Scientific Research

Double Stone Algebra's Ideal and Congruence Ideal

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Abstract: Rough set theory is a new mathematical tool to deal with vagueness and uncertainty data, which can analyse and deal with imprecise and incomplete information. Rough set theory algebraic analysis is the study of the most active branch. In [6], it gives an ideal and congruence ideal of stone algebra. So in this paper, underlying the ideals of double Stone algebra, we have given the ideal and congruence ideal of double Stone algebra.

Keyword: double stone algebra; ideals; congruence relation; congruence ideals

1. Background

Rough Set theory has been put ford by Poland mathematician Z. Pawlak in 1982. Eighties, many Poland scholars researched consistently in Rough Set theory and its application. Many study on Rough Set theory's mathematic attributions and logic systems. At the same time, they also developed some application systems. But because first study results has been published by Poland Language on "Bulletin of the Polish Academy of Sciences: Mathematics" or "Bulletin of the Polish Academy of Sciences: Technical Sciences" The study is limited to Eastern European countries. At that time it did not attract the attention of international computer industry. Until the late eighties, Rough set theory has gradually attracted the attention of scholars around the world.

Rough set theory is a new mathematical tool to deal with vagueness and uncertainty data, which can analyse and deals with imprecise and incomplete information. Since rough set theory was introduced by Polish mathematician Pawlak in the early 1980s, it has been greatly developed in both theory and application. It was successfully applied in the fields of machine learning, knowledge acquisition, decision analysis, knowledge discovery, expert systems, decision support systems, inductive reasoning and pattern recognition.

Both rough set theory and neural networks show advantages in dealing with various imprecise and incomplete knowledge. However, they are quite different. Neural networks often have complex structures when dimensions of input data are high while rough sets have a large advantage on decreasing redundancy among the input data. Rough set have a weak tolerance and generalization performance whereas neural networks have a better capability on anti-jamming performance, self-organizing and generalization.

Rough set theory, algebraic analysis is the study of the most active branch. In this paper, underlie the ideals of double Stone algebra, we have given the congruence ideal of double Stone algebra.

2. Basic Method

Definition1 A information table is a object set Q and mapping $f: Q \to R^n$ corresponds the set of the table's attribution. Here f leads a equivalent relation "$\approx$" on Q, $\omega \approx \omega' \Leftrightarrow f(\omega) = f(\omega')$. Let Λ is a division of this equivalent relation, and Q is a complete

Boolean algebra which is generated by Λ. Among the notion which set up by f, the element of Q sets an important role, For any sebset A of Q, we can define upper approximation and lower approximation:

$$\underline{R}X = \bigcup\{Y \in U/R \mid Y \subseteq X\}, \overline{R}X = \bigcup\{Y \in U/R \mid Y \cap X \neq \phi\}$$

We can also define them as

$$\underline{R}X = \{x \in U \mid [x]_R \subseteq X\}; \overline{R}X = \{x \in U \mid [x]_R \cap X \neq \varphi\}$$

In a rough algebra, let U is a filed, R is an equational relation on U, for $\forall X \subseteq U$, we use $\underline{R}X$ and $\overline{R}X$ respectively X 's upper approximation and lower approximation in approximation space (U, R). $[x]_R$ mines the equivalence class R which is decide by x. For $\forall X, Y \subseteq U$, if $\underline{R}X = \underline{R}Y$ and $\overline{R}X = \overline{R}Y$, then we can X, Y rough equivalent, denoted $X \approx Y$ apparently $\approx$ is an equational relation on $P(U)$, $\approx$'s equational relation is called rough set. All rough set Constitute a class called $\Re^0$, then $\Re^0 = \{[X]_\approx \mid X \subseteq U\}$, $\Re^0$ is a complete Stone Algebra.

The attributions of Stone algebra:

1) $a \wedge a^* = a \wedge a^{**} = 0$

2) $a \wedge b = 0 \Leftrightarrow a \le b^*$

3) $a \le a^{**}$

4) $a^{***} = a^*$

5) $a \wedge b = 0 \Leftrightarrow a \le b^*$

6) $a \le b \Rightarrow b^* \le a^*$

7) $(a \vee b)^* = a^* \wedge b^*$

8) $a^{**} \wedge b^{**} = (a \wedge b)^{**}$

9) $(a^{**} \vee b^{**})^{**} = (a \vee b)^{**} = a^{**} \vee b^{**}$

10) $(a \vee a^*)^* = 0$

Definition 2 Let $(L;\vee,\wedge,*,0,1)$ is abounded distributive pseudo-complemented lattice, if for $\forall a \in L$, $a^* \vee a^{**} = 1$, then L is a Stone algebras, $a^* \vee a^{**} = 1$ is called a Stone identity.

Lemma3 Let $\Re^0$ is a rough set class which is determined by the approximation space (U, R), then algebra $(L;\vee,\wedge,*,0,1)$ is a complete, atomic Stone Algebra.

Lemma4 Lattice $(L;\cap,\cup,*,0,1)$, here $0 = (\varphi,\varphi)$, $1 = (U,U)$ is a complete, atomic Stone Algebra. and isomorphism with $(L;\cap,\cup,*,0,1)$

Definition5 If a distributive lattice L is not only a Stone algebra, but also a dual Stone algebra, then L is a double Stone algebra。

Lemma6 Let $(L;\vee,\wedge,*,+,0,1)$ is a double Stone algebra, then

1) $(L;\vee,\wedge,0,1)$ is abounded distributive lattice;

2) $a \wedge x = 0$ if and only if $x \le a^*$;

3) $a \vee x = 1$ if and only if $x \ge a^+$;

4) $a^* \vee a^{**} = 1$;

5) $a^+ \wedge a^{++} = 0$。

Definition7 Let L is a double Stone algebra. $a \in L$, if $\exists b \in L$, brought $a \vee b = 1$, $a \wedge b = 0$ .Then we call a is a Boolean complement element, and b is called a's Boolean complement. The set of L 's Boolean complement elements called the center of, denoted cen(L).

Lemma8 Let L is a double Stone algebra. $a \in L$, then the conditions is equivalence: a is a Boolean complement element; a^* is a's is a Boolean complement; $a \vee a^* = 1$; $a = a^{**}$; $a^* = a^+$.

Proof: we can fin the proof $(1) \Rightarrow (2), (2) \Rightarrow (3)$, $(3) \Rightarrow (4)$ in [6] $(4) \Rightarrow (5)$

$$a^* \vee (a^+ \wedge a) = (a^* \vee a^+) \wedge (a^* \vee a) = a^* \vee a^+ = a^*$$

$$\therefore a^* \ge a^+;$$

$$a^+ \vee (a^* \wedge a) = (a^* \vee a^+) \wedge (a^+ \vee a) = a^* \vee a^+ = a^+$$

$$\therefore \text{V 故 } a^* = a^+.$$

$$(5) \Rightarrow (1) \quad a^* = a^+,$$

$$a \vee a^* = a \vee a^+ = 1 ,$$

$$a \wedge a^* = a \wedge a^+ = 0,$$

So a is a Boolean complement element

Lemma A type $(2,2,1,0,0)$ of algebra V is a dual distributive pseudo-complemented lattice's necessary and sufficient condition is it satisfies:

1) $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice

2) $x \vee (x \vee y)^+ = x \vee y^+$

3) $x \vee 1^+ = x$

4) $1^{++} = 1$

Proof: necessary condition any dual distributive pseudo-complemented lattice satisfies (1), (3), (4), so we need to proof (2), Because of $(x \vee y) \vee (x \vee y)^+ = 1$, namely $y \vee (x \vee (x \vee y)^+) = 1$, thus $x \vee (x \vee y)^+ \ge x \vee y^+$. Other side $(x \vee y^+) \vee (x \vee y) = 1$, then $(x \vee y^+) \ge (x \vee y)$, so $x \vee (x \vee y)^+ = x \vee y^+$ .sufficient condition: Let $(L;\vee,\wedge,+,0,1)$ is a bounded distributive lattice which satisfies: then we will proof "+" is a dual pseudo-complemented of L, which it satisfies: $a \vee x = 1$ if and only if $x \ge a^+$.For each $a \in L$, on one side, $a \vee a^+ = a \vee (a \vee 1^+)^+ = a \vee (1^+)^+ = a \vee 1 = 1$;Another side, if $a \vee b = 1$, then $b \vee a^+ = b \vee (b \vee a)^+ = b \vee 1^+ = b$, so $b \ge a^+$.

3. Congruence

Definition 9 Let L is a double Stone algebra, $I = (d]$ is a lattice ideal of L, define $a^0 = d \wedge a^*$, $a' = d \wedge a^+$ then "o", "′"are called ideal of I 's induced pseudo-complemented and induced dual pseudo-complemented.

Lemma10 Let L is a double Stone algebra, induced dual pseudo-complemented of ideal $I = (d]$ "′"satisfies: $x' \wedge x'' = 0$ $(\forall x \in I)$ necessary and sufficient condition is d satisfies $d \wedge d^+ = 0$。

Inference 11 Let L is a double Stone algebra, $I = (d]$ is a lattice ideal of L, the induced dual pseudo-complemented "′"satisfies $x' \wedge x'' = 0$ $(\forall x \in I)$ necessary and sufficient condition is $d \in cen(L)$.

Definition12 Let L is a double Stone algebra, if the induced dual pseudo-complemented "′"of lattice ideal

$I = (d]$ satisfies $x' \wedge x'' = 0\ (\forall x \in I)$, then I is a Double Stone Algebra's Idea of L.

Inference 13 $I = (d]$ is a Double Stone Algebra's necessary and sufficient condition is $d \in cen(L)$.

Inference 14 If $(a],(b]$ are all Double Stone Algebra's Idea of L, then $(a]\wedge(b] = (a \wedge b], (a]\vee(b] = (a \vee b]\ (a^*], (a^+], (b^*], (b^+]$, are all Double Stone Algebra's Idea of L.

The proof we can use the definition of main ideal $(a] = \{x \mid x \in L, x \le a\}, (b] = \{y \mid y \in L, y \le b\}$ .Definition1 5 A congruence relation θ of double Stone algebra L which satisfies the equivalence relation about the replacemental attributions about $\vee, \wedge, *, +$, the lattice congruence relation of L which is the equivalence relation about the replacemental attributions about $\vee, \wedge$.And when it satisfies $a \equiv b(\theta)$, then $a^* \equiv b^*(\theta)$, $a^+ \equiv b^+(\theta)$. We use the set $C(L)$ means, All congruence relation on L.

Definition16 Let $\theta \in C(L)$, the congruence class which the element O in called the core of θ, denoted $\ker(\theta)$, so $\ker(\theta) = \{x \in L \mid x \equiv 0(\theta)\} \ker(\theta)$.

Definition17 About the idea I of a Double Stone Algebra, if there exist a congruence relation θ, its core is I, then I called congruence idea.

First we should discuss the conditions a Double Stone Algebra's idea $I = (d]$ is a congruence idea.

Lemma18 Let L is a double Stone algebra, $d \in L$, define a binary relation θ on L:

$$x \equiv y(\theta) \Leftrightarrow x \vee d = y \vee d \text{, then}$$

1） θ is a lattice congruence relation on L

2） $(d]$ is the core's smallest congruence on θ

3） θ is a congruence relation on L necessary and sufficient condition is d is a Boolean complement element.

Proof: (1) clearly.

We can find the proof of (2) in [6].

3) sufficient condition: if d is a Boolean complement element, next proof θ keep the replacement attribution of *，+.

If $a \equiv b(\theta)$, then $a \vee d = b \vee d$, thus

$$a^* \wedge d^* = b^* \wedge d^*, a^+ \wedge d^+ = b^+ \wedge d^+ \text{, then}$$

$$a^* \vee d = (a^* \wedge d^*) \vee d = (b^* \wedge d^*) \vee d = b^* \wedge d$$

$$a^+ \vee d = (a^+ \wedge d^+) \vee d = (b^+ \wedge d^+) \vee d = b^+ \wedge d$$

so $a^* \equiv b^*(\theta)$, $a^+ \equiv b^+(\theta)$. necessary condition: if θ is a congruence relation on L

$$\because d \vee d = 0 \vee d \therefore d \equiv 0(\theta)$$

$$\text{Then } d^* \equiv 1(\theta) \text{, so}$$

$$d \vee d^* \equiv 0 \vee 1 = 1(\theta),$$

$$\therefore (d \vee d^*) \vee d = 1 \vee d = 1 \text{ then}$$

$$d \vee d^* = 1 \text{，则} d \wedge d^* = 0.$$

Similarly provable $d \vee d^+ = 1$, $d \wedge d^+ = 0$.

Thus d is a Boolean complement element.

Therewith we will give the necessary and sufficient condition of congruence idea is a Double Stone Algebra's idea $I = (d]$.

Theorem19 A Double Stone Algebra's idea $I = (d]$ is a congruence idea's necessary and sufficient condition is d is a Boolean complement element.

Proof: sufficient condition: if d is a Boolean complement element, then θ is a congruence relation and $x \equiv y(\theta) \Leftrightarrow x \vee d = y \vee d$ is the core ($(d]$)'s smallest congruence on θ, thus, $(d]$ is a congruence relation. necessary condition: if $(d]$ is a congruence idea, then $(d]$ is a core of $\theta \in C(L)$, this is $d \equiv 0(\theta)$, thus $d^* \equiv 0^* = 1(\theta)$, （$d^+ \equiv 0^+ = 1(\theta)$）, $d \vee d^* = 1 (d \vee d^+ = 1)$, $d \wedge d^* = 0 (d \wedge d^+ = 0)$. d is a Boolean complement element.

Lemma20 If $(a],(b]$ are all congruence ideas of a Double Stone Algebra, then $(a]\wedge(b] = (a \wedge b]$, $(a]\vee(b] = (a \vee b]$, are all congruence ideas of a Double Stone Algebra L.

The proof we also need to use the definition of main idea.

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