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A New 2 + 1-Dimensional Integrable Variable Coefficient Toda Equation

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Abstract

In this paper, a new integrable variable coefficient Toda equation is proposed by utilizing a generalized version of the dressing method. At the same time, we derive the Lax pair of the new integrable variable coefficient Toda equation. The compatibility condition is given, which insures that the new Toda equation is integrable. To further analyze the character of the Toda equation, we derive one soliton solution of the obtained Toda equation by using separation of variables.

Keywords

The Generalized Dressing Method, Variable Coefficient Toda, Separation of Variables

1. Introduction

Integrable variable coefficient equations describe the real world in many fields of physical and engineering sciences. Many researchers are devoted to discussing these equations by utilizing different methods ref. [1]-[6]. In ref. [7] [8], Dai and Jeffrey extended the dressing method to a generalized version for solving nonlinear evolution equations associated with matrix spectral problems and variable coefficient cases, in which a key is that variable coefficient dressing operators are transformed to different variable coefficient ones. By using the generalization, we have studied integrable variable coefficient coupled Hirota equation in ref. [9]. In ref. [10] [11], integrable variable coefficient Manakov model and cylindrical NLS equation are discussed in detailed, respectively. In ref. [12], we developed the generalized dressing method to the discrete system and an integrable variable coefficient Toda equation is researched. Recently, the dressing method is extended to a matrix Lax pair for Camassa-Holm equation in ref. [13], in which interactions between soliton and cuspon solutions of the system are studied. The dressing method as nonlinear superposition in Sigma models has been researched by Dimitrios Katsinis *et al.* in ref. [14]. Multi-lump solutions of KP equation with integrable boundary are discussed in ref. [15] by using the generalized dressing method. Nabelek *et al.* in ref. [16] studied Kaup-Broer system and derived its solutions.

In the present paper, we extend the generalized dressing method to discrete operators similar to ref. [12]. Through direct calculations, we derive a new integrable variable coefficient Toda equation

$$\chi_{yy} - \chi_{n,tt} - \Delta n(n-1)\left(e^{\chi_{n-1}-\chi_n} - 1\right) = 0, \tag{1.1}$$

where, the coefficient is related to n, $\Delta = E - E^{-1}$. Equation (1.1) is an extension of the well known two-dimensional Toda equation. We will construct one soliton solution of (1.1).

The present paper is organized as follows. In Section 2, we obtain a new integrable variable coefficient Toda equation based on the generalized dressing method. In Section 3, as an application, we derive one soliton solution of (1.1) by utilizing the separation of variables.

2. Integrable Variable Coefficient Toda Equation

In this section, we first summarize the variable coefficient version of the dressing method. We extend the generalized version of the dressing method to discrete systems and derive different integrable cylindrical Toda lattice equations by choosing different operators.

First, we consider three linear differential difference operators ref. [12]

$$\begin{aligned}
\mathbf{F}(n,m,t,y)\psi_n &= \sum_{m=-\infty}^{\infty} F(n,m,t,y)\psi_m, \\
\mathbf{K}_+(n,m,t,y)\psi_n &= \sum_{m=n}^{\infty} K_+(n,m,t,y)\psi_m, \\
\mathbf{K}_-(n,m,t,y)\psi_n &= \sum_{m=-\infty}^{n} K_-(n,m,t,y)\psi_m.
\end{aligned} \tag{2.1}$$

Similar to the generalized dressing method application to continuous system, we introduce the triangular factorization about the operator "$\mathbf{F}$"

$$\mathbf{I} + \mathbf{F} = (\mathbf{I} + \mathbf{K}_+)^{-1}(\mathbf{I} + \mathbf{K}_-), \tag{2.2}$$

where $\mathbf{I}$ is the identity operator, $\mathbf{K}_+(n,m,t,y) = 0$ for $m < n$ and $\mathbf{K}_-(n,m,t,y) = 0$ for $m > n$. It is assumed that

$$\sup \sum_{m=n_0}^{\infty} \left|K_\pm(n,m,t,y)\right|\psi_m < \infty, \quad \sup \sum_{m=n_0}^{\infty} \left|F(n,m,t,y)\right|\psi_m < \infty,$$

for all $n_0 > -\infty$. For convenience, we denote $F(n,m,t,y) = F(n,m)$, $K_\pm(n,m,t,y) = K_\pm(n,m)$. The discrete Gelfand-Levitan-Marchenko (GLM) equation can be obtained from (2.2), which reads in ref. [12]

$$F(n,m)+K_+(n,m)+\sum_{s=n}^{\infty}K_+(n,s)F(s,m)=0. \tag{2.3}$$

We introduce two differential-difference operators $\mathbf{M}_1$ and $\mathbf{M}_2$ defined by

$$\mathbf{M}_1=\partial_t+\partial_y-n\mathbf{E}, \tag{2.4}$$

$$\mathbf{M}_2=\partial_t-\partial_y+n\mathbf{E}^{-1}, \tag{2.5}$$

where $\mathbf{E}$ is the shift operator of the discrete variable n, defined by $\mathbf{E}^k f(n)=f(n+k)$, $k\in Z$, t and y are continuous variables.

The dressing operators $\mathbf{N}_1$ and $\mathbf{N}_2$ can be derived from the relations

$$\mathbf{N}_1(\mathbf{I}+\mathbf{K}_+(n,m))-(\mathbf{I}+\mathbf{K}_+(n,m))\mathbf{M}_1=0, \tag{2.6}$$

$$\mathbf{N}_2(\mathbf{I}+\mathbf{K}_+(n,m))-(\mathbf{I}+\mathbf{K}_+(n,m))\mathbf{M}_2=0. \tag{2.7}$$

Similar to a theorem ref. [7] for continuous systems, it can be proved that $\mathbf{N}_1$ and $\mathbf{N}_2$ are differential-difference operators. For sake of simplicity, we denote $\mathbf{K}_+(n,m)=\mathbf{K}(n,m)$.

We write the dressing operators

$$\mathbf{N}_1=\mathbf{M}_1+\mathbf{D}_1, \tag{2.8}$$

$$\mathbf{N}_2=\mathbf{M}_2+\mathbf{D}_2. \tag{2.9}$$

Acting on function φ_n on (2.6) and with aid of (2.8), which is reduced to

$$\begin{aligned}&M_1K(n,m)\varphi_n+D_1K(n,m)\varphi_n+D_1\varphi_n-K(n,m)M_1\varphi_n\\&=\sum_{m=n}^{\infty}K_t(n,m)\varphi_m+\sum_{m=n}^{\infty}K_y(n,m)\varphi_m-n\sum_{m=n+1}^{\infty}K(n+1,m)\varphi_m\\&\quad+D_1\sum_{m=n}^{\infty}K(n,m)\varphi_m+D_1\varphi_n+\sum_{m=n+1}^{\infty}K(n,m-1)\varphi_m,\end{aligned}$$

from which, comparing coefficient of φ_n, we have

$$K_t(n,n)+K_y(n,n)+D_1K(n,n)+D_1=0. \tag{2.10}$$

Letting $D_2=d_1E^{-1}$, with aid of (2.7) and (2.9), we have

$$\begin{aligned}&M_2K(n,m)\varphi_n+D_2K(n,m)\varphi_n+D_2\varphi_n-K(n,m)M_2\varphi_n\\&=\sum_{m=n}^{\infty}K_t(n,m)\varphi_m-\sum_{m=n}^{\infty}K_y(n,m)\varphi_m+n\sum_{m=n-1}^{\infty}K(n-1,m)\varphi_m\\&\quad+d_1\sum_{m=n-1}^{\infty}K(n-1,m)\varphi_m+d_1\varphi_{n-1}-n\sum_{m=n-1}^{\infty}(m+1)K(n,m+1)\varphi_m,\end{aligned}$$

from which, comparing coefficient of φ_{n-1}, we have

$$nK(n-1,n-1)-nK(n,n)+d_1K(n-1,n-1)+d_1=0, \tag{2.11}$$

and we derive

$$d_1=n\frac{K(n,n)-K(n-1,n-1)}{1+K(n-1,n-1)}. \tag{2.12}$$

The following theorem in ref. [7] is an extension of original dressing method, which can yield a wide range of integrable variable-coefficient nonlinear evolution equations.

Theorem: If the operators $\mathbf{M}_1$ and $\mathbf{M}_2$ satisfy a relation

$$[\mathbf{M}_1,\mathbf{M}_2]=\rho_1\mathbf{M}_1+\rho_2\mathbf{M}_2, \tag{2.13}$$

where ρ_1,ρ_2 are arbitrary functions of x,y,n, then their corresponding dressing operators will satisfy the relation

$$[\mathbf{N}_1,\mathbf{N}_2]=\rho_1\mathbf{N}_1+\rho_2\mathbf{N}_2. \tag{2.14}$$

Proof: According to (2.6), (2.7) and (2.13), we can give simple proof as follows through simple calculation. In fact,

$$\begin{aligned}[\mathbf{N}_1,\mathbf{N}_2](\mathbf{I}+\mathbf{K}_+)&=\mathbf{N}_1(\mathbf{I}+\mathbf{K}_+)\mathbf{M}_2-\mathbf{N}_2(\mathbf{I}+\mathbf{K}_+)\mathbf{M}_1\\&=(\mathbf{I}+\mathbf{K}_+)\mathbf{M}_1\mathbf{M}_2-(\mathbf{I}+\mathbf{K}_+)\mathbf{M}_2\mathbf{M}_1\\&=(\mathbf{I}+\mathbf{K}_+)[\mathbf{M}_1,\mathbf{M}_2]\\&=(\rho_1\mathbf{N}_1+\rho_2\mathbf{N}_2)(\mathbf{I}+\mathbf{K}_+).\end{aligned}$$

Actually, variable-coefficient Toda equations are obtained from (2.14). From (2.14), we derived

$$d_{1t}+d_{1y}+(n+d_1)(1-E^{-1})D_1=0, \tag{2.15}$$

$$D_{1y}-D_{1t}-nEd_1+(n-1)d_1=0. \tag{2.16}$$

Letting

$$u_n=\frac{1+K(n,n)}{1+K(n-1,n-1)},\quad D_1=v_n, \tag{2.17}$$

then the above Equations (2.15) and (2.16) are reduced to

$$v_{n,y}-v_{n,t}-\Delta n(n-1)(u_n-1)=0, \tag{2.18}$$

$$u_{n,y}+u_{n,t}+u_n(v_n-v_{n-1})=0. \tag{2.19}$$

According to (2.19), we assume that

$$u_n=\mathrm{e}^{\chi_{n-1}-\chi_n},\quad v_n=\chi_{n,t}+\chi_{n,y}, \tag{2.20}$$

then (2.18) is reduced to a new integrable variable coefficient Toda equation

$$\chi_{n,yy}-\chi_{n,tt}-\Delta n(n-1)\left(\mathrm{e}^{\chi_{n-1}-\chi_n}-1\right)=0. \tag{2.21}$$

Let $\xi=y+t$, $\eta=y-t$, then the above equation is reduced to a new 2 + 1 dimensional Toda lattice equation

$$4\chi_{n,\xi\eta}-\Delta n(n-1)\left(\mathrm{e}^{\chi_{n-1}-\chi_n}-1\right)=0. \tag{2.22}$$

The above equations are new and different to classical Toda lattice equation in ref. [17] [18] [19] [20] [21]. Because the coefficient of equation is related to *n*, this is an important physical meaning.

3. Explicit Solution of Integrable Variable Coefficient Toda Equation

In this section, we shall use the generalized dressing method to construct explicit solutions of the variable coefficient Toda Equation (2.21). Using the relation $[M_1,F]=0,[M_2,F]=0$, we have

$$F_t(n,m)+F_y(n,m)-nF(n+1,m)+(m-1)F(n,m-1)=0, \tag{3.1}$$

$$F_t(n,m)-F_y(n,m)+nF(n-1,m)-(m+1)F(n,m+1)=0. \tag{3.2}$$

Assume that (3.1) and (3.2) have N-soliton solutions in the form of separation of variables

$$F(m,n)=\sum_{j=1}^{N} f_j(n,t,y)g_j(m,t,y), \tag{3.3}$$

moveover, we suppose that

$$K(m,n)=\sum_{j=1}^{N} k_j(n,t,y)g_j(m,t,y). \tag{3.4}$$

Substituting (3.3) and (3.4) into the GLM (2.3) yields that

$$K(n,n)=-(f_1,f_2,\cdots,f_N)L^{-1}(g_1,g_2,\cdots,g_N)^{\mathrm{T}}, \tag{3.5}$$

where L is defined by

$$L_{jl}=\delta_{jl}+\sum_{s=n}^{\infty} g_j(t,y,s)f_l(t,y,s), \quad 1\le j,l\le N,$$

and δ_{jl} is Kronecker's delta.

In what follows, we will obtain one soliton solution of (2.21). First, we give separation of variables solutions for $N=1$ in (3.3) and (3.4),

$$F(m,n)=f_1g_1=\mathrm{e}^{pt+qy+nw+\eta_0}\mathrm{e}^{mw}, \quad K(m,n)=k_1g_1=k_1\mathrm{e}^{mw}. \tag{3.6}$$

From (3.5), we derive

$$K(n,n)=\frac{\mathrm{e}^{pt+qy+2nw+\eta_0}-\mathrm{e}^{pt+qy+(2n+2)w+\eta_0}+\mathrm{e}^{2pt+2qy+4nw+2\eta_0}}{1-\mathrm{e}^{2w}}, \tag{3.7}$$

with $p=chw, q=-shw$, using (2.17), we have

$$u_n=\frac{1-\mathrm{e}^{2w}+\mathrm{e}^{pt+qy+2nw+\eta_0}-\mathrm{e}^{pt+qy+(2n+2)w+\eta_0}+\mathrm{e}^{2pt+2qy+4nw+2\eta_0}}{1-\mathrm{e}^{2w}+\mathrm{e}^{pt+qy+2(n-1)w+\eta_0}-\mathrm{e}^{pt+qy+2nw+\eta_0}+\mathrm{e}^{2pt+2qy+4(n-1)w+2\eta_0}}. \tag{3.8}$$

Under transformation $u_n=\mathrm{e}^{\chi_{n-1}-\chi_n}$, we derive one soliton solution of (2.21)

$$\chi_n=\chi_0-\ln(u_1u_2\cdots u_n). \tag{3.9}$$

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Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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Authors' Contributions

Yanan Huang and Ting Su do derivation and calculations. Junhong Yao mainly draw soliton solution picture.

Conflicts of Interest

There is no competition of interests among authors.

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