Advances in Historical Studies, 2014, 3, 130-154
Published Online June 2014 in SciRes. http://www.scirp.org/journal/ahs
http://dx.doi.org/10.4236/ahs.2014.33013

Einstein's 1905 "Revolutionary" Paper on Quanta as a Manifest and Detailed Example of a "Principle Theory"

Antonino Drago

University of Pisa, Pisa, Italy
Email: drago@unina.it

Received 18 March 2014; revised 20 April 2014; accepted 2 May 2014

Abstract

In the last times some scholars tried to characterize Einstein's distinction between "constructive"—i.e. deductive-theories—and "principle" theories, the latter ones being preferred by Einstein. Here this distinction is qualified by an accurate inspection on past physical theories. Some previous theories are surely non-deductive theories. By a mutual comparison of them a set of features—mainly the arguing according to non-classical logic—are extracted. They manifest a new ideal model of organising a theory. Einstein's paper of 1905 on quanta, qualified by him as presenting a "principle theory", is interpreted according to this model of theory. Some unprecedented characteristic features are manifested. At the beginning of the same paper Einstein declared one more dichotomy about the kind of mathematics in theoretical physics. These two dichotomies are recognized to constitute the foundations of theoretical physics. With respect to these dichotomies the choices by Einstein in the paper on quanta are the alternative choices to Newton's ones. This fact gives reason to the "revolutionary" nature that Einstein attributed to his paper.

Keywords

Quanta, Einstein, Principle Theory, Kinds of Organization, Kinds of Mathematics

1. Introduction

In the last times some scholars suggested formulations of both relativity and quantum mechanics (Brown, 2005; Clifton, Bub, & Halvorson, 2003; Bub, 2005) according to the model of a "principle theory" which Einstein preferred to the model of a "constructive theory" (Einstein, 1934)[1]. This distinction among the various physical

[1]For an illustration of Einstein's distinction see the excellent paper by Klein (Klein, 1967). One more detailed illustration, but less accurate in demarking the distinction, is given by Miller (Miller, 1981: pp. 123-142).

How to cite this paper: Drago, A. (2014). Einstein's 1905 "Revolutionary" Paper on Quanta as a Manifest and Detailed Example of a "Principle Theory". *Advances in Historical Studies, 3,* 130-154. http://dx.doi.org/10.4236/ahs.2014.33013

theories was introduced by the same Einstein, who intended the notion of “principle” not as concerning some *a priori* principles, but rather as having a “heuristic” character[2], i.e. directly induced from the experimental data.

A first question studied in the literature is the anticipations of this distinction. Recently Lorentz has been accredited to have preceded Einstein (Frisch, 2006)[3]. But it seems an unknown fact that Einstein’s suggestion was surely preceded by a Poincaré’s similar one (Poincaré, 1903: ch. “Optique et Electricité”; Poincaré, 1905: ch. VII).

A second question is how to characterize Einstein’s rough distinction between “constructive” theories and “principle” theories by means of further characteristic features. For instance, someone added the distinction between Kitcher’s “top-down view” and Salmon’s “bottom-up view” (Flores, 1999)[4].

But it was not remembered that this kind of distinction has a very long life. In Section 2 we will see that it predates since the times of the two most authoritative philosophers of the ancient times, i.e. Plato, who offered some suggestions for defining a model of a non-deductive theory, and Aristotle, who accurately theorised the “apodeictic” (deductive) theory, as entirely drawn from few a priori axioms. The celebrated and almost unique scientific theory of ancient times, i.e. Euclid’s *Elements*, was presented according to the latter model. Subsequently, in modern times the most authoritative scientific theory, i.e. Newton’s *Principia*, reiterated this model of a systematic organization of a scientific theory. Hence, along two millennia this model was commonly considered as the unique one.

But an accurate inspection on past theories will show in Sections 3 and 4 that some theorists of classical physics anticipated a similar distinction to Einstein’s and moreover some ones presented theories according to a non-deductive model, which corresponds to what Einstein’s intended for a “principle” theory. In Sections 5-8 a mutual comparison of all these theories will suggest several characteristic features; the crucial one is the use of the non-classical logic instead of the classical one. The essential novelty of the present paper is the presentation of this logical distinction as being very relevant to accurately define the above distinction. This logical feature is of an elementary nature, so that in the following no previous competence in mathematical logic is required to the reader.

Through this accurate characterization of the new model it will be easy in Sections 9-12 to accurately define Einstein’s 1905 paper as an example of a “principle” theory. Moreover, Einstein declared—since the beginnings of the same paper—a dichotomy about the kind of mathematics in theoretical physics. These two dichotomies are recognized to constitute the foundations of theoretical physics (Sections 13 and 14). In Section 15 Einstein’s appraisal of the “revolutionary” character of his paper will be qualified in terms of choices on these foundational dichotomies; Einstein’s choices in this theory will be recognised to be an alternative to those of Newton’s mechanics. Einstein’s paper results to be the first, substantially accurate presentation of a theory according to these choices. In the final Section 16 historical considerations will be added to qualify the revolutionary role played by this paper in the history of theoretical physics. An **Appendix** will list all Einstein’s propositions which qualify his arguments as pertaining to an alternative logic to the classical one.

2. Aristotle’s Model for a Scientific Theory and the Philosophical Alternative to It

At present, although each scientific researcher knows that his activity follows unforeseeable paths, in scientific education the best presentation of a scientific theory is considered a unbending deductive organization of it, usually adopted for a context of justification of the results by dismissing “the context of discovery”. This deductive way of presenting the results constitutes almost an obligation; rare persons avoid it, at the cost of unfavourable

[2]A qualification of the other Einstein “principle” theory, i.e. special relativity, was offered by (Brown, 2005: p. 72): “The principle of relativity… is not to be conceived as a “complete system” [i.e. as included in a deductive theory], in fact, not a system at all, but merely as a heuristic principle… we are not dealing here with a [deductive] “system”… which the individual laws are implicitly contained and from they can be found by deduction alone, but only with a principle that (similar to the second of the theory of heat) permits the reduction of certain laws to others”.

[3]It is interesting that about this distinction Lorentz openly advocated a “theoretical pluralism” which the present paper suggests in more detailed way.

[4]The author adds that Newton too made distinction between “the frameworks of principles which govern all possible forces (i.e., the laws of motion of the *Principia* and their Corollaries) from theories which describe physical interaction *within* this framework, like the theory of universal gravitation”; Flores calls them respectively “framework theories” and “interaction theories”. But this appraisal attributes to the latter ones a subordinate role, as particular cases of the former ones. Rather, it is remarkable that Newton wrote late his book on his first physical works in optics (Newton, 1704) since unsuccessfully he tried to organize also this theory in deductive terms from *a priori* principles (Shapiro, 1984). For this reason, at the end of the book he listed 31 Queries, i.e. all the problems which remained unsolved by his researches. A distinction which is similar to the above mentioned one was presented by (Lakatos, 1978: pp. 24-42).

feelings in the audience.

The origin of this habit is in the two most important scientific works of all times. The first one, Euclides' *Principles*, belongs to the ancient times. Remarkably, it was written just after Aristotle's suggestion of an ideal model of a scientific theory, called by him an "apodeictic" one. The mathematician Beth summarised it in the following way:

> 12. ARISTOTLE'S THEORY OF SCIENCE
> The essentials of Àristotle's theory of science may be compressed into the following definition of the term "deductive"—or, as Aristotle says "apodeictic"—"science":
> A *deductive science* is a system S of sentences, which satisfies the following postulates:
> (I) Any sentence *belonging* to *S* must refer to a specific domain of real entities;
> (II) Any *sentence* belonging to *S* must be true;
> (III) If certain sentences belong to *S*, any logical consequence of these sentences must belong to *S*;
> (IV) There are in *S* a (finite) number of terms, such that
> (a) the *meaning* of these terms is so obvious as to require no further explanation;
> (b) any other term occurring in *S* is definable by means of these terms;
> (V) *There* are in *S* a (finite) number of sentences, such that
> (a) *the* truth of these sentences is so obvious as to require no further proof;
> (b) *the* truth of any other sentence belonging to *S* may be established by logical inference starting from these sentences.
> *The* postulates (I), (II), and (III) will be called respectively the reality, the truth, and the *deductivity* postulate. The postulates (IV) and (V) together constitute the so-called evidence postulate; the fundamental terms and sentences, referred to in postulates (IV) and (V), are called the principles of the science under consideration.
> It is known that Aristotle's doctrines met with a fierce and systematic opposition from the School of Megara…; these objections, though taken quite seriously in Antiquity—later authors, such as Prantl and Zeller, dismiss them abusively as trivial fallacies—could not prevent the triumph of Attic philosophy (Beth, 1959: p. 47ff)[5].

Indeed the subsequent scholars did not remark the fact that the same Aristotle actually suggested more than apodeictic science; he did not organise his *Physics* by drawing in a deductive way from few axioms, but by studying several problems to which he offered solutions[6]. Moreover, the teacher of Aristotle, Plato, had suggested a different attitude:

> I understand, he said, that you are speaking of the province of geometry and the sister arts.
> And when I speak of the other division of the intelligible, you will understand me to speak of that other sort of knowledge which reason herself attains by the power of dialectic, using the hypotheses not as first principles, but only as hypotheses—that is to say, as steps and points of departure into a world which is above

[5]It is instructive to recall the parallel historical search in modern Mathematics and Logic for identifying an appropriate way of organizing a theory. In 1899 Hilbert improved Euclides' deductive method of organizing geometry to an axiomatic organization, which he suggested to be the final model for organising whatsoever scientific theory. But in 1931 Gödel's theorems stopped this program. While the large majority of subsequent mathematicians did not lowered their confidence in the axiomatic method, some scholars suggested a philosophical meaning of Gödel's theorems, as proving that the deductive model is insufficient for representing an entire mathematical theory (Beth, 1950; p. 102; van Heijenoort, 1967: p. 480; Kreisel, 1989; Hintikka, 1989). Already some previous logicians and mathematicians have attempted to find out an alternative to Aristotle's ideal. The most impressive writing is Lukasiewicz' solemn speech about his inner motivation for producing a new theory of mathematical logic (i.e. many-valued logic). The Polish logician "confesses" to be fighting "a spiritual war", that is an "ideal struggle for the liberation of the human spirit" from the "logical coercion given by Aristotle's science as a system of principles and theorems connected by logical relationships. The concept came from Greece and has reigned supreme... The creative mind revolts against this concept of science... A system of three-valued logic... destroys the former concept of science, based on necessity... Logic is a free product of man, like a work of art" (Lukasiewicz, 1970: pp. 84-85). However, his new logic missed the target. In the years '50s several times a similar view was stressed by Beth in more moderate but more accurate terms: "there are two types of science..., rational science" and "empirical science", as well as there are "two fundamentally divergent types of scientific method" (Beth, 1959; § 1.2, p. 136). He scrutinized past theories in order to discover an example of an alternative model; but he recognized little more than a philosophical divergence, without achieving an accurate distinction in scientific terms.

[6]In fact, accurate inspections on his works show that he suggested a mere preference, not a final decision for the apodictic science (Cambiano, 1967; Sainati, 1990). Moreover, elsewhere Aristotle's definition of a logical theory is at variance with his celebrated model: "... to find a *method* by which... we shall avoid saying anything self-contradictory" (Aristotle, 350 B.C.; 100a18; here and in the following quotations, emphasis is added; it will be justified later).

> hypotheses, in order that she may soar beyond them to the first principle of the whole; and clinging to this and then to that which depends on this, by successive steps she descends again without the aid of any sensible object, from ideas, through ideas, and in ideas she ends.
> … I understand you to say that knowledge and being, which the science of dialectic contemplates, are clearer than the notions of the arts [i.e. the deductive sciences], as they are termed, which proceed from hypotheses only… And the habit which is concerned with geometry and the cognate sciences I suppose that you would term understanding, and not reason, as being intermediate between opinion and reason (Plato, 360 B.C.: book VI, 511).
> … geometry and the like—they only dream about being, but never can they behold the waking reality so long as they leave the hypotheses which they use unexamined, and are unable to give an account of them. For when a man knows not his own first principle, and when the conclusion and intermediate steps are also constructed out of he knows not what, how can he imagine that such a fabric of convention can ever become science? (Plato, 360 B.C.: book VII, 533 e).

But no scientific theory apparently applied his ideas. Archimedes, who elaborated all his theories by starting from problems without invoking a priori principles, was commonly considered as a mere discoverer of a multitude of practical results, without theoretical relevance.

3. The Long Search for an Alternative to Aristotle's Model for a Scientific Theory

In order to discover more evidence for an alternative model to the apodeictic one, I improved the above suggestions. Here I bound myself to quickly present in a chronological order the most relevant facts I discovered.

After Aristotle, his model was considered along two millennia as almost the unique model for a scientific theory. Galilei was well aware of this model, since in his youth he studied it (Wallace, 1984)[7]. But never he attempted to state principles, the inertia principle either. Moreover, he wrote the last two books in a dialogical form, i.e. according to Plato's philosophy of science. Hence, he was aware of a choice about a dichotomy. But he was dubious and at last he did not choose. However, since Galilei's accurate suggestion of an experimental method, the experimental evidence was introduced into the postulate Va (evidence) of the model of Aristotle's theory, whose inner structure remained unchanged.

After him, the first authoritative theory of modern science (Newton, 1687) was built according to Aristotle's model of a deductive science. Its great influence on theoretical physicists chased out all Galileo's doubts; rather it confirmed Descartes' basic tenet for the deductive model as the unique one. Thus, Newton's theory gave the seal of the certainty to the ancient resolution of this scientific-philosophical question.

However half a century after Newton, D'Alembert suggested a distinction ("rational/empirical") among the scientific theories. Its main argument was that the "rational" illustration of the development of a deductive scientific theory is hopelessly incomplete, for it "leaves always some holes" (D'Alembert, 1754: p. 501)—truly, a philosophical anticipation of Goedel's theorems. He suggested that a theory has rather to be faithfully built from empirical facts; in particular, he wanted to reduce the status of a principle (e.g., the Newton's second principle of mechanics) from a self-evident proposition to a merely empirical proposition. But he was unable to present a scientific theory according to a clear alternative model of organization.

Lazare Carnot improved D'Alembert viewpoint by writing some lucid words in his books on mechanics[8].

> Among the philosophers [read: scientists] who are occupied with research into the laws of motion, some make of mechanics an experimental science, the others, a purely rational science; that is to say, the former in comparing the phenomena of nature break them up […] in order to know what they have in common, and thereby reduce them to a small number of principal facts […]; the others begin with hypotheses, then reasoning in consequence upon their suppositions, come to discover the laws which bodies follow in their motions, if these hypotheses conform to nature, then comparing the results with the phenomena, and finding them to be in accord, conclude from this that their hypothesis is exact (Carnot L., 1783: p. 102; see also Carnot L., 1803a: p. xvii).

A century ago both Poincaré and Einstein, although ignoring previous suggestions for an alternative model,

[7]The author extends the import of this fact till up to consider Galileo as an Aristotelian. For a different appraisal see (Coppola & Drago, 1984).

[8]A quick illustration of this alternative formulation of classical mechanics is given by (Drago, 2004).

characterised through some characteristic features two different kinds of organization of a physical theory; (Poincaré, 1903; Einstein, 1934; Esposito, 1997). They pointed out some instances of physical theories for each of these two models; in particular, both scholars considered thermodynamics as the theory of reference for the alternative model. I summarize the characteristic features suggested by Einstein (Einstein, 1934) through the following **Table 1**.

Some decades ago a Symposium discussed the same problem, i.e., which is an alternative to Aristotle's ideal; the authors investigated some ancient and modern theories (Hintikka, Gruender, & Agazzi, 1981: vol. II). However, no decisive result was achieved. In previous Section 1 I recalled some more recent appeals to the "principle" theories, however also them did not decisively improved their definition.

In conclusion, in past times several scholars maintained that, although the deductive one was commonly considered as the unique possible model, the problem of which is the appropriate kind of organization of a scientific theory is an important problem concerning not only the genetic stage of a theory or some instrumental applications of a theory. This general problem was considered as properly pertaining to the science. Indeed, each scientist formulating a theory as a system has to decide which kind of its organization he wants to apply.

4. Some Instances of Theories Organized in an Alternative Way

Moreover, I discovered that in the past time some scientists authors presented their scientific theories by organizing them in a different way from the Aristotelian model of a deductive theory.

The above mentioned Lazare Carnot gave instances of this theoretical attitude when founded anew the three mathematized sciences of his time, i.e. geometry, calculus and mechanics respectively on the following problems: Given some elements of a geometric figure to obtain all elements; Whether infinitesimals do pertain to reality; Which are the invariants during an impact of bodies (Carnot L., 1803; Carnot L., 1813; Carnot L., 1783). The organization of this last theory was characterized as a "problematic" organization by his major historian (Gillispie, 1971: p. 87); I prefer call it a problem-based organization (PO).

Moreover, L. Carnot suggested to his son Sadi the basic ideas of the thermodynamic theory (Gillispie, 1971: ch. III D), i.e. the first theory which defied the monopoly of Newtonian mechanics upon theoretical physics. This new theory was organized in a same way of the previously mentioned ones which rejected the deductive model (it is not a chance that the title of the book starts with the word *Reflections*…). It investigates upon a problem, i.e. the maximum efficiency in the energy conversions, argues in a non-deductive way, which was often misinterpreted as an informal way; e.g., even at present time the main argument of thermodynamics, i.e. S. Carnot's *ad absurdum* argument, is considered by most physicists as an engineers' argument; instead, its theoretical import is the highest one, since it covers all possible energy conversions and moreover represents the most profound result of the thermodynamic theory (Mach, 1986: ch. XIX).

My further historical studies found out that in past centuries some more logical, mathematical, physical and chemical theories have been shaped according to the alternative model of organization (Drago, 1990; Drago, 2012). The more relevant are the following ones: Lavoisier's theory of the elements of matter (Lavoisier, 1789), Avogadro's theory of atoms (Avogadro, 1811), Lobachevsky's theory on non-Euclidean parallel lines

Table 1. Constructive theories and principle theories according to Einstein.

CONSTRUCTIVE THEORIES	***PRINCIPLE THEORIES***
Based on assumptions speculative in character; they gave result to a set of relatively simple formulas for explaining relatively complex phenomena	No hypothetical elements, but general properties of natural phenomena (for instance, a perpetual mobile is impossible); then all phenomena are explained through a suitable mathematical formalism
Most of the above assumptions concern the elements constituting the object at issue	Without constitutive hypotheses
Synthetic method	Analytic method
Completeness [+]adaptability, lucidity	Logical perfection, certainty[§] of the fundamental principles, claim to be valid for all phenomena of the same kind
Statistical mechanics (Classical mechanics?)	Thermodynamics, Special relativity

Notes: [+]The following Goedel's theorems (van Heijenoort, 1967) denied this feature. [§]Yet, elsewhere he wrote also that the principles are fictitious.

(Lobachevsky, 1840)[9], Einstein's special relativity (Einstein, 1905), Planck second theory on quanta (Planck, 1913), Kolmogorov's theory of intuitionist formal logic (Kolmogorov, 1924/5), Markov's theory of constructive real numbers (Markov, 1962)[10]. All they are commonly recognised as "revolutionary" theories, since their importance in the history of science cannot be underestimated.

5. A Basic Logical Feature of a Problem-Based Theory: The Doubly Negated Propositions

A mutual comparison of the above mentioned theories allows to extract some characteristic features which are sufficient to characterize the new model of organization of a scientific theory. Let us illustrate the more relevant ones.

Each PO theory deals with a problem so great to appear at a first sight a metaphysical one. The theory consists in looking for a new scientific method capable to reduce the problem to a scientific one and then solve it.

For instance, S. Carnot's thermodynamics puts the problem of the conversions of the heat—whose nature was unknown—in work. He deals with the scientific problem of the maximum efficiency of this kind of machine. In order to solve it, the theory is aimed at discovering a new scientific method for theorizing on how the heat engines work at best.

A PO theory starts through some propositions which play the role of guiding principles; I call them *methodological principles*. The best-known proposition of this kind is the "Impossibility of a perpetual motion"[11] (the exact scientific meaning of the word "perpetual" is "that has no end", as Stevin puts it; Dugas, 1950: p. 131). Gillispie remarked that this methodological principle played the role of a "demonstrative and tutelary principle" in L. Carnot's mechanics (Gillispie, 1971: p. 99). Along a century it played a crucial role in the theoretical development of thermodynamics.

When we drop the two negations of the above proposition, we obtain the proposition: "Every motion has an end time"[12]; which lacks scientific evidence, because we are unable to state the exact moment of the end of a motion—e.g. of the Earth, of a motor-car in a highway—, since our calculation needs to *a priori* know the friction function, which instead is unknown before the motion is occurred. On the other hand, the innumerable experiences of a motion which effectively come to an end are not enough to generalize them in an affirmative version of the above proposition, for this proposition would represent a universal scientific law directly contradicting the principle of inertia.

In the following I list, for brevity's sake without comments, some DNPs that I recognized in the original texts of some other physical or chemical theories: "It is impossible that matter can be divided to infinity [≠matter is divided to a finite degree]" (19th Century chemists); "We call element that which we cannot be further broken down [≠it is simple]" (Lavoisier); "Nothing is created [=not conserved] [≠all is preserved]" (Lavoisier); "It is patently absurd to suppose that forces God gave to matter may be destroyed by man's action [≠ forces are preserved]" (J.P. Joule); "It is not true that heat is not equal to work [≠heat is equal to work]" (thermodynamicists)[13].

Notice that in classical logic the law of the double negation, $A \leftrightarrow A$ (read: *not-not-A*), holds true, i.e. each member of the equality implies the other one. Instead, each of the previous doubly negated propositions cannot be changed into its positive proposition since the latter one lacks of operative evidence. In such cases the double

[9]The two theories on parallel lines of respectively Lobacevsky and Bolyai started a revolution in the two-thousand-year-old tradition of Mathematics. Each of these theories wants to solve a universal problem: How many parallel lines exist? Since their arguing is developed through a list of theorems, they manifest in a lucid way the alternative way of arguing as well as the alternative organization. See (Drago, 2002; Drago & Perno, 2004; Drago, 2007).

[10]One can add, Marx' theory of economics and Freud's psychoanalysis (Freud, 1925).

[11]Here and in the following, emphasis is added for manifesting to the reader the two negations within a proposition. Among the historians of science only Klein (Klein, 1982) suspected the relevance of scientists' linguistic expressions.

[12]The word "impossible" is composed by a negation and the modal word "possible", which is a mere enhancement of the negation. Indeed, the above proposition is at all equivalent to the proposition "It is not the case that every motion has no end". Notice, that however a modal word by itself alone is equivalent to a doubly negated proposition; for instance "possible" is equivalent to "it is not true tat it is not…" This intuitive translation is confirmed by the well-known formal translation from the S4 model of modal logic to the intuitionist logic.

[13]One can add instances belonging to mathematical and logical theories: "Infinitesimals are not chimerical beings" (Carnot L., 1782: p. 182) (the scientific meaning of the word "chimerical" is "not real") [≠ infinitesimals are real beings]. "The [hyperbolic] hypothesis is not contradictory [≠it is true]..." (Lobachevsky, 1840: prop. 23) "The use of the principle of the excluded middle never leads to a contradiction [≠is consistent]"; (Kolmogorov, 1924/5: p. 461). "I see no reasonable basis for rejecting it [=the resulting affirmative predicate]" (Markov, 1962: p. 5).

negation law fails. According to authoritative logicians (Prawitz & Melmnaas, 1968; Prawitz, 1976; Dummett, 1977), this fact, more than the failure of the law of the excluded middle, characterizes intuitionist logic and, in general, the various kinds of non-classical logic[14].

6. The Role Played by the DNPs inside a Physical Theory

Let us focus again the attention on S. Carnot's theory. He deliberately devoted the first part of his book to arguments stating his main results without making use of mathematics, which was relegated by him to a footnote. In fact, this part includes a great number of DNPs, which all together capture the contents of the book, so that their sequence is enough for reconstructing the whole logical thread of his thinking (Drago & Pisano, 2000).

This logical phenomenon was ignored by all the above-mentioned scientists. Rather, some scientists, when meeting DNPs, felt an uncomfortable situation; this feeling is revealed by their *ad hoc* inventions of affirmative words which replace the DNPs at the cost of a distortion of their meanings; e.g. thermodynamics' textbooks declare heat to be *equivalent* to work, without never explaining what means this word; actually it replaces a DNP ("it is <u>not</u> true that they are <u>not</u> equal")[15]. In stating the inertia principle in an apodeictic way Newton's crucial verb was "perseveres", which is animistic in nature, i.e. it does not pertain to experimental science; rather he had to state a DNP, as L. Carnot did: "A body… by itself does <u>not</u> <u>change</u>…" (Carnot L., 1803a: p. 49).

In 1936 Birkhoff and von Neumann discovered a non-classical logic in quantum mechanics (Birkhoff & von Neumann, 1936). Originally, this theory was a PO theory, since it dealt with a universal problem, i.e. the problem of how measure at a same time two conjugate magnitudes constituting the state of a physical system. Quantum mechanics was characterized by the proposition: "It is <u>impossible</u> to measure at the same time two conjugate magnitudes with <u>infinite</u> accuracy [≠ two conjugate magnitudes are measured at the same time with finite accuracy]". Notice that the proposition in square brackets proves to be false owing to the possibility of achieving infinite accuracy in only one magnitude; on the other hand, by dropping one negation in the former one, the resulting proposition is again false, owing to the experimental facts supporting the indeterminacy principle. Hence, the logic of quantum mechanics was not a novelty since it is the same logic shared by all the PO theories (Drago & Venezia, 2002).

However, quantum mechanics introduced a historical novelty. Whereas in classical physics non-classical logic could be expunged by—as we saw in the above—a suitable distortion of some physical notions, in quantum mechanics this operation was no longer possible. E.g. in a typical classical theory, i.e. thermodynamics, in a first time the relationship between heat and work is declared an "equivalence", without explaining the meaning of this word; then it is dismissed by using only Maxwell's relationships on entropy and the other state variables. Instead in quantum theory the DNP expressing the typical problem of the theory does not concern some lateral magnitudes but the magnitudes defining the state of the system which needs to be defined exactly. Hence, for the first time the use of a non-classical logic in the theoretical development of quantum mechanics resulted to be unavoidable (Drago, 1991a).

Let us now consider which model of a scientific theory is appropriate to the occurrence of DNPs inside the illustration of a theory. Notice that in a full deductive scientific theory, which is managed by classical logic, the propositions of both the axioms and their derivations are all equivalent to their respective doubly negated versions; hence, a DNP, being not equivalent to the affirmative version, can neither be derived from the axioms nor work as an axiom; it is at variance with the deductive, i.e. Aristotelian, model of a scientific theory. This fact shows that a scientific theory including DNPs must be organized according to an alternative model.

Indeed in the texts presenting PO theories a former part argues by means of *ad absurdum* theorems[16], which are linked together in a chain till up a final one. In fact, S. Carnot's thermodynamics argues through *ad absur-*

[14]In the past this point was misinterpreted owing to a widespread opinion that, as a common slogan says, two negations affirm. But this slogan wordy expresses the classical law of the double negation, whereas several kinds of non-classical logic are possible when this law fails. Moreover, it is a traditional "dogma" of the Anglo-Saxon linguists that a double negation is a Latinate which corresponds to a unfrankly speaking; so that it has to be suppressed in all cases (Horn, 1989: pp. 84 ff., 296-308; Horn, 2002: p. 79ff.). Moreover, it is a widespread prejudice that the non-classical logic cannot be applied to the reality. Instead several kinds of non-classical logic are at present applied to for instance computer science.

[15]For two centuries in Mathematics an infinitesimal was defined as a number that *tends* to zero, without never explaining what is a number's tension. Instead, in non-standard analysis it is correctly defined through a DNP; it is a number which is <u>lesser than</u> any other real number <u>greater than</u> zero (each previous comparative word is negative since it <u>differentiates</u> from the equality).

[16]It is a widespread opinion to consider each *ad absurdum* proof as reducible to a direct proof; see e.g. (Gardiès, 1991). But the conclusion of such a kind of proof is a DNP, which can be changed in an affirmative proposition by only using the double negation law of classical logic.

dum theorem till up to the celebrated one-reiterated also by present textbooks—, determining the maximum efficiency in *all* heat/work conversions (Drago & Pisano, 2000). Hence, one more logical feature of a PO theory is the composition of the DNPs in *ad absurdum* proofs.

7. A Same Logical Predicate Concluding a Problem-Based Theory

Since the demarcation of the non-classical logic from the classical one was established not before some decades ago, no surprise if an author of a PO theory did not rigorously followed this new way of thinking. Moreover, we cannot hope either that we can neatly recognize in each such theory the new ideal model of a scientific theory, because we have to take in account that it was the product of the ingenuousness of its founder, who was more concerned with fitting the experimental data in an original theory than to outline the new model through his theory. Indeed, some authors of the above-mentioned ones wrote texts which are less structured in logical terms; e.g. L. Carnot's calculus, Lavoisier's chemistry, Galois' theory of groups, Klein's Erlanger program, Einstein's theory of special relativity all lack of *ad absurdum* theorems. The latter theory was presented by a paper which lacks of this kind of argument just when the author made a mistake (he added and subtracted the velocity from the light speed; Einstein, 1905b: § 2).

However, some presentations of PO theories present even the following feature. Let us list the universal conclusions of some PO theories. The last *ad absurdum* proof obtains a doubly negated predicate of a universal validity on all the cases considered by the theory. By addressing for more information the reader to a specific paper on this point (Drago, 2012), here I will quote only two instances. S. Carnot: "…no change of temperature inside the bodies employed for obtaining the motive power of heat occurs without a change in the volume" (Carnot S., 1824: p. 23).

We remark that this conclusion is formalised by the logical formula: $x\, f(x)$.

Some other authors achieved a conclusion of a different kind. For ex., Avogadro: "[All] The proportions among the quantities in the combinations of the substances do not seem depend other than both the relative number of molecules which combine themselves and the number of the composed molecules which result from them", i.e., "All the proportions among the quantities in the combinations of the substances depend"[17] (Avogadro, 1811: p. 58).

The above proposition is formalised by the following formula: $\forall x\, f(x)$.

It is a remarkable fact that in intuitionist logic the latter formula is equivalent to the former formula (Dummett, 1978: p. 29); we will represent it by A^I.

We obtain a very important conclusion: a PO theory, in the search for a new method capable to solve a basic problem, argues according to a sequence of DNPs composing a chain of *ad absurdum* arguments, aimed at eventually obtaining an instance of the typical predicate A^I.

In my opinion, it is an unfortunate fact the common depreciation of Leibniz' philosophy of science as a metaphysical one. Really, he was the last philosopher who was able to constructively work in the foundations of modern science. In particular, his analysis on the basic ideas of human knowledge suggested to him two labyrinths of human reason, i.e. the dual notion of infinity (either the actual one or the potential one) and an opposition between law and free will. We see that the latter opposition represents in subjective and moral terms a dichotomy between two general attitudes, which in science correspond to respectively the apodeictic ideal—in other words, the context of demonstration which produces compulsory laws—, and a research for a new scientific method capable to solve a crucial problem–in other words, the context of discovery (Drago, 1994).

Furthermore, in correspondence to the latter opposition, Leibniz stated:

> Two are the principles of the human mind: the principle of non-contradiction and the principle of the sufficient reason, [that is] nothing is without reason, or everything has its reason, although we are not always capable of discovering this reason (Leibniz, 1686).

The proposition "Nothing is without reason" is a DNP; in the quotation the same Leibniz explains why; indeed, one wants to conclude "Everything has a reason", but Leibniz underlines that not always we have sufficient evidence for stating this reason; hence, only the doubly negated proposition holds true; being it different

[17]Notice that also in this case the modal word "seems" merely enhances the negation to which it is associated. The subsequent sentence ("Hence, there exist simple relationships [of mutual dependence] also among the volumes of [all] the gaseous substances and the number of the simple or composed molecules which compose them") constitutes the celebrated "Avogadro's law" on the molecular constitution of whatsoever kind of matter. An analysis of the paper through its DNPs is given by (Drago & Oliva, 1999).

from the affirmative one it is DNP. No more sharp dichotomy could be suggested about human thinking.

The former proposition ("Nothing is without reason"; for short, PSR) is formalised by calling *x* "event" and *f*(*x*) "it is connected to some events"; the following formula is obtained: *x f*(*x*); which is the same formula for all final predicates of the PO theories, i.e. A^I. Hence, this Leibniz' principle, being itself a DNP, appears to be the philosophical principle addressing an author of a PO theory in his non-classical arguing for achieving the final conclusion.

8. The Logical Change of the Final Predicate

The universal nature of the final predicate suggests to the author to have led his inductive arguing to its greatest potentiality. For this reason he estimates that his arguing had collected all possible evidence for solving the stated problem by means of the final predicate. Hence, the corresponding affirmative predicate may be plausibly considered as a new hypothesis from which the subsequent part of the theory may be deductively developed in classical logic.

In fact, in the original texts of the PO theories an author changes the final predicate in the corresponding affirmative predicate. For instance; S. Carnot, after having in his book stated his celebrated theorem, develops a theory to the affirmative proposition "There exists a maximum efficiency in each heat engine" which is the affirmative proposition corresponding to the DNP concluding the *ad absurdum* theorem. He then applies it to obtain gas' laws and new appraisals on the efficiencies of current heat engines. In other terms, in the latter part of his book, he considers the affirmative predicate A corresponding to A^I as an hypothesis, from which he deduces according to classical logic a lot of theorems (Drago, 2012).

It is remarkable that in the history of Mathematics the presentation of the first instance of a PO theory—Lobachevsky's non-Euclidean geometry—overtly declared a change in the theoretical role played by a crucial proposition: "[My plausible supposition of two parallel lines] can likewise be *admitted* [as usually Euclides' principle is; hence it may work as an affirmative principle for developing the following, deductive part of my theory] without leading to any contradiction in the results and *founds* [in a deductive way] a new geometry" (Lobachevsky, 1840: proposition 22; emphasis added). We see that this change summarises a change of the organisation of his theory. The same occurred in the history of theoretical physics; the presentation of a crucial instance of a PO theory—Einstein's special relativity. In his celebrated paper Einstein wrote: "We will raise this conjecture [i.e. a DNP] (the substance of which will be hereafter called the "[affirmative axiom] principle of relativity") (Einstein, 1905b: p. 892). As a fact, both authors originated two celebrated scientific revolutions.

Let us now recall how Leibniz's presented the entire proposition of the principle of sufficient reason. He too in a first time stated a universal DNP which subsequently he changed (although afterwards he remarked that this sentence may be unsupported by sufficient evidence) in the corresponding affirmative sentence "everything has its reason". Let us call this change PSR°. It also is represented by a change of the intuitionist predicate A^I in the affirmative predicate A, i.e. the same logical formula A^I =>> A formalising the final move of a PO theory.

One more information about this change comes from Markov's presentation of PO theory on computable real numbers. In the course of it, Markov presents a different predicate form the previous ones, ∃*x f*(*x*), which is i) decidable and ii) results from an *ad absurdum* proof (Markov, 1962: p. 6). He suggests that we are allowed to change it in the corresponding affirmative predicate ∃*x f*(*x*) (Markov's principle) although he admits that this change is not accepted by the intuitionists; but he justifies it by an argument which substantially is the PSR°. This fact suggests that also in general the PRS° has to be restricted by these two constraints; i) the predicate results from an indirect proof; ii) the indirect proofs have to rely on operative-hence decidable-propositions, just that each scientific theory of nature does.

In conclusion, the strategy of the logical development of a PO theory is summarised by Leibniz' first two sentences of his version of the PSR.

I conclude that since two centuries some authors presented their scientific theories by making an effective and substantially consistent use of non-classical logic; it is related to the whole context of the theory, i.e. a PO organization of the theory. Hence a well-defined alternative organisation to the Aristotelian one based upon deductive arguing does exist, as implicitly manifested by these theories[18].

[18]Notice that mathematical logic is commonly formalized as a theory according to the Aristotelian organization, i.e. as a pyramid in which truth is injected from the top to the bottom of an infinite list of theorems. Yet, the founder of the intuitionism, Brouwer, did not trust in axioms (Brouwer, 1908) and he considered as impossible to formalize according to the deductive model his logic. When subsequently his disciple Heyting offered a formalization, he stated that it however was insufficient to grasp the entire content of the intuitionist logic (Heyting, 1960: p. 102).

Since the distinction between the two kinds of mathematical logic was established not before the years '70s, of course the previous scholars could declare nothing more than philosophical hints qualifying this distinction; to which no theoretical physicist was obliged to adhere. Moreover, Einstein was not steady in his theoretical preference for the "principle" theory; in the latter years he presented his previous philosophical viewpoint as an "opportunistic" one (Einstein, 1949: pp. 683-684).

9. Introduction to the Physical Relevance of Einstein's Paper

The first Einstein' paper on quanta (Einstein, 1905a) offered a proof for the existence of energy quanta, through a theoretical argument (of both a thermodynamic and statistical nature) which leads to successfully interpret three experimental effects, including the photoelectric effect.

Of this paper there exist some translations (Arons & Peppard, 1965; Boorse & Motz, 1966: pp. 544-557; Ter Haar, 1967: pp. 91-107)[19] and several commentaries (Rosenfeld, 1936; Jammer, 1966: pp. 28-30; Kuhn, 1978: ch. VII; El'yashevich, 1979: pp. 556-563; Pais, 1979: pp. 871-873; Stachel, 1989: pp. 134-148; Rynasiewiecz & Renn, 2006; Norton, 2006).

Dorling listed a series of, if not defective, unclear points[20]. But he decided to "assume… [that] there exists some fairly straightforward reformulation of Einstein's argument which avoids those particular difficulties" (Dorling, 1971: p. 7). I agree, because these defects do not weaken the fundamental results.

About this Einstein's paper the historian Klein remarked:

> It is of course…. the theory of relativity, with which Einstein's name is uniquely linked by the public and by most of the community of scientists. Einstein's work on relativity has generated millions of words of comment and exposition on all levels of discourse. Comparatively little has been written about his probings, over a period of a quarter of a century, into the theory of radiation and its significance for our understanding of the physical world. And yet the boldness and clarity of Einstein's insight show forth as characteristically in these studies as in his more famous investigations into the nature of space and time (Klein, 1963: p. 60).

Indeed, Einstein warned that this paper was his "most revolutionary paper" (Einstein, 1905c). Why? In the lack of an Einstein's answer, let us list historians' answers. According to Stachel it was revolutionary "…in challenging the unlimited validity of Maxwell's theory of light and suggesting the existence of light quanta" (Stachel, 1986: p. 134). The historian Klein gives more emphasis to the paper: Einstein's "arguments, at one fundamentally simple and incredibly daring, demonstrate the essential features of Einstein's whole approach to physics" (Klein, 1963: p. 61)". It took more than twenty years for its ideas to be worked into the structure of physics, and in the process that structure was essentially and radically changed" (Klein, 1963: p. 59). Recently, another scholar wrote that Einstein's arguments were "miraculous" inasmuch as they centred the solution of the problems although their support was weak (Norton, 2006: title). In all cases these historians see the paper under the light of the subsequent development of the theoretical physics, i.e. the birth of quantum mechanics.

Rather I want to consider the paper under the light of the previous analysis on the past OP theories.

10. A Quick Illustration of the Paper's Contents

Let us first consider what Einstein was saying in this paper.

Einstein starts by posing a problem: Is continuous electromagnetism valid for all phenomena? He answers in the negative; he suggests a physical situation whose description by means of the continuous formulas of Maxwell's electromagnetism leads to a mathematical divergence and hence to "no definite energy distribution" (p. 369 I)[21].

[19]I will refer to the first translation.

[20]One historian remarked that the treatment is limited to Wien's law, because Einstein did not consider aggregates of photons: "So Einstein's conclusion seems to be that if radiation in free space consists of quanta, it should obey Wien's rather than Planck's distribution law. He offers no explanation of this odd conclusion at his point" (Brush, 1983: p. 112). But in the introduction of the paper of the following year, Einstein declared that at time of his previous paper he was dubious of Planck's theory (Einstein, 1906: p. 199). In addition, Einstein could disregard aggregates of photons, since in the range of low frequencies, Boltzmann statistics is a good approximation to Bose-Einstein statistics. Furthermore, the re-construction of Einstein's argument, shown in the following, will prove that he deliberately chose Wien's law in order to extend his interpretation of electromagnetism in discrete terms to the classical domain. At last, one year later he obtained Planck's law by correcting Planck's calculation of complexions.

[21]I quote from Arons & Peppard's translation by merely writing the number of pages in round parentheses.

In order to discover a discrete theory of electromagnetic phenomena he directs his attention to the case study of black body radiation, where previously Planck introduced in a theoretical way the unprecedented idea of energy quanta. He wants to rather argue directly from experimental facts. He first of all showed that the Planck formula alone is enough to deduce Avogadro's number. Encouraged by this success to pursue on this new path, he then exploits thermodynamics and statistical mechanics to theorize on radiation. His theoretical arguments result in stating an analogy: radiation energy behaves, even at a microphysical level, according to the same formal law $S = S(V)$ of an ideal gas.

From the new hypothesis suggested by the analogy (i.e. independent quanta of energy) Einstein makes some deductions which he tests with the experimental data concerning three effects. Successful interpretations of these phenomena are obtained. Hence, his results changed the notion of quanta from a mathematical hypothesis—as it was in Planck's writings—to a directly testable notion.

Previously quoted appraisals on the revolutionary nature of Einstein's paper are well justified. The paper constituted a shock for the scientific community since it furnished a firm experimental basis for quanta, a notion implying a revolutionary conception of theoretical physics as a whole. Indeed, at that time "… very few were willing to follow him in accepting the startling idea of light quanta on the strength of deductions that were based on the [dubious] statistical interpretation of the second law of thermodynamics" (Klein, 1963: p. 61). Later, even when presenting Einstein for membership to the Prussian Academy, Planck wrote a negative appraisal on this paper ("he… missed the target of his speculations" (quoted in Seelig, 1956: p. 143)). In 1917, the same Millikan who confirmed the quanta hypothesis through experiments on the photo-electric effect wrote: "I shall not attempt to present the basis for such an assumption, for, as a matter of fact, it had almost none at that time" (Millikan, 1917: 238)[22]. However, although slowly, "it was primarily the photoelectric effect [as interpreted by Einstein's paper] to which physicists referred as an irrefutable demonstration of the existence of photons and which thus played an important part in the conceptual development of quantum mechanics" (Jammer, 1966: p. 33).

11. The New Kind of Organisation, Implicitly Presented by the Paper

In the above it was recalled that Einstein, together with his contemporaries Poincaré and Lorentz, made explicit a dichotomic option on the kind of the organisation of a physical theory. Although Einstein and all other scientists presenting a PO theory have been unable to formally qualify this alternative, all they, by wanting to solve problems; have to argue (no matter if unwarily) outside the deductive arguments drawing conclusions from few axioms; at the best, according to the PO model of a theory.

In fact, in the title of the 1905 paper Einstein characterises his theory as "heuristic". By this word is commonly meant that the paper does not present principles-axioms, from which deductions are derived. Indeed, he showed the existence of light quanta without using the formal apparatus of his previous papers on statistical mechanics and Maxwell's electromagnetic theory either.

The point is illustrated by a philosopher of science:

> The structure of this paper of Einstein's does not fit in easily with popular twentieth-century conceptions of scientific method. The traditional way of forcing it into the hypothetical-deductive mould is to avoid any mention of the first two-thirds of the paper. Thus the reader of Weyl (1931), von Laue (1947), Messiah (1958), Heisenberg (1959) or Eisberg (1961) is left with the impression that Einstein merely postulated the existence of photon in order to explain the photo-electric effect (Dorling, 1971: p. 6)[23].

Even the logical reasoning of this paper has been questioned.

> Commentators often characterize the arguments of the paper as creating an analogy [read: a theoretically little relevant result] between radiation and classical ideal gas of material particles. But to Einstein, the connection was more than a weak argument, as is an analogy; it was a "far reaching formal relationship".

[22]Already in the previous year he wrote that Einstein's suggestion of quanta was "a bold, not to say a reckless, hypothesis"; because "Einstein's photoelectric equation… cannot in my judgment be looked upon at present as resting upon any sort of a satisfactory theoretical foundation" (Millikan, 1916).

[23]This author makes use of the word "deduction" in a misleading way in both the title and the text. For instance, he calls a "deduction from phenomena" a process obtaining "a new explanatory theory from certain empirical generalisations taken as phenomena together with certain more general assumptions whose validity is taken for granted" (p. 1). Rather it is usual to call this an inference process which is the opposite process to the deductive one.

(Klevgard, 2008: p. 14)[24].

"But—the historian Klein asks—, how seriously was one to take this conclusion? Did it really amount to anything more than an analogy, with the "as if" the essential phrase in its proposition?... According to him, "The conclusion… was to be taken seriously, and Einstein immediately exploited this "suggestion" as to the nature of radiation, tenuous as it might (and did) seem to others, pressing it in directions that might yield experimentally verifiable consequences" (Klein, 1963: pp. 70-71).

But, which "formal relationship" is this? The question leads to the more general question of the model of the organization of a scientific theory.

Previous reading of Einstein's paper shows a different structure from the deductive one. A more accurate inspection of the paper shows that the theory is organised according to the same PO model that we previously found in S. Carnot's theory and some other theories in physics, mathematics and logic.

Indeed, also Einstein's paper presents a basic problem, which is located in the (middle of the) "Introduction": Is the continuous electromagnetic theory valid in all physical situations? May the mathematics of this consolidated theory lead to contradictions?

Dorling summarised the subsequent mathematical deductions–pertaining to the Sections 3-6 of Einstein's exposition-by means of a diagram[25] (Dorling, 1971: p. 2). However, he recognised that two, in actual fact conclusive, logical steps are no more than analogies. Hence, an analysis of the development of Einstein's arguments has to go beyond his mathematical calculations, in order to make apparent *all* the logical steps leading to such analogies.

In only five pages of the theoretical part of Einstein's paper I recognised 55 DNPs. They are listed in the Appendix. Admittedly, not all DNPs are clear and essential to Einstein's arguments. However, their high frequency represents an absolutely different way of arguing from a deductive one. A further verification comes from the appropriateness of the DNPs for presenting a PO theory; the reader easily verifies that the sequence of the DNPs listed in the Appendix is sufficient on its own to provide the logical thread of Einstein's presentation[26].

An even more important verification is the author's use of the only the kind of proof which is suitable for concluding an argument relying on DNPs, i.e. the *ad absurdum* proof, like S. Carnot' proof in thermodynamics. In Einstein's paper this kind of proof concerns the failure of Maxwell's theory in a special case (the conclusion is the universal DNP no. 18); this proof implicitly states that the discrete in the mathematical treatment of light is <u>not</u> <u>im</u>possible[27].

However, let us remark that for the first time Einstein argued about the microscopic world, which at that time was unknown; there, it was at all possible to imagine things that are absurd in the macroscopic world. For cause Einstein's universal theoretical conclusion, i.e. the DNP 52, is not drawn from an *ad absurdum* proof, but from an analogy[28]; only analogies allowed the introduction of novelties into the theoretical physics of the microscopic world.

Dorling put forward some probabilistic arguments in order to avoid the final analogy of Einstein's paper. These arguments are all *ad absurdum* proofs, as is S. Carnot's. Hence they conform more closely Einstein's presentation to the typical way of arguing of a PO theory.

In conclusion, no more precise distinction from deductive reasoning could be suggested than the arguing in the intuitionist logic. All the above facts show that *Einstein not only chose an alternative model of organisation of a theory, but also that his inductive reasoning conformed substantially to the alternative way of non-classical reasoning which is characteristic of a PO theory.*

---

[24]For instance, Rosenfeld wrote: "… il se contente d'ailleurs d'attribuer un caractère "heuristique" (Rosenfeld, 1936: p. 170).

[25]However, the analogy starts since the end of the Section 4, not at the end of the Section 6.

[26]Also Einstein's paper on special relativity (Einstein, 1905b) presents DNPs; e.g. the word "equivalent" (which means "It is <u>not</u> true that it is <u>not</u> equal…"≠ is equal) or the word <u>in</u>-<u>dependent</u> (≠free), or the two words "... <u>without</u> <u>dividing</u>..." (Drago, 2010a). It is not a contrary evidence to Einstein's use of non-classical logic what Stachel (Stachel, 2005: pp. xv-lxxii, p. xxxv), wrote: "For Einstein the process of thinking was a solitary activity primarily non-verbal in nature. As a secondary stage, it was necessary for him to transform his results of this primary process into forms communicable to others". Owing to this two stages process, it is well plausible that unwarily Einstein thought in non-classical logic and then was lea to choose his verbal expressions in the common classical logic.

[27]A further misinterpreted point concerns the *ad absurdum* proof and the analogies; both are under-evaluated by most theoretical physicists as weak kinds of arguments.

[28]Maybe this conclusion was intended by Einstein as drawn from only his deductive arguing through mathematical calculations. However, before the final conclusion, he apperceived a jump of this mathematical development to an analogy, since he wrote: "From this we further conclude that…." where "further" here apparently means "in addition".

12. Einstein's Arguing in Detail

Let us recall how a PO theory proceeds. After having stated the problem, it develops through a sequence of units of reasoning. Each unit concludes (more or less explicitly) a proposition which for the next unit plays the role of a methodological principle. In this way, the units all together form a logical sequence.

In Einstein's paper we recognise a sequence of four units of reasoning. Section 1 presents the first one, which solves (negatively) the previous problem. Indeed, Einstein studies a particular physical situation (electrons and molecules interacting through radiation). By assuming the universal validity of electromagnetism, the classical equipartition theorem of statistical mechanics leads to an infinity; which is an absurdity in both experimental and theoretical terms. In other words, in this unit he argues through an *ad absurdum* proof.

He explicitly concludes that this unacceptable divergence represents a stumbling block for the advancement of theoretical physics; implicitly, he concludes that it is not impossible that continuous electromagnetism fails. Of course, this proposition does not imply the corresponding affirmative proposition, i.e. a continuous electromagnetism is disproved, because the previous result is too specific;

Before the year 1905 Planck had already suggested theoretical arguments, including a first notion of energy quantum, which have obtained the experimental law of the black-body distribution of energy. But Einstein is dubious about Planck's theoretical method, since he wants to argue directly from experimental data, as he declared later in the paper: "In the following we shall consider the experimental facts concerning blackbody radiation without invoking a model [or rather, any model, including Planck's] for the emission and propagation of the radiation itself" (p. 368 col. II).

Then Einstein's abruptly begins Section 2; he leaves as implicit the following question: In which specific field of macroscopic phenomena arise the contradiction? Or, in constructive terms: In which physical situations is discrete electromagnetism possible? Einstein exploits no more than Planck's formula, not his theory, for deriving from it a typical discrete result, i.e. the Avogadro number, which along half a century played the role of the crucial notion of the atomic worldview. From this result concerning a number of great import his implicit conclusion is the following: It is not impossible to obtain a discrete electromagnetism.

In Sections 3 and 4 a third unit of reasoning about radiation suggests, by using thermodynamic arguments, a formula $S = S(V)$ which is the same formula for an ideal gas; he explicitly obtains an *analogy*, which may be formulated through the following proposition: it cannot be excluded that radiation is composed, like a gas, of discrete objects (i.e. quanta)[29].

A fourth unit of reasoning can be recognized in Sections 5-6. By following Boltzmann's approach to statistical mechanics, Einstein obtains an analogy between the microscopic distribution of gas particles in space and that of the elements of radiation; i.e., the spatial distribution of quanta is not different from the spatial distribution of the mutually independent particles of a gas.

The above sequence of units of reasoning cannot represent a deductive theory, as Newton's mechanics does, also because the final conclusion is an analogy, i.e. no more than a probable hypothesis; by means of it Einstein acquires a merely plausible solution to the problem he stated at the beginning of the paper. Einstein indicated the nature of this conclusion (an analogy) in the title of his paper, qualifying his theory as a non-deductive one[30]. But his term "heuristic" is insufficient for qualifying his arguing, because Einstein's reasoning is more complex than a merely heuristic process for guessing a rule concerning some experimental data; indeed, he produced a complex argumentation of theoretical import; indeed, the above-mentioned analogy is the final conclusion of the sequence of the four units of reasoning. Moreover, let us remark that the analogy ("From this we further conclude that: Monochromatic radiation of low density (within the range of validity of Wien's radiation formula) behaves thermodynamically as thought it consisted [= so that is not true that it does not consists ≠ so that it consists]…") is equivalent the DNP in square brackets. Hence, it constitutes an appropriate way for concluding a PO theory[31].

As the next logical step, each author of a past PO theory translated the hypothetical proposition of the conclusion into the corresponding affirmative proposition, from which he then deduced all the relevant theoretical predictions regarding experiments. Also in Einstein's paper, at the beginning of Sections 7-9 on the applications of

[29](Rynasiewicz & Renn, 2006: pp. 11-12) accurately circumscribe the validity of this analogy.

[30](Klein, 1982) noticed and commented—but in a mere intuitive way—the word "heuristic".

[31]Moreover, a final indirect proof is implicitly expressed by the theory as a whole: It is impossible that quanta do not exist; otherwise several phenomena cannot be understood.

the previous theory, the final conclusion is explicitly changed into the affirmative proposition: "… let us assume that…. both the incident and emitted light consist of energy quanta…" (Notice that the Introduction anticipated this proposition in affirmative terms). In the subsequent Sections 7-9 he deduces from the proposition of the existence of quanta some specific formulas which are successfully tested with respect to three new electromagnetic phenomena. His "heuristic" (actually PO) theory is now completed.

In the following **Table 2**, I will summarize the logical sequence of the steps in his reasoning. In Italics I add some implicit propositions in order to make more clear which is the logical structure of the paper according to a PO theory.

All the above, considered from the point of view of the foundations of a theory, shows that Einstein's "heuristic" presentation substantially conformed to the model of a PO theory, which is alternative to the Newtonian one. Hence, Einstein's theory introduced theoretical physics *a dichotomic option concerning the kind of organization.*

13. The Two Kinds of Mathematics Illustrated by Einstein's Paper

The main subject of Einstein's theory is a mathematical question which is declared since the introduction of the paper. Let us analyse it.

By taking in account the physical theories of previous times he stresses that in theoretical physics there exists a conflict between the use of two different kinds of mathematical formalisms, i.e. "continuous spatial functions" (which allows the "subdivision [e.g. of a wave] into arbitrarily small parts") and the use of "finite" mathematics to describe discrete material points (e.g. in gas theory). Let us quote:

Table 2. A summary of Einstein's paper.

(Introduction) [*A dichotomic option of the kind of mathematics? From this problem concerning the whole of theoretical physics to the same problem in electromagnetism*] Recognition of either continuous or discontinuous descriptions of physical reality. In particular, [*this is the problem*] is Maxwell's continuous electromagnetic theory of universal validity? If not, the electromagnetic theory is also discrete and thus one may hypothesize energy quanta although [at present time] without the support of any evidence.
(Section 1) [*A first unit of reasoning* (*including an indirect proof*) *on the relationship between Maxwell's theory and statistical mechanics*] In order to solve the above problem, the application of *Maxwell's continuous electromagnetism* plus the equipartition theorem of statistical mechanics to the case of molecules and electrons interacting through radiation; the calculations lead to a distribution [Rayleigh-Jeans'], whose formula gives a divergence; this result does not correspond to experience [*i.e., it is an absurdity. Implicit conclusion: The hypothesis of a discrete electromagnetism is* not *impossible*].
(Section 2): [*A second unit of reasoning for obtaining an introductory result*] The formula of Planck's law [*already obtained by means of the mathematical notion of quanta*] produces an exact evaluation of the Avogadro constant. [*Implicit conclusion: It is not impossible to theorize a discrete electromagnetism*].
(Section 3): [*A third unit of reasoning obtains an analogy between the two formulas concerning the thermodynamic behaviours of respectively radiation and an ideal gas*] From the well-established Wien's law some thermodynamic arguments lead to define the entropy S for the radiation of a particular frequency v.
(Section 4): [*The analogy*] From this definition is obtained for the radiation the function $S = S(V)$, which is the same as the functions $S = S(V)$ for an ideal gas [*Implicit logical conclusion: It is not impossible that some electromagnetic phenomena are discrete*].
(Section 5): [*A fourth unit of reasoning obtains an analogy between the microscopic behaviours of the two physical systems at issue*] Probabilistic definition of Boltzmann's formula for the entropy of a gas. Deduction of the function $S = S(V)$ for the molecules of a gas.
(Section 6): [*Analogical conclusion of a universal nature*] The probability of occurrence of the elements in space, obtained by the previous function is the same as that obtained by the formula $S = S(V)$ of Section 4 for radiation. Conclusion: "Monochromatic radiation of low density… behaves thermodynamically *as thought* it consisted of a number of independent energy quanta…" (emphasis added), i.e. like the particles of a gas. [*The implicit logical conclusion: It is not impossible that quanta exist as independent entities*].
(Sections 7, 8 and 9): [*Three experimental verifications*] The previous conclusion is changed into the corresponding affirmative proposition, i.e. independent energy quanta do exist. The mathematical deductions from it are successfully tested on three electromagnetic phenomena [*The argument of the entire paper implicitly constitutes an ad absurdum proof for the existence of quanta*].

> A profound formal distinction exists between the theoretical concepts which physicists have formed regarding gases and other ponderable bodies and the Maxwellian theory of electromagnetic processes in so-called empty space. While we consider the state of a body to be completely determined by the positions and velocities of a very large, yet finite, number of atoms and electrons, we make use of continuous spatial functions to describe the electromagnetic state of a given volume, and a finite number of parameters cannot be regarded as sufficient for the complete determination of such a state. According to the Maxwellian theory, [for instance, the] energy is to be considered a continuous spatial function in the case of all purely electromagnetic phenomena including light, while the energy of a ponderable object should, according to the present conceptions of physicists, be represented as a sum carried over [a finite number of] the atoms and electrons. [Indeed] The energy of a ponderable body cannot be subdivided into arbitrarily many or arbitrarily small parts, while the energy of a beam of light from a point source (according to the Maxwellian theory of light or, more generally, according to any wave theory) is continuously spread over an ever increasing volume (Einstein, 1905 a: p. 367).

Here Einstein is considering the discrete not as a property of a physical system, i.e. particles instead of the waves, but as the mathematics pertaining to the foundations of theoretical Physics. He does not considers discrete Mathematics as an instrumental role, or as an astute theoretical tool for providing a provisional description of nature, which eventually will become a continuous representation one. Rather, he deals with a foundational problem, i.e. how to choose between the discrete and the continuum[32].

At his time this point was an "utterly heretical proposal" as a historian (Klein, 1980: p. 167) characterised his hypothesis of a discrete electromagnetism. Indeed he advanced this hypothesis in a timid way

> The wave theory of light, which operates with continuous spatial functions, has worked well in the representation of purely optical phenomena and will probably never be replaced by another theory. It should be kept in mind, however, that the optical observations refer to time averages rather than instantaneous values. In spite of the complete experimental confirmation of the theory as applied to diffraction, reflection, refraction, dispersion, etc., it is still conceivable that the theory of light which operates with continuous spatial functions may lead to contradictions with experience when it is applied to the phenomena of emission and transformation of light [where the discrete is at stake].

What are the consequences of this "profound formal distinction" about the use of two different mathematical formalisms in theoretical physics? Above Einstein's words lead to suspect that there exist some electromagnetic phenomena—of the interaction of radiation with matter-which cannot be explained by means of continuous mathematics; hence, certain basic physical notions are discrete (quanta of energy). In addition, Einstein suggests, although with humility (previously he used the term "*conceivable*"; now he starts a sentence with the words "It *seems* to me…" and adds "… are *more readily* understood…"; emphasis added) an alternative theoretical view-i.e. a discontinuous electromagnetism—as capable to interpret some new phenomena:

> It seems to me that the observations associated with blackbody radiation, fluorescence, the production of cathode rays by ultraviolet light, and other related phenomena connected with the emission or transformation of light are more readily understood if one assumes that the energy of light is discontinuously distributed in space. In accordance with the assumption to be considered here, the energy of a light ray spreading out from a point source is not continuously distributed over an increasing space but consists of a finite number of energy quanta which are localized at points in space, which move without dividing, and which can only be produced and absorbed as complete units.

In Section 1 Einstein starts his argumentation by presenting an astonishing mathematical result about a physical situation where radiation, molecules and electrons mutually interact. The classical treatment according to continuous Maxwell's electromagnetism plus the equipartition theorem of the basically discrete statistical mechanics leads to a divergence in calculations, which constitutes in both experimental and theoretical terms an absurdity. Einstein took seriously this absurdity. Historian Klein lucidly remarked:

[32]Some historians recall that according to Boltzmann a theoretical physicist has to be able to reduce all mathematical notions to the corresponding discrete notions see (Boltzmann, 1874; Dugas, 1959; for instance, let us recall that the method of finite elements for solving Poisson's differential equation came from this idea). On this point (Kuhn, 1978: Ch. II, fn. 43) is oddly careless. But this idea is not the same as Einstein's, who considered the difference in the kind of mathematics as a very dichotomy (to be decided in agreement with the specific kind of phenomena to be dealt with).

> The "ultraviolet catastrophe, recognised and presented as a failure of classical physics, actually made its first appearance in the paper of Einstein's" (Klein, 1963: p. 64).
> [It implied a] Dualism… [which] was probably noticed by others besides Einstein, but there is no record that anyone else suggested removing it in the drastic way that Einstein then proposed (I am not even aware that anyone else was disturbed by the dualism at that time, and yet it was already a major theme in Einstein's own work) (Klein, 1963: p. 62)[33].

Indeed, the relationship between mathematics and physics was ignored by previous theoretical physicist, apart from Mach and Poincaré. More in general, Einstein gave much attention to the foundations of the physical theories. In 1924 he wrote: "From the early years all my scientific efforts have been directed to deepening the foundations of physics" (Quoted from Klein, 1980: p. 182).

The above mentioned historian remarked also that since then

> [The] Existence of this distinction [in the kind of mathematics]… marked a fundamental inhomogeneity in the foundations of physics (Klein, 1970: p. 241). …[or, rather,] a dichotomy in the foundations of physics… (Klein, 1980: p. 167).

Notice that the word "dichotomy" suggests, beyond the existence of discrete quanta of light, an unavoidable dilemma between two different attitudes in the mathematical description of nature. In other words, Einstein showed that *in the entire theoretical physics there exists a dichotomic option concerning which kind of mathematics a theorist building a theory has to follow*.

14. The Formal Qualification of Einstein's Choice of the Kind of Mathematics

Einstein's paper referred to the two mathematical alternatives by means of notions which are similar to those of ancient Greeks, i.e. the continuous and the discrete. Not before sixty years after Einstein's paper, the alternatives were accurately formalized in Mathematics by means of the different technical tools pertaining to the two different kinds of infinity—i.e. actual and potential. On the one hand, there exists the classical continuous mathematics, which makes—more or less freely-use of actual infinity in defining both mathematical objects and mathematical techniques[34]; on the other hand, there exists the constructive mathematics, which defines any mathematical notion by means of a finite algorithm relying on the discrete, hence by means of potential infinity only[35] (Bishop, 1967; pp. 1-10). Under the light of this recent view on mathematics foundations, Einstein's dichotomy has to be considered as opposing two entire conceptions of Mathematics.

The resulting technical differences between the two kinds of Mathematics are more subtle than the old differences between the continuous and the discrete. Whereas the ancient Greeks avoided to use the number π as incommensurable with the discrete, it pertains to constructive mathematics since it is defined by an approximation process. Moreover, the albeit restricted techniques of constructive mathematics obtain continuous functions (for instance, the analytical functions), Taylor's and Fourier's expansions, integral operation on uniformly continuous functions, differential equations and most of their solutions; in sum, they include so many notions to cover almost the entire practice of the usual mathematics in theoretical physics. However, since the set of algorithms of approximation is denumerable, the numbers which are definable in such mathematics are no more than a denumerable infinity, i.e. one order of infinity lesser than that of the classical real number. Moreover constructive mathematics is unable to decide in general the single points or the global properties, since its limit operation is no more than an approximation process, which does not therefore necessarily achieve a sharp result; e.g. a single real number.

These subtle differences between constructive mathematics and classical mathematics are relevant for the foundations of a theory[36]. Constructive mathematics may reject an entire formulation of a physical theory since its basic notions and/or the principles are formalized in a non-constructive way; for instance, Carathéodory's

[33]In fact, already in 1900 Rayleigh had achieved the same result, but he had suggested eliminating it by means of a theoretical trick (Rayleigh, 1900).

[34]Notice that Cauchy's reform did not rejected the actual infinity from calculus. Already Du Bois Raymond remarked that actual infinity plays an essential role in Cauchy's definition of a limit; only by appealing to it one obtains a single point from a sequence of intervals (Kogbetlianz, 1968: App. 2).

[35]For a plain introduction, see (Drago & De Martino, 2002).

[36]In the past it was questioned whether the constructive mathematics is relevant to theoretical physics. But the question was put in a misleading way, by inquiring whether constructive mathematics can obtain or not an advanced result of present theoretical physics. See for instance in *J. Phil Logic*, years 1993-1997, the discussion among Hellman, Bridges and Billinge.

thermodynamics is rejected since its second principle requires to be capable to manage all the points in any neighbours of a single point; as the same author advised, this task is physically impossible (Drago, 1986; Drago, 1991c).

Let us inquire whether Einstein's theory on quanta makes use of mathematical notions or tools which do not belong to constructive mathematics. The calculations of Sections 1, 3, 4, 6 include derivatives and integrals; since these operations concern uniformly continuous functions, they have their counter-parts in constructive mathematics (Bishop, 1967; pp. 133,190). But in Section 4 the mathematical technique aimed at obtaining a maximum on a set of functions, is not constructive; indeed, the exact equality of the values of two variables cannot be achieved constructively; even less the exact equality between two functions (i.e. two totalities of infinite values) cannot. However, one may obviate this difficulty by offering a substitutive argument of a thermodynamics nature; in point of fact, already it was suggested by Pais, who was motivated by simplicity reasons. (Pais, 1979: p. 873 col. I)[37].

In conclusion, *provided that one substitutes according to Pais the technique for obtaining the function* $S = S(V)$ *for radiation, Einstein, although unaware of the constructive mathematics, conformed his calculations to this kind of mathematics.*

15. Qualification of the Revolutionary Role Played by Einstein's Paper in the History of Theoretical Physics

At the end of the 19th century some physical theories presented so radical differences to suspect contradictions among their formalisms. Almost alone among the physicists of his time, Einstein faced this problem by choosing an original strategy. Rather than to establish between a couple of them either a contradictory relationship (involving the rejection of one of these theory), or an inclusive relationship (as Maxwell successfully did when he included geometrical optics in electromagnetism), he looked for a relationship of "reconciliation"[38].

In fact, his paper on special relativity reconciled mechanics with electromagnetism. He replaced the old invariance group of Newton's mechanics by Lorenz' group, discovered in the electromagnetic theory. Also the paper on quanta reconciled theories—i.e. electromagnetism, thermodynamics and statistical mechanics—by changing the mathematics of the former theory[39]. As first move, Einstein took in account the "profound formal distinction" in the kind of mathematics presented by the various physical theories, i.e. he considered it as a possible option. In the "Introduction" to his paper he stated the problem, i.e. which is the suitable mathematics for electromagnetism. Then, from the current theoretical physics he derived a mathematical formula which is in contradiction with experiments (ultraviolet catastrophe). In this way his seemingly philosophical problem—i.e. to reconcile three physical theories—was translated into an unprecedented problem of theoretical physics, how to construct a new electromagnetic theory relying on discrete mathematics. The following Sections of the paper solved the problem in a "heuristic" way.

A posteriori we recognize that, having to solve an apparently intractable problem, he had to construct a theory of a PO kind. In the above we saw that in fact his presentation conforms to the PO model, although he reasoned through DNPs in an implicit way only, composed only one *ad absurdum* proof and concluded at the minimum level (an analogy).

All in all, Einstein's paper on quanta manifests in a basically clear way the above two basic choices. *Since they are the alternative ones to Newton's choices, the main characteristic feature of Einstein's paper was to represent a very "revolution"*. Already two studies on this paper by the historian Klein intuitively stressed this revolutionary nature of Einstein's paper:

> Einstein was well aware that all of this marked the beginning of a new era in physics, and he indicated that awareness by referring to his work in the title of his paper as offering "a heuristic viewpoint". He saw that thoroughgoing changes in the foundations were needed…" (Klein, 1961: p. 477).
>
> [Indeed,] His deepest concern, expressed in the opening sentences of his paper, was the very foundation of science (Klein, 1979: p. 135).

[37]He suggests a typical textbook computation for obtaining $S = S(V)$ for a perfect gas; the employed differential operations on uniformly continuous functions are all constructive.

[38]Einstein made use of this word in the 1905 paper on special relativity, when he stated that the two theories at issue, Newton mechanics and Maxwell's electrodynamics, were "not irreconciliable" (Einstein, 1905b: p. 891) and also in the next paper on quanta (Einstein, 1906: p. 199).

[39]Previous Planck's strategy was at all different (Drago, 2010b; Drago, 2010c).

16. Some Historical Considerations

Previous analysis suggests four historical considerations.

A first historical consideration concerns the controversial question of who discovered quanta. Planck suggested a merely local and instrumental change in the kind of mathematics without evaluating its import for theoretical physics. Hence, in the first years of the Century he had no awareness of what differences the quanta implied in the basic choices. Moreover, until the 20's he had not lost hope of explaining the black body behaviour within a classical framework. His greatest merit is rather to have qualified a different foundational issue, i.e. the universal constants and the related system of absolute units. But they play no more than a preliminary role to the new theoretical physics. Instead, Einstein was almost fully aware of the foundational change that the theoretical physics of his time needed to undergo.

Under the light of this foundational viewpoint, the priority in the discovery of quanta has to be attributed to Einstein, although it is necessary to recognize that Planck suggested substantial improvements in the classical theoretical framework. In this respect one can reiterate Mendeleev's words:

> An Englishman, named Mayow, who lived a whole century before Lavoisier (in 1666), understood certain phenomena of oxidation in their true aspect, but was not able to develop his views with clearness, or support them by conclusive experiments; he cannot therefore be considered, like Lavoisier, as the founder of contemporary chemical learning. Science is a universal heritage, and therefore it is only just to give the highest honour in science, not to those who first enunciate a certain truth, but to those who are first able to convince others of its authenticity and establish it for the general welfare (Mendeleev, 1905: pp. 17-18).

In conclusion, on this question I agree with Kuhn's suggestion of attributing the priority of Einstein, but I suggest more stringent arguments, those pertaining to the foundations of physics.

As a second consideration let us remark that before the births of Einstein's two revolutionary theories–on quanta and special relativity-the two alternative choices in the foundations of a physical theory were either ignored or obscured as specific variants of well-known theories; e.g., Lagrange's mechanics, actually a PO theory, was commonly considered a mathematical enlargement of Newton's deductive mechanics[40]; or S. Carnot's thermodynamics, a clear PO theory, was considered as a phenomenological or Baconian theory. But around the year 1900 new experimental phenomena (Michelson's disproof of ether, the failure of Wien's law, etc.) and some years later, the mathematical result of the ultraviolet catastrophe showed that the traditional notions (absolute space and time, forces and continuous light, etc.) were manifestly inadequate to interpret the new phenomena. Hence theoretical physicists have to enlarge their theoretical views[41]. The two fundamental aspects stressed by Einstein's paper make clear the nature of the crisis occurring in theoretical physics around the year 1900.

As a third consideration let us now compare the fundamental choices of Einstein's theory on quanta with those of the previous physical theories. By the year 1905, three theoretical physicists (L. Carnot, S. Carnot and the same Einstein) had already founded their theories (respectively mechanics, thermodynamics and special relativity) on the same two alternative choices; but the foundation of each of these theories was unsatisfactory. Although the first one, L. Carnot's mechanics, stated a well-defined problem and moreover consistently made use of an elementary mathematics (which is included in constructive mathematics), it made a confused use of non-classical logic, so that even the mathematical development has to be interpreted and reconstructed (Drago, Manno & Mauriello, 2001; Drago, 2004; Bellini, Drago, & Mauriello, 2007). The second one, S. Carnot's thermodynamics, presented a very clear PO organisation; it was based upon a well-defined problem and, moreover, was consistently developed through a sequence of *ad absurdum* proofs composed by DNPs (by the way, S. Carnot was the most perspicuous of all past physicists in reasoning in such a way; Drago & Pisano, 2000); this theory is also consistent in the use of elementary (and hence, constructive) mathematics (although in a long footnote, which however was not essential to his final result, he applied calculus in a non-constructive way). (Drago & Pisano, 2005). However, the theory was partly invalidated by its use of a subsequently discarded hypothesis (caloric).

The third theory, special relativity, was presented by a paper whose development conforms to a PO model in the first pages only; the same holds true of its use of constructive mathematics (Di Matteo & Drago, 2006; Di Matteo & Drago, 2007; Drago, 2010a). Hence, in the history of physics Einstein's paper introducing quanta

[40]Even one of the profound critics of classical mechanics, E. Mach, shared this opinion (Mach, 1907: Chap. IV, Sect. III). About Lagrange' mechanics as a PO theory see (Capecchi & Drago, 2005; Drago, 2009).

[41]For a general view of this turning point see (Drago, 2013).

presents the first valid theory which was founded in a substantially adequate way on both the alternative choices.

As a fourth historical consideration I remark that the crisis of the first years of the 20th Century in the foundations of physics was so deep that theoretical physicists were lacking of the theoretical tools for solving the fundamental problems in the direction of the theories to be constructed. Indeed, they needed not only some new notions, or new techniques (groups of transformations for founding special relativity), but also the alternative choices to non-Newtonian ones; or, even more in general, the formalizations of the two dichotomic options. In the above sections we saw that very early Einstein took into account the two dichotomies when was successful in founding his theory by virtue of his great ingenuity, which even at the present time seems to someone "miraculous"[42]. But since the alternative choices have been formalised not before some decades later in constructive mathematics and intuitionist logic, he lacked the suitable formal tools for answering the problems in a direct way.

No surprise if, after his "revolutionary" theory Einstein did not persist in manifesting the foundational choices. About his unusual way of arguing in his paper Dorling wrote: "Such form of argument [analogy]… does not as far I am aware occur in any of his other major writings" (Dorling, 1971: p. 6). He did not found his further theory (general relativity) in an operative way nor did he devote attention to discrete mathematics or constructive mathematics, although in the 20's some mathematicians-among which H. Weyl, who was well-known to Einstein-began to define it.

References

Aristotle (350 B.C.). *Topics*.

Arons, A. B., & Peppard, M. B. (1965). Einstein's Proposal of the Photon Concept—A Translation of the *Annalen der Physik* Paper of 1905. *American Journal of Physics, 33,* 367-374. http://dx.doi.org/10.1119/1.1971542

Avogadro, A. (1811). Essai d'une manière de déterminer les masses relatives des molecules élémentaires des corps, et les proportions selon lesquelles elles entrent dans ces combinaisons. *Journal de Physique, Chimie et Histoire Naturelle,* 58-76.

Bellini, E., Drago, A., & Mauriello, G. (2007). Ricostruzione della meccanica di Lazare Carnot come alternativa fondazionale alla meccanica newtoniana. In M. Leone, B. Preziosi, & N. Robotti (Eds.), *L'eredità di Fermi, Majorana ed altri temi. Atti del XXIV Congresso della Società Italiana della Storia della Fisica e dell'Astronomia* (pp. 113-118). Napoli: Bibliopolis.

Beth, E. W. (1950). Fundamental Features of Contemporary Theory of Science. *British Journal for the Philosophy of Science*, *1,* 89-102.

Beth, E. W. (1959). *Foundations of Mathematics*. New York: Harper.

Bishop, E. (1967). *Foundations of Constructive Mathematics* (pp. 1-10). New York: McGraw-Hill. (Manifesto, in Idem)

Birkhoff, G., & von Neumann, J. (1936). The Logic of Quantum Mechanics. *The Annals of Mathematics, 37,* 823-843. http://dx.doi.org/10.2307/1968621

Boltzmann, L. (1874). *Theoretical Physics and Philosophical Problems*. Boston, MA: Reidel.

Boorse, H. A., & Motz, L. (Eds.) (1966). *The World of the Atom*. New York: Basic Book.

Brouwer, L. E. J. (1908). The Unreliability of the Logical Principles. In A. Heyting (Ed.), *Collected Works* (pp. 107-111). Amsterdam: North-Holland.

Brown, J. (2005). *Physical Relativity. Space-Time Structure from a Dynamical Perspective*. Oxford: Oxford U.P.

Brush, S. (1983). *Statistical Physics and the Atomic Theory from Boyle and Newton to Landau and Onsager*. Princeton, NJ: Princeton U.P.

Bub, J. (2005). Quantum Mechanics Is about Quantum Information. *Foundations of Physics, 35*, 541-560. http://dx.doi.org/10.1007/s10701-004-2010-x

Cambiano, G. (1967). Il metodo ipotetico e le origini della sistemazione euclidea della geometria. *Rivista di Filosofia, 58,* 115-149.

Capecchi, D., & Drago, A. (2005). On Lagrange's History of Mechanics. *Meccanica, 40,* 19-33. http://dx.doi.org/10.1007/s11012-004-2198-z

Carnot, L. (1783). *Essai sur le Machines en Général*. Dijon: Defay.

Carnot, L. (1803a). *Principes fondamentaux de l'équilibre et du mouvement*. Paris: Deterville.

[42]See the title of (Norton, 2006).

Carnot, L. (1803b). *Géométrie de Position*. Paris: Duprat.

Carnot, L. (1813). *Réflexions sur la métaphysique du calcul infinitésimal*. Paris: Courcier.

Carnot, S. (1978). *Réflexions sur la puissance motrice du feu (1824)*. Edition critique par Fox, Robert. Paris: Vrin.

Clifton, R., Bub, J., & Halvorson, H. (2003). Characterising Quantum Thoery in Terms of Information Theoretic Constraints. *Foundations of Physics, 33,* 1561-1591. http://dx.doi.org/10.1023/A:1026056716397

Coppola, A., & Drago, A. (1984). Galilei ha inventato il princìpio d'inerzia. Proposta di una nuova interpretazione della sua opera. In S. D'Agostino, & S. Petruccioli (Eds.), *Atti V Congresso Nazionale di Storia della Fisica* (pp. 319-327). Roma: Accademia dei XL.

D'Alembert, J. le R. (1754). Elémens. In J. le R. D'Alembert, & D. Diderot (Eds.), *Encyclopédie Française* (Vol. 17, p. 501). A Livourne: de l'Imprimerie des éditeurs.

Di Matteo, A. L., & Drago, A. (2006). La nascita della relatività ristretta. Poincaré non ha preceduto Einstein. *Atti della Fondazione Giorgio Ronchi, 61,* 97-113.

Di Matteo, A. L., & Drago, A. (2007). La nascita della relatività è da attribuire ad Einstein, non a Poincaré. In M. Leone, B. Preziosi, & N. Robotti (Eds.), *L'eredità di Fermi, Majorana ed altri temi. Atti del XXIV Congresso della Società Italiana della Storia della Fisica e dell'Astronomia* (pp. 155-159). Napoli: Bibliopolis.

Dorling, J. (1971). Einstein's Introduction to Photons: Argument by Analogy or Deduction from Phenomena? *British Journal for the Philosophy of Science, 22,* 1-8. http://dx.doi.org/10.1093/bjps/22.1.1

Drago, A. (1986). Relevance of Constructive Mathematics to Theoretical Physics. In E. Agazzi et al. (Eds.), *Logica e Filosofia della Scienza, oggi* (Vol. II, pp. 267-272). Bologna: CLUEB.

Drago, A. (1990). I quattro modelli della realtà fisica. *Epistemologia, 13,* 303-324.

Drago, A. (1991a). Alle origini della meccanica quantistica: Le sue opzioni fondamentali. In G. Cattaneo, & A. Rossi (Eds.), *I fondamenti della meccanica quantistica. Analisi storica e problemi aperti* (pp. 59-79). Cosenza: Editel.

Drago, A. (1991b). *Le due opzioni*. Molfetta, BA: La Meridiana.

Drago, A. (1991c). The Alternative Content of Thermodynamics: Constructive Mathematics and Problematic Organization of the Theory. In K. Martinas, L. Ropolyi, & P. Szegedi (Eds.), *Thermodynamics: History and Philosophy. Facts, Trend, Debates* (pp. 329-345). Singapore: World Scientific.

Drago, A. (1994). The Modern Fulfillment of Leibniz' Program for a *Scientia generalis*. In H. Breger (Ed.), *VI Internationale Kongres: Leibniz und Europa* (pp. 185-195). Hannover, and also in (1994). Leibniz's *Scientia Generalis* Reinterpreted and Accomplished by Means of Modern Scientific Theories. In C. Cellucci et al. (Eds.), *Logica e Filosofia della Scienza. Problemi e Prospettive* (pp. 35-54). Pisa: ETS.

Drago, A. (2002). The Introduction to Non-Euclidean Geometries by Bolyai through an Arguing of Non-Classical Logic. *Communication to International Conference on Bolyai 2002*, Budapest.

Drago, A. (2004). A New Appraisal of Old Formulations of Mechanics. *American Journal of Physics, 72,* 407-409. http://dx.doi.org/10.1119/1.1625924

Drago, A. (2005). A.N. Kolmogoroff and the Relevance of the Double Negation Law in Science. In G. Sica (Ed.), *Essays on the Foundations of Mathematics and Logic* (pp. 57-81). Milano: Polimetrica.

Drago, A. (2007). There Exist Two Models of Organization of a Scientific Theory. *Atti della Fondazione Giorgio Ronchi, 62,* 839-856.

Drago, A. (2009). The Lagrange's Arguing in Méchanique Analytique. In G. Sacchi Landriani, & A. Giorgilli (Eds.), *Sfogliando la Méchanique Analytique. Giornata di Studio su Louis Lagrange* (pp. 193-214). Milano: LED.

Drago, A. (2010a). La teoria delle relatività di Einstein del 1905 esaminata secondo il modello di organizzazione basata su un problema. In E. Giannetto, G. Giannini, & M. Toscano (Eds.), *Relatività, Quanti, Caos e altre rivoluzioni della Fisica* (pp. 215-224). Rimini: Guaraldi.

Drago, A. (2010b). La storia della teoria del corpo nero: Le premesse fondazionali di Planck. In E. Giannetto, G. Giannini, & M. Toscano (Eds.), *Relatività, Quanti, Caos e altre rivoluzioni della Fisica, Atti del XXVII Congresso della Società Italiana della Storia della Fisica e dell'Astronomia* (pp. 253-260). Rimini: Guaraldi.

Drago, A. (2010c). Storia del corpo nero: La strategia effettiva di Planck rivelata dalle sue scelte fondamentali. *Atti della Fondazione Giorgio Ronchi, 65,* 619-634.

Drago, A. (2012). Pluralism in Logic: The Square of Opposition, Leibniz' Principle of Sufficient Reason and Markov's Principle. In J.-Y. Béziau, & D. Jacquette (Eds.), *Around and beyond the Square of Opposition* (pp. 175-189). Basel: Birkhaueser. http://dx.doi.org/10.1007/978-3-0348-0379-3_12

Drago, A. (2013). The Relationship between Physics and Mathematics in the XIXth Century: The Disregarded Birth of a

Foundational Pluralism. In E. Barbin, & R. Pisano (Eds.), *The Dialectic Relations between Physics and Mathematics in the XIXth Century* (pp. 159-179). Berlin: Springer. http://dx.doi.org/10.1007/978-94-007-5380-8_8

Drago, A., Manno, S. D., & Mauriello, A. (2001). Una presentazione concettuale della meccanica di Lazare Carnot. *Giornale di Fisica, 42,* 131-156.

Drago, A., & De Martino, P. (2002). Introduzione alla matematica costruttiva. *Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche (Napoli), 69,* 37-49.

Drago, A., & Oliva R. (1999). Atomism and the Reasoning by Non-Classical Logic. *HYLE, 5,* 43-55.

Drago, A., & Perno, A. (2004). La teoria geometrica delle parallele impostata coerentemente su un problema (I). *Periodico di Matematiche., ser. VIII, 4,* 41-52.

Drago, A., & Pisano, R. (2000). Interpretazione e ricostruzione delle *Réflexions* di Sadi Carnot mediante la logica non classica. *Giornale di Fisica, 41*, 195-215. [English Translation (2004). Interpretation and Reconstruction of *Réflexions* by Sadi Carnot through Non-Classical Logic. *Atti della Fondazione Giorgio Ronchi, 59,* 615-644. Abstract in *Bulletin of Symbolic Logic, 8,* 130- 131.]

Drago, A., & Pisano, R. (2005). La nota matematica nelle *Réflexions sur la Puissance Motrice du Feu* di Sadi Carnot: Interpretazione del calcolo col metodo sintetico. *Quaderni di Storia della Fisica, 13,* 37-57.

Drago, A., & Venezia, A. (2002). A Proposal for a New Approach to Quantum Logic. In C. Mataix, & A. Rivadulla (Eds.), *Fisica Cuantica y Realidad. Quantum Physics and Reality* (pp. 251-266). Madrid: Facultad de Filosofía, Universidad Complutense de Madrid.

Dugas, R. (1950). *Histoire de la Mécanique*. Neuchâtel: Griffon.

Dugas, R. (1959). *La théorie physique au sens de Boltzmann*. Neuchatel: Griffon.

Dummett, M. (1978). *Elements of Intuitionism*. Oxford: Claredon.

Einstein, A. (1905a). Ueber einen die Erzeugung der Verwandlung des Lichtes betreffenden heuristisch Gesichtpunkt. *Annalen der Physik, 17,* 132-148. Reprinted in J. Stachel (Ed.) (1989), *Collected Papers of Albert Einstein* (Vol. 2, 149-165). Princeton, NJ: Princeton U.P.

Einstein, A. (1905b). Zur Elektrodynamik bewegter Körper. *Annalen der Physik, 17,* 891-921. http://dx.doi.org/10.1002/andp.19053221004

Einstein, A. (1905c). Letter to Conrad Habicht, April 14th.

Einstein, A. (1906). Zur Theorie del Lichterzeugung und Lichtabsorption. *Annalen der Physik, 20,* 199-206 (199).

Einstein, A. (1934). *Mein Weltbild*. Amsterdam: Querido [English Translation: (1954) *Ideas and Opinions*. New York: Crown], 17.

Einstein, A. (1949). Autobiographical Notes. In P. A. Schilpp (Ed.), *Albert Einstein, Philosopher-Scientist*. The Library of Living Philosophers, Open Court, La Salle IL.

El'yashevich, M. A. (1979). Einstein's Part in the Development of Quantum Concepts. *Soviet Physics Uspekhi, 22,* 555-575 (556-563).

Esposito, G. (1997). "Teorie di principi" e "teorie costruttive" in Einstein: Una reinterpretazione. In P. Tucci (Ed.), *Atti XVI Conv. Naz. Storia della Fisica* (pp. 425-456). Milano. http://www.brera.unimi.it/sisfa/atti/atti1996.html

Flores, F. (1999). Einstein's Theory of Theories and Types of Theoretical Explanation. *International Studies in Philosophy of Science, 13,* 123-134. http://dx.doi.org/10.1080/02698599908573613

Frisch, M. (2006). Mechanics, Principles, and Lorentz' Cautious Realism. *Studies in History and Philosophy of Modern Physics, 36,* 659-679. http://dx.doi.org/10.1016/j.shpsb.2005.07.002

Gardiès, J.-L. (1991). *Le raisonnement par l'absurde*. Paris: PUF.

Gillispie, C. C. (1971). *Lazare Carnot Savant*. Princeton, NJ: Princeton U.P.

van Heijenoort, J. (1967). Gödel's Theorem. In *Encyclopedia of Philosophy*. London: Macmillan.

Heyting, A. (1960). *Intuitionism*. Amsterdam: North-Holland.

Hintikka, J. (1989). Is There Completeness in Mathematics After Gödel? *Philosophical Topics, 17,* 69-90. http://dx.doi.org/10.5840/philtopics19891724

Hintikka, J., Gruender, C. D., & Agazzi, E. (Eds.) (1981). *Pisa Conference on the History and Philosophy of Science*. Boston, MA: Reidel.

Horn, L. R. (1989). *The Natural History of Negation*. Chicago, IL: Chicago U.P.

Horn, L. R. (2002). The Logic of Logical Double Negation. In K. Yasuhiko (Ed.), *Proceedings of the Sophia Symposium on Negation* (pp. 79-112). Tokyo: Sophia University.

Jammer, M. (1966). *The Conceptual Development of Quantum Mechanics*. New York: McGraw-Hill.

Klein, M. J. (1961). Max Planck and the Beginning of Quantum Theory. *Archive for History of Exact Sciences, 1,* 459-479. http://dx.doi.org/10.1007/BF00327765

Klein, M. J. (1963). Einstein's First Paper on Quanta. *The Natural Philosopher, 2,* 59-86.

Klein, M. J. (1967). Thermodynamics in Einstein's Thought. *Science, 157,* 509-516. http://dx.doi.org/10.1126/science.157.3788.509

Klein, M. J. (1970). *Paul Ehrenfest. The Making of a Theoretical Physicist*. Amsterdam: North-Holland.

Klein, M. J. (1979). Einstein and the Development of Quantum Physics. In A. P. French (Ed.), *Einstein. A Centenary Volume* (pp. 133-151). Cambridge, MA: Harvard U.P.

Klein, M. J. (1980). No Firm Foundations: Einstein and the Early Quantum Theory. In H. Woolf (Ed.), *Some Strangeness in the Proportion. A Centennial Symposium to Celebrate the Achievements of Albert Einstein* (pp. 161-185). Reading, PA: Addison-Wesley.

Klein, M. J. (1982). Some Turns of Phrase in Einstein's Early Papers. In A. Shimony (Ed.), *Physics as Natural Philosophy* (pp. 364-373). Cambridge, MA: MIT Press.

Klevgard, P. A. (2008). *A Fresh Approach to Quantum Mechanics and Relativity*. Chula Vista, CA: Aventine P.

Kogbetliantz, K. G. (1968). *Fundamentals of Mathematics from an Advanced Point of View*, New York: Gordon and Breach.

Kolmogorov, A. N. (1924/1925). On the Principle "Tertium Non Datur". *Mat. Sb., 32,* 646-667. [English Translation in: van Heijenoorth, J. (Ed.) (1967). *From Frege to Gödel* (pp. 416-437). Cambridge, MA: Harvard U.P.]

Kreisel, G. (1989). Logic Aspects of Axiomatic Method. In H.-D. Ebbinghaus et al. (Eds.), *Logic Colloquium'87* (pp. 183-217). Amsterdam: North-Holland.

Kuhn, T. S. (1978). *Block-Body Theory and Quantum Discontinuity, 1894-1912* (Chap. VII). Oxford: Claredon.

Lakatos, I. (1978). A Renaissance of Empiricism in the Recent Philosophy of Mathematics. In J. Worral, & G. Currie (Eds.), *Philosophical Papers* (Vol. 2, pp. 24-42). Cambridge: Cambridge U.P.

Leibniz, G. W. (1686). Letter to Arnaud, July 14th. In L. Couturat (Ed.), *Opuscules et Fragments Inedits* (p. 402). Paris.

Lobachevsky, I. N. (1840). *Geometrische Untersuchungen zur Theorie der Parallelinien*. Berlin: Finkl. [English Translation as an Appendix to Bonola, R. (1950) *Non-Euclidean Geometry*. New York: Dover.]

Lukasiewicz, J. (1970). *Selected Works*. Amsterdam: North-Holland.

Mach, E. (1986). *The Principles of Heat*. Boston, MA: Reidel.

Mach, E. (1907). *The Science of Mechanics: A Critical and Historical Account of Its Development*. Chicago, IL: Open Court.

Markov, A. A. (1962). On Constructive Mathematics. *Trudy Mathematicieskie Institut Steklov,67,* 8-14. [English Translation in (1971) *American Mathematical Society Translations, 98,* 1-9.]

Mendeleev, D. I. (1905). *Principles of Chemistry*. New York: Longmans.

Miller, A. I. (1981). *Albert Einstein's Special Theory of Relativity*. Reading, MA: Addison-Wesley.

Millikan, R. A. (1916). A Direct Photoelectric Determination of Planck's "h". *Physical Review, 7,* 355-388. http://dx.doi.org/10.1103/PhysRev.7.355

Millikan, R. A. (1917). *The Electron*. Chicago, IL: Chicago U.P.

Newton, I. (1687). *Philosophiae Naturalis Principia Mathematica*. London: Streater J.

Newton, I. (1704). *Optiks*. London: Innys W.

Norton, J. D. (2006). Atoms, Entropy, Quanta: Einstein's Miraculous Argument of 1905. *Studies in History and Philosophy of Modern Physics, 37,* 71-100.

Pais, A. (1979). Einstein and Quantum Theory. *Reviews of Modern Physics, 41,* 863-913 (871-873).

Plato (360 B.C.). *Republic.* [English Translation: The Internet Classic Archive. http://classics.mit.edu/Plato/republic.html]

Poincaré, H. (1903). *La Science et l'Hypothèse*. Paris: Hermann.

Poincaré, H. (1905). *La Valeur de la Science*. Paris: Flammarion.

Prawitz, D. (1977). Meaning and Proofs. *Theoria, 43,* 6-39.

Prawitz, D., & Melmnaas, P. E. (1968). A Survey of Some Connections between Classical, Intuitionistic and Minimal Logic. In A. Schmidt, K. Schutte, & H. J. Thiele (Eds.), *Contributions to Mathematical Logic* (pp. 215-229). Amsterdam: North-Holland.

Rayleigh, J. W. S. (1900). Remarks upon the Law of Complete Radiation. *Philosophical Magazine, 49,* 539-540. http://dx.doi.org/10.1080/14786440009463878

Rosenfeld, L. (1936). La première phase de l'évolution de la théorie des quanta. *Osiris, 2,* 149-196. http://dx.doi.org/10.1086/368459

Rynasiewiecz, R., & Renn, J. (2006). The Turning Point of Einstein's *Annus Mirabilis*. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics, 37,* 5-35. http://dx.doi.org/10.1016/j.shpsb.2005.12.002

Sainati, V. (1990). La teoria aristotelica dell'apodissi. *Teoria, 10,* 3-47.

Seelig, C. (1956). *Albert Einstein. A Documentary Bibliography*. London: Staples.

Shapiro, A. E. (1984). Experiment and Mathematics in Newton's Theory of Color. *Physics Today, 37,* 34-42. http://dx.doi.org/10.1063/1.2916400

Stachel, J. (1989). Einstein Early Work on the Quantum Hypothesis. In J. Stachel (Ed.), *Collected Papers of Albert Einstein* (Vol. 2, pp. 134-148). Princeton, NJ: Princeton U.P.

Stachel, J. (2005). Introduction to the Centenary Edition. In J. Stachel (Ed.), *Einstein (2005) Einstein Miraculous Year*. Princeton, NJ: Princeton U.P.

Ter Haar, D. (1967). *The Old Quantum Theory*. New York: Pergamon P.

Wallace, W. A. (1984). *Galileo and His Sources. The Heritage of the Collegio Romano in Galileo's Science*. Princeton, NJ: Princeton U.P.

Appendix: The List of the Doubly Negated Propositions in Einstein's Paper

For facilitating the reader I refer to the English translation by (Arons & Peppard, 1965); however I checked the appropriateness of it by comparing the propositions quoted in the following with those of the original paper in German language. About the main characteristic features of the DNPs and the way to identify them in a text, see the paper (Drago, 2005). In the following, modal words are wave underlined and some words, such as in-de-pendent, are split with a hyphen in order to highlight the two negations. In square bracket are added the words of the corresponding affirmative proposition to a DNP. Tree points inside square brackets […] means a discontinuity in the quotation of the original text.

(***Introduction***) 1. […] While we consider the state of a body to be completely determined by the positions and velocities of a very large, yet finite, number of atoms and electrons, we make use of continuous spatial functions to describe the electromagnetic state of a given volume, and a finite number of parameters cannot be regarded [≠is] as sufficient for the complete determination of such a state.

2. According to the Maxwellian theory, energy is to be considered [≠is] a continuous spatial function in the case of all purely electromagnetic phenomena including light,

3. while the energy of a ponderable object should, according to the present conceptions of physicists, be [≠is] represented as a sum carried over [discrete objects, i.e.] the atoms and electrons.

4. The energy of a ponderable body cannot be subdivided [≠is elementary]

5. into arbitrarily many or arbitrarily small [=without bounds] [≠precise] parts,

6. while the energy of a beam of light from a point source (according to the Maxwellian theory of light or, more generally, according to any wave theory) is continuously spread over an ever increasing [without a bound] [≠precise] volume.

7. The wave theory of light, which operates with continuous spatial functions, has worked well in the representation of purely optical phenomena and will probably never be replaced by another theory [≠will be always true].

8. […] In spite of the complete experimental confirmation of the theory as applied to diffraction, reflection, refraction, dispersion, etc., it is still conceivable [≠is true] that

9. the theory of light which operates with continuous spatial functions may lead [≠leads] to contradictions with experience when it is applied to the phenomena of emission and transformation of light.

10. It seems to me [≠it is true] that

11. the observations associated with blackbody radiation, fluorescence, the production of cathode rays by ultraviolet light, and other related phenomena connected with the emission or transformation of light are more readily [=with less difficulty] [≠in fact] understood if one assumes that the energy of light is discontinuously distributed in space.

12. In accordance with the assumption to be considered here, the energy of a light ray spreading out from a point source is not continuously distributed [≠is not a discrete distribution] over an increasing space but consists of a finite number of energy quanta which are localized at points in space,

13. which move without dividing [≠as a whole],

14. and which can only be [≠is] produced and absorbed as complete units.

(***Section* 1**) 15. […] they can collide with each other like [≠equal to] molecules in the kinetic theory of gases.

16. (footnote 1) This assumption is equivalent [≠equal] to the suggestion that the average kinetic energies of gas molecules and electrons are equal to each other at thermal equilibrium.

17. […] According to the present view regarding the origin of light, the radiation in the space we are considering ([…]) must be [≠is] identical with the blackbody radiation—at least if oscillators of all the relevant frequencies are considered to be present.

18. […] The kinetic theory of gases asserts that the average kinetic energy of an oscillator electron must be [≠is] equal to the average kinetic energy of a translating gas molecule.

19. […] Therefore, in the case we are considering, dynamic equilibrium is possible only [= in no other case than] when [≠when] each oscillator has the average energy E.

20. […] Herr Planck has derived the condition for the dynamical equilibrium in this case under the supposition that the radiation can be considered [≠is] a completely random process.

21. (footnote 3) The radiation is then as disordered as conceivable if…

22. […] If the radiation energy of frequency v is not continually [≠discretely] increasing or decreasing,

23. the following relations must obtain [≠obtain] [formula].

24. These relations, found to be the conditions of thermodynamic equilibrium, not only fail to coincide with

experiment, but also state that in our model there can be [≠there is] no talk of a definite energy distribution between ether and matter…

(***Section* 2**) 25. […] We wish to show in the following that Herr Planck's determination of the fundamental constants is, to a certain extent, in-dependent [≠free] of his theory of blackbody radiation.

26. […] In the following we shall consider the experimental facts concerning blackbody radiation without invoking a [dubious] model [≠directly] for the emission and propagation of the radiation itself.

(***Section* 3**) 27. […] We assume that the observable properties of the radiation are completely determined when the radiation density $\rho(\nu)$ is given for all frequencies (footnote 3). This assumption is an arbitrary one as long as it is not controverted [≠confirmed] by experiment.

28. Since radiations of different frequencies are to be [≠are] considered

29. in-dependent [≠free] of each other when there is no transfer of heat or work,

30. the entropy of the radiation can be [≠is] represented by[formula]

31. Φ can be [≠is] reduced to a function of a single variable

32. through formulation of the condition that the entropy of the radiation is un-altered [≠constant] during adiabatic compression between reflecting walls.

33. […] In the case of blackbody radiation, therefore, $\partial\varphi/\partial\rho$ is in-dependent [≠free] of ν.

34 [...] Therefore one can derive [≠derives] the law of blackbody radiation from the function φ, and inversely, one can derive the function φ by integration keeping in mind the fact that φ vanishes when ρ=0.

(***Section* 4**) 36. […] We shall base our analysis on this formula, keeping in mind that our results are only [= in no other case] valid [≠valid] within certain limits.

37. […] This equation shows that entropy of a monochromatic radiation of a sufficient low density varies with the volume in the same manner as [≠equally to] the entropy of an ideal gas or a dilute solution.

(***Section* 5**) 38. […] In particular, "cases of equal probability' have frequently been hypothetically established [≠established]

39. when the theoretical models being utilized are definite enough to permit [≠so to give] a deduction rather than a conjecture.

40. I will show in a separate paper that the so-called "statistical probability" is fully adequate for the treatment of thermal phenomena, and I hope that by doing so I will eliminate a logical difficulty which obstructs the application of [≠apply the] Boltzmann's Principle.

41. […] If it is reasonable to speak [≠We speak] of the probability of the state of a system,

42. and furthermore if every entropy increase can be [≠is] understood as a transition to a state of higher probability, then the entropy S_1 of a system is a function of W_1 the probability of its instantaneous state.

43. If we have two noninteracting systems S_1 and S_2, we can write [≠write]…

44. […] The last equation says that the states of the two systems are in-dependent [≠free] of each other.

45. […] We consider a number (n) of movable [= which can be moved] [≠in motion] points (e.g., molecules) confined in a volume v_0.

46. Besides these points, there can be [≠is] in the space any number of other movable points of any kind….

47. …We shall not assume anything concerning the law in accordance with which the points move in this space except that with regard to [≠we shall assume] this motion,

48. no part of the space (and no direction within it) can distinguished from any other [≠is unite].

49. Further, we take the number of these movable points to be so small that we can disregard interactions between them [≠are interaction free].

50. […] Let us imagine transferring all n movable points into a volume v (part of the volume v_o) without anything else being changed in the system [≠all constant].

51. […] It is noteworthy that in the derivation of this equation… no assumption has to be [≠is] made as to a law of motion of molecules.

(***Section* 6**) 52. […] From this we further conclude that: Monochromatic radiation of low density (within the range of validity of Wien's radiation formula) behaves thermodynamically as thought [≠as] it consisted

53. of a number of in-dependent [≠free] energy quanta of magnitude $R\beta\nu/N$.

54. […] If the entropy of monochromatic radiation depends on volume as thought [≠as] the radiation were a discontinuous medium consisting of energy quanta of magnitude $R\beta\nu/N$,

55… the next obvious step is to investigate whether the laws of emission and the transformation of light are also of such a nature that they can be [≠are] interpreted as explained by considering light to consist of such energy quanta […].