{\Delta Y}$ is Y), THEN ($\widetilde{RGF}$ is RGF)" (4.64)

In other words, we use fuzzy conditional inference of the following type [7] [8]:

$$\begin{array}{l} \text{Ant 1: If } \Delta z \text{ is } Z \text{ and } \Delta x \text{ is } X \text{ and } \Delta y \text{ is } Y \text{ then } rgf \text{ is } RGF \\ \text{Ant 2: } \Delta z \text{ is } Z' \text{ and } \Delta x \text{ is } X' \text{ and } \Delta y \text{ is } Y' \\ \text{---} \\ \text{Cons: } rgf \text{ is } RGF'. \end{array} \tag{4.65}$$

where $Z, Z' \subseteq U_Z$, $X, X' \subseteq U_X$, $Y, Y' \subseteq U_Y$ and $RGF, RGF' \subseteq U_{RGF}$.

Now for fuzzy sets (4.47), (4.53), (4.59) and (4.62) a *binary relationship* for the fuzzy conditional proposition of the type (4.64) and (4.65) for fuzzy logic we use so far is defined as

$$\begin{aligned} &R\left(A_1(z,x,y), A_2(rgf)\right) = \left[Z \times U_Z \cap X \times U_X \cap Y \times U_Y\right] \\ &\to RGF \times U_{RGF} = \int_{U_Z \times U_X \times U_Y \times U_{RGF}} \left(\left[\mu_{\Delta z}(u_z)/u_z \wedge \mu_{\Delta x}(u_x)/u_x \wedge \mu_{\Delta y}(u_y)/u_y\right]\right. \\ &\left.\to \mu_{RGF}(u_{rgf})/u_{rgf}\right)/(u_z, u_x, u_y, u_{rgf}) \end{aligned} \tag{4.66}$$

Given (2.10) and since we consider that $CardU_Z = CardU_X = CardU_Y = CardU_{RGF}$, then expression (4.66) looks like

$$\begin{aligned} &\left[\mu_{\Delta z}(u_z)/u_z \wedge \mu_{\Delta x}(u_x)/u_x \wedge \mu_{\Delta y}(u_y)/u_y\right] \to \mu_{RGF}(u_{rgf})/u_{rgf} \\ &= \begin{cases} (1 - \left[\mu_{\Delta z}(u_z) \wedge \mu_{\Delta x}(u_x) \wedge \mu_{\Delta y}(u_y)\right] \cdot \mu_{RGF}(u_{rgf}), \left[\mu_{\Delta z}(u_z) \wedge \mu_{\Delta x}(u_x) \wedge \mu_{\Delta y}(u_y)\right] > \mu_{RGF}(u_{rgf}), \\ 1, \left[\mu_{\Delta z}(u_z) \wedge \mu_{\Delta x}(u_x) \wedge \mu_{\Delta y}(u_y)\right] \le \mu_{RGF}(u_{rgf}). \end{cases} \end{aligned} \tag{4.67}$$

where $\left[\mu_{\Delta z}(u_z) \wedge \mu_{\Delta x}(u_x) \wedge \mu_{\Delta y}(u_y)\right]$ is $\min\left[\mu_{\Delta z}(u_z), \mu_{\Delta x}(u_x), \mu_{\Delta y}(u_y)\right]$. It is well known that given a *unary relationship* $R\left(A_1(z', x', y')\right) = Z' \cap X' \cap Y'$ one can obtain the consequence $R\left(A_2(rgf)\right)$ by applying compositional rule of inference (CRI) to $R\left(A_1(z', x', y')\right)$ and $R\left(A_1(z,x,y), A_2(rgf)\right)$ of type (4.66):

$$\begin{aligned} &R\left(A_2(rgf)\right) = Z' \cap X' \cap Y' \circ R\left(A_1(z,x,y), A_2(rgf)\right) \\ &= \int_{U_Z \times U_X \times U_Y} \left[\mu_{\Delta z'}(u_z) \wedge \mu_{\Delta x'}(u_x) \wedge \mu_{\Delta y'}(u_y)\right]/(u_z, u_x, u_y) \\ &\circ \int_{U_Z \times U_X \times U_Y \times U_{RGF}} \left[\mu_{\Delta z}(u_z) \wedge \mu_{\Delta x}(u_x) \wedge \mu_{\Delta y}(u_y)\right] \to \mu_{RGF}(u_{rgf})/(u_z, u_x, u_y, u_{rgf}) \\ &= \int_{RGF} \bigcup_{z \in U_Z, y \in U_y, x \in U_x} \left\{\left[\mu_{\Delta z'}(u_z) \wedge \mu_{\Delta x'}(u_x) \wedge \mu_{\Delta y'}(u_y)\right]\right. \\ &\left.\wedge\left(\left[\mu_{\Delta z}(u_z) \wedge \mu_{\Delta x}(u_x) \wedge \mu_{\Delta y}(u_y)\right] \to \mu_{RGF}(u_{rgf})\right)\right\}/u_{rgf} \end{aligned} \tag{4.68}$$

But for practical purposes we will use another *Fuzzy Conditional Rule* (*FCR*)

$$\begin{aligned} &R\left(A_1(z,x,y), A_2(rgf)\right) = (P \times U \to V \times S) \cap (\neg P \times U \to V \times \neg S) \\ &= \int_{U \times V} \left(\mu_P(u) \to \mu_S(v)\right) \wedge \left((1 - \mu_P(u)) \to (1 - \mu_S(v))\right)/(u,v) \end{aligned} \tag{4.69}$$

where $P = Z \cap X \cap Y$ and $U = U_Z = U_X = U_Y$, therefore from (4.69) we are getting

$$R\left(A_1\left(z,x,y\right),A_2\left(rgf\right)\right)=\left(\mu_P\left(u\right)\to\mu_S\left(v\right)\right)\wedge\left(\left(1-\mu_P\left(u\right)\right)\to\left(1-\mu_S\left(v\right)\right)\right)$$
$$=\begin{cases}\left(1-\mu_P\left(u\right)\right)\cdot\mu_S\left(v\right),\mu_P\left(u\right)>\mu_S\left(v\right),\\ 1,\mu_P\left(u\right)=\mu_S\left(v\right),\\ \left(1-\mu_S\left(v\right)\right)\cdot\mu_P\left(u\right),\mu_P\left(u\right)<\mu_S\left(v\right).\end{cases} \tag{4.70}$$

As was already mentioned above, *FCR* from (4.70) gives more reliable results.

4.5. Example

To build a binary relationship matrix of type (4.69) we us use a conditional clause of type (4.64):

P = "IF (Z is "ideal surface") AND (X is "ideal surface") AND (Y is "ideal surface"), THEN (RGF is "perfect")" (4.71)

To build membership functions for fuzzy sets Z, X and Y we use (4.47), (4.53) and (4.59) respectively.

In (4.47) the membership functions for fuzzy set Z (for instance from Table 5) would look like:

$$\mu_P\left(u\right)=\mu_Z\left(\text{“ideal surface”}\right)\wedge\mu_X\left(\text{“ideal surface”}\right)\wedge\mu_Y\left(\text{“ideal surface”}\right)$$
$$=1/0+0.75/1+0.5/2+0.25/3+0/4 \tag{4.72}$$

Same membership functions we use for fuzzy sets X and Y. Note, that the membership function for fuzzy set RGF from Table 5 is also the same

$$\mu_{RGF}\left(\text{“perfect”}\right)=1/0+0.75/1+0.5/2+0.25/3+0/4 \tag{4.73}$$

Given (4.70), (4.72) and (4.73) we have $R\left(A_1\left(z,x,y\right),A_2\left(rgf\right)\right)$ shown in Table 6.

Suppose that the current value of *"horizontal derivation"*, represented by a fuzzy set X' from (4.53), is defined as

$\mu_{X'}\left(\text{“high”}\right)=0/0+0.25/1+0.5/2+0.75/3+1/4$ and (Z' is "ideal surface") AND (Y' is "ideal surface") and

$$\mu_{P'}\left(u\right)=\mu_{Z'}\left(\text{“ideal surface”}\right)\wedge\mu_{X'}\left(\text{“high”}\right)\wedge\mu_{Y'}\left(\text{“ideal surface”}\right)$$
$$=0/0+0.25/1+0.5/2+0.25/3+0/4.$$

After applying CRI from (4.68) we get the following

$R\left(A_2\left(rgf\right)\right)=0/0+0.25/1+0.5/2+0.25/3+0/4$ and after membership function normalization we are getting $\mu_{RGF'}\left(\text{“in doubt”}\right)=0/0+0.5/1+1/2+0.5/3+0/4$,

Table 6. Binary relationship matrix of a current example.

$p\to rgf$	**1**	**0.75**	**0.5**	**0.25**	**0**
1	1	0	0	0	0
0.75	0	1	0.125	0.0625	0
0.5	0	0.125	1	0.125	0
0.25	0	0. 0625	0.125	1	0
0	0	0	0	0	1

which means (4.61) is not fully satisfactory and an Entity has to re-orient itself to successfully "dock" to another one.

All elements of a predicates based logical sentences ((4.38), for instance) like structure such as "*FLRS*", "*FTB*", "*FFBS*", "*SMALLER*", *"LARGER"*, *"HIGHER"*, *"LOWER"* etc. from an Appendix, could be presented by above-described technique.

The summary of this presentation:

1) The AIA strategic targeting could be based on an *axiomatic geometry* and *extended objects* representation.

2) The AIA intermediary behavior trigger could be based on both (4,36), (4.37) between entities fuzzy distance and "Noth-East-South-West" orientation estimates.

3) The AIA tactical behavior could be defined by fuzzification of an element of a predicate based logical sentences, very limited subset of which are presented in an APPENDIX.

5. Conclusion

In this work it was shown that the AIA strategic targeting could be based on approximate geometric behavior of extended objects, described by fuzzy predicates. The axiom system of Boolean Euclidean geometry was fuzzified and formalized in the language of fuzzy logic, presented in [1]. Based on the same fuzzy logic, we formulated a special form of positional uncertainty, namely positional tolerance that arises from geometric constructions with extended primitives. We also addressed Euclid's first postulate, which lays the foundation for consistent geometric reasoning in all classical geometries by considered extended primitives and gave a fuzzification of Euclid's first postulate by using the same fuzzy logic. Fuzzy equivalence relation *"Extended lines sameness"* is introduced. We also use the fuzzy logic from [1] for fuzzy conditional inference determination, elements of which were *"Degree of indiscernibility"* and *"Discernibility measure* "of extended points. We also presented some logical principles of AIA orientation, which will allow an implementation of its fuzzy "tactical" decision making.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Tserkovny, A. (2017) A t-Norm Fuzzy Logic for Approximate Reasoning. *Journal of Software Engineering and Applications*, **10**, 639-661. https://doi.org/10.4236/jsea.2017.107035

[2] Genesereth, M. and Nilsson, N.J. (1989) Logical Foundations of Artificial Intelligence. Morgan Kaufmann Publishers, Los Altos. https://doi.org/10.1016/0004-3702(89)90073-8

[3] Tserkovny, A. (2016) Fuzzy Logic for Incidence Geometry. *The Scientific World*

Journal, **2016**, Article ID 9057262. https://doi.org/10.1155/2016/9057263

[4] Wilke, G. (2014) Equality in Approximate Tolerance Geometry. *Proceedings of the 7th IEEE International Conference Intelligent Systems* (*IS'14*), Springer International, Warsaw, 365-376. https://doi.org/10.1007/978-3-319-11313-5_33

[5] Aliev, R.A. and Tserkovny, A. (1988) The Knowledge Representation in Intelligent Robots Based on Fuzzy Sets. *Doklady Mathematics*, **37**, 541-544.

[6] Aliev, R.A., Mamedova, G.A. and Tserkovny, A.E. (1991) Fuzzy Control Systems. Energoatomizdat, Moscow.

[7] Fukami, S., Mizumoto, M. and Tanaka, K. (1980) Some Considerations of Fuzzy Conditional Inference. *Fuzzy Sets and Systems*, **4**, 243-273. https://doi.org/10.1016/0165-0114(80)90014-7

[8] Aliev, R.A. and Tserkovny, A. (2011) Systemic Approach to Fuzzy Logic Formalization for Approximate Reasoning. *Information Sciences*, **181**, 1045-1059. https://doi.org/10.1016/j.ins.2010.11.021

[9] Schockaert, S., De Cock, M. and Kerre, E.E. (2008) Modelling Nearness and Cardinal Directions between Fuzzy Regions. *Proceedings of 2008 IEEE 16th International Conference on Fuzzy Systems* (*FUZZ-IEEE*), Hong Kong, 1-6 June 2008, 1548-1555.

Appendix

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(SMALLER(O,Tar)) \Rightarrow T(DIF(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(SMALLER(O,Tar)) \Rightarrow T(DAB(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(SMALLER(O,Tar)) \Rightarrow T(DAL(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(SMALLER(O,Tar)) \Rightarrow T(DAR(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(SMALLER(O,Tar)) \Rightarrow T(DOT(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(ATLEFT(O,Tar)) \Rightarrow T(DAL(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(ATRIGHT(O,Tar)) \Rightarrow T(DAR(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(ATLEFT(O,Tar)) \Rightarrow F(DAR(O,Tar))$$

$$T(RGF(O)) \wedge T(RGF(Tar)) \wedge T(ATRIGHT(O,Tar)) \Rightarrow F(DAL(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \wedge T(ATLEFT(O,Tar)) \Rightarrow T(DAL(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \wedge T(ATRIGHT(O,Tar)) \Rightarrow T(DAR(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \wedge T(ATLEFT(O,Tar)) \Rightarrow F(DAR(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \wedge T(ATRIGHT(O,Tar)) \Rightarrow F(DAL(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \Rightarrow F(DOT(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \Rightarrow F(DUN(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \Rightarrow F(DIF(O,Tar))$$

$$T(FLRS(O)) \wedge T(FLRS(Tar)) \Rightarrow F(DAB(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \wedge T(HIGHER(O,Tar)) \Rightarrow T(DOT(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \wedge T(HIGHER(O,Tar)) \Rightarrow F(DUN(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \wedge T(LOWER(O,Tar)) \Rightarrow F(DOT(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \wedge T(LOWER(O,Tar)) \Rightarrow T(DUN(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \Rightarrow F(DIF(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \Rightarrow F(DAB(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \Rightarrow F(DAL(O,Tar))$$

$$T(FTB(O)) \wedge T(FTB(Tar)) \Rightarrow F(DAR(O,Tar))$$

$$T(FFBS(O)) \wedge T(FFBS(Tar)) \wedge T(INFRONT(O,T)(O,Tar)) \\ \Rightarrow T(DIF(O,Tar))$$

$$T(FFBS(O)) \wedge T(FFBS(Tar)) \wedge T(BEHIND(O,T)(O,Tar)) \\ \Rightarrow T(DAB(O,Tar))$$

$$T\left(FFBS\left(O\right)\right)\wedge T\left(FFBS\left(Tar\right)\right)\wedge T\left(INFRONT(O,T)\left(O,Tar\right)\right)$$
$$\Rightarrow F\left(DAB\left(O,Tar\right)\right)$$

$$T\left(FFBS\left(O\right)\right)\wedge T\left(FFBS\left(Tar\right)\right)\wedge T\left(BEHIND(O,T)\left(O,Tar\right)\right)$$
$$\Rightarrow F\left(DIF\left(O,Tar\right)\right)$$