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Analytical Solution of Two Extended Model Equations for Shallow Water Waves by He's Variational Iteration Method

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Abstract

In this paper, we consider two extended model equations for shallow water waves. We use He's variational iteration method (VIM) to solve them. It is proved that this method is a very good tool for shallow water wave equations and the obtained solutions are shown graphically.

Keywords: He's Variational Iteration Method, Shallow Water Wave Equation

1. Introduction

Clarkson *et al.* [1] investigated the generalized short water wave (GSWW) equation

$$u_t - u_{xxt} - \alpha u u_t - \beta u_x \int^x u_t \mathrm{d}x + u_x = 0, \quad (1)$$

where α and β are non-zero constants.

Ablowitz *et al.* [2] studied the specific case $\alpha = 4$ and $\beta = 2$ where Equation (1) is reduced to

$$u_t - u_{xxt} - 4u u_t - 2u_x \int^x u_t \mathrm{d}x + u_x = 0, \quad (2)$$

This equation was introduced as a model equation which reduces to the KdV equation in the long small amplitude limit [2,3]. However, Hirota *et al.* [3] examined the model equation for shallow water waves

$$u_t - u_{xxt} - 3u u_t - 3u_x \int^x u_t \mathrm{d}x + u_x = 0, \quad (3)$$

obtained by substituting $\alpha = \beta = 3$ in (1).

Equation (2) can be transformed to the bilinear forms

$$\left[D_x \left(D_t - D_t D_x^2 + D_x \right) + \frac{1}{3} D_t \left(D_s + D_x^3 \right) \right] f \cdot f = 0, \quad (4)$$

where s is an auxiliary variable, and f satisfies the bilinear equation

$$D_x \left(D_s + D_x^3 \right) f \cdot f = 0, \quad (5)$$

However, Equation (3) can be transformed to the bilinear form

$$D_x \left(D_t - D_t D_x^2 + D_x \right) f \cdot f = 0, \quad (6)$$

where the solution of the equation is

$$u(x,t) = 2(\ln f)_{xx}, \quad (7)$$

where $f(x, t)$ is given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \quad (8)$$

where ε is a bookkeeping non-small parameter, and $f_n(x,t)$, n = 1, 2, ⋯ are unknown functions that will be determined by substituting the last equation into the bilinear form and solving the resulting equations by equating different powers of ε to zero.

The customary definition of the Hirota's bilinear operators are given by

$$D_t^n D_x^m a.b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m a(x,t) \, b(x',t') \,|\, x' = x, t' = t. \quad (9)$$

Some of the properties of the D-operators are as follows

$$\frac{D_t^2 f \div f}{f^2} = \iint u_{tt} \mathrm{d}x \mathrm{d}x, \quad \frac{D_t D_x^3 f \cdot f}{f^2} = u_{xt} + 3u \int x u_t \mathrm{d}x',$$
$$\frac{D_x^2 f \cdot f}{f^2} = u, \quad \frac{D_x^4 f \cdot f}{f^2} = u_{2x} + 3u^2, \quad (10)$$
$$\frac{D_t D_x f \cdot f}{f^2} = \ln\left(f^2\right)_{xt}, \quad \frac{D_x^6 f \cdot f}{f^2} = u_{4x} + 15 u u_{2x} + 15u^3,$$

where

$$u(x,t) = 2\left(\ln f(x,t)\right)_{xx}, \quad (11)$$

Also extended model of Equation (2) is obtained by the operator D_x^4 to the bilinear forms (4) and (5)

$$\left[D_x\left(D_t - D_t D_x^2 + D_x + D_x^3\right) + \frac{1}{3}D_t\left(D_s + D_x^3\right)\right] \quad (12)$$
$$f \cdot f = 0,$$

where s is an auxiliary variable, and f satisfies the bilinear equation

$$D_x\left(D_s + D_x^3\right) f.f = 0, \quad (13)$$

Using the properties of the D operators given above, and differentiating with respect to x we obtain the extended model for Equation (2) given by

$$\begin{aligned} &u_t - u_{xxt} - 4uu_t \\ &-2u_x \int^x u_t \mathrm{d}x + u_x + u_{xxx} + 6uu_x = 0, \end{aligned} \quad (14)$$

In a like manner, we extend Equation (3) by adding the operator D_x^4 to the bilinear forms (6) to obtain

$$D_x\left(D_t - D_t D_x^2 + D_x + D_x^3\right) f.f = 0, \quad (15)$$

Using the properties of the D operators given above we obtain the extended model for Equation (3) given by

$$\begin{aligned} &u_t - u_{xxt} - 3uu_t \\ &-3u_x \int^x u_t \mathrm{d}x + u_x + u_{xxx} + 6uu_x = 0, \end{aligned} \quad (16)$$

In this paper, we use the He's variational iteration method (VIM) to obtain the solution of two considered equations above for shallow water waves. The variational iteration method (VIM) [4-10] established in (1999) by He is thoroughly used by many researchers to handle linear and nonlinear models. The reliability of the method and the reduction in the size of computational domain gave this method a wider applicability. The method has been proved by many authors [11-15] to be reliable and efficient for a wide variety of scientific applications, linear and nonlinear as well. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. For concrete problems, a few numbers of approximations can be used for numerical purposes with high degree of accuracy. The VIM does not require specific transformations or nonlinear terms as required by some existing techniques.

2. Basic Idea of He's Variational Iteration Method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \quad (17)$$

where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \begin{pmatrix} Lu_n(\tau) + N\tilde{u}_n(\tau) - \\ g(\tau) \end{pmatrix} \mathrm{d}\tau \quad (18)$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the $n\,th$ approximation and $\tilde{u}_n$ is considered as a restricted variation $\delta\tilde{u}_n = 0$.

3. VIM Implement for First Extended Model of Shallow Water Wave Equation

Now let us consider the application of VIM for first extended model of shallow water wave equation with the initial condition of:

$$u(x,0) = \frac{(c-1)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \quad (19)$$

Its correction variational functional in x and t can be expressed, respectively, as follows:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \Bigg(&\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ &- 4u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} - 2\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau}\mathrm{d}\delta \\ &+ \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} + 6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\Bigg)\mathrm{d}\tau \end{aligned} \quad (20)$$

where λ is general Lagrangian multiplier.

After some calculations, we obtain the following stationary conditions:

$$\lambda'(\tau) = 0 \quad (21a)$$

$$1 + \lambda(\tau)\big|_{\tau=t} = 0 \quad (21b)$$

Thus we have:

$$\lambda(t) = -1, \quad (22)$$

As a result, we obtain the following iteration formula:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Bigg(&\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ &- 4u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} - 2\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau}\mathrm{d}\delta \\ &+ \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} + 6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\Bigg)\mathrm{d}\tau \end{aligned} \quad (23)$$

We start with the initial approximation of $u(x,0)$ given by Equation (19). Using the above iteration formula (23), we can directly obtain the other components as follows:

$$u_0(x,t)=\frac{(c-1)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \quad (24)$$

$$u_1(x,t)=\frac{1}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)(c+1)^2}$$
$$\left(\frac{1}{2}(c-1)\left(c\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)+\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\right.\right. \quad (25)$$
$$\left.\left.+2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\sqrt{\frac{c-1}{c+1}}tc\right)\right),$$

$$u_2(x,t)=u_1(x,t)-\int_0^t\left(\frac{\partial u_1(x,\tau)}{\partial\tau}-\frac{\partial^3 u_1(x,\tau)}{\partial x^2\partial\tau}\right.$$
$$-4u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial\tau}-2\frac{\partial u_1(x,\tau))}{\partial x}\int_0^x\frac{\partial u_1(\delta,\tau)}{\partial\tau}\mathrm{d}\delta \quad (26)$$
$$\left.+\frac{\partial u_1(x,\tau)}{\partial x}+\frac{\partial^3 u_1(x,\tau)}{\partial x^3}+6u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x}\right)\mathrm{d}\tau$$

In **Figure 1** we can see the 3-D result of first extended model of shallow water wave equation by VIM.

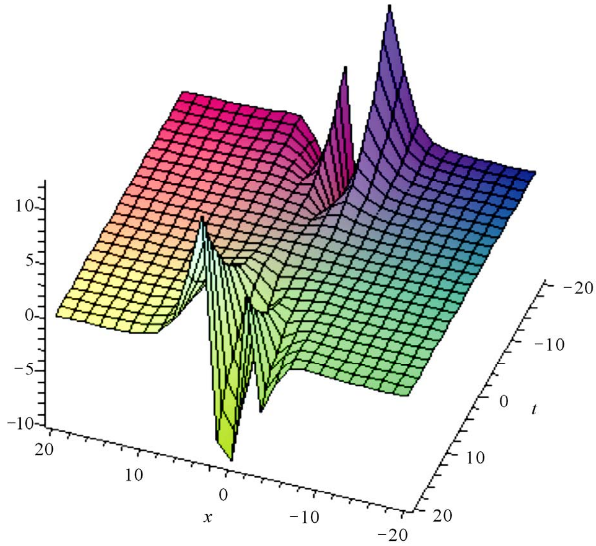

Figure 1. For the first extended model of shallow water wave equation with the first initial condition (24) of Equation (14), when $c = 2$.

4. VIM Implement for Second Extended Model of Shallow Water Wave Equation

At last we consider the application of VIM for second extended model of shallow water wave equation with the initial condition given by Equation (19).

Its correction variational functional in x and t can be expressed, respectively, as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\left(\lambda\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial^3 u_n(x,\tau)}{\partial x^2\partial\tau}\right.$$
$$-3u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial\tau}-3\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x\frac{\partial u_n(\delta,\tau)}{\partial\tau}\mathrm{d}\delta \quad (27)$$
$$\left.+\frac{\partial u_n(x,\tau)}{\partial x}+\frac{\partial^3 u_n(x,\tau)}{\partial x^3}+6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)\mathrm{d}\tau$$

where λ is general Lagrangian multiplier.

After some calculations, we obtain the following stationary conditions:

$$\lambda'(\tau)=0 \quad (28)$$

$$1+\lambda(\tau)\big|_{\tau=t}=0 \quad (29)$$

Thus we have:

$$\lambda(t)=-1, \quad (30)$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)$$
$$-\int_0^t\left(\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial^3 u_n(x,\tau)}{\partial x^2\partial\tau}-3u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial\tau}\right.$$
$$-3\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x\frac{\partial u_n(\delta,\tau)}{\partial\tau}\mathrm{d}\delta+\frac{\partial u_n(x,\tau)}{\partial x} \quad (31)$$
$$\left.+\frac{\partial^3 u_n(x,\tau)}{\partial x^3}+6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)\mathrm{d}\tau$$

We start with the initial approximation of $u(x,0)$ given by Equation (19). Using the above iteration formula (31), we can directly obtain the other components as follows:

$$u_0(x,t)=\frac{(c-1)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \quad (32)$$

$$u_1(x,t)=\frac{1}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)(c+1)^2}$$
$$\left(\frac{1}{2}(c-1)\left(c\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)+\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\right.\right. \quad (33)$$
$$\left.\left.+2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\sqrt{\frac{c-1}{c+1}}t\right)c\right),$$

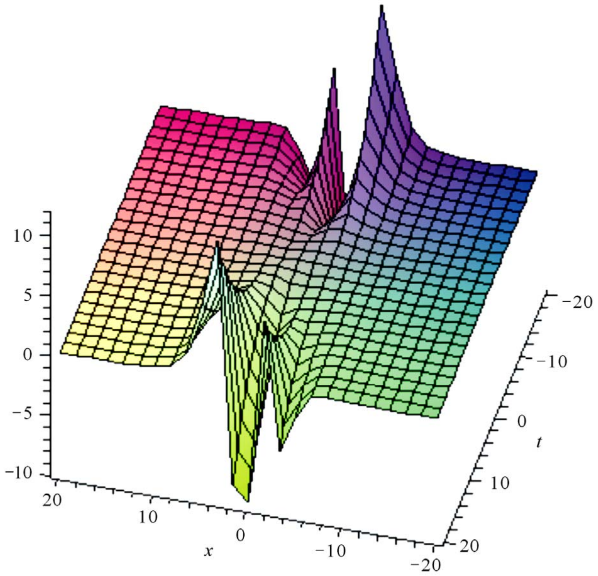

Figure 2. For the second extended model of shallow water wave equation with the first initial condition (32) of Equation (16), when c = 2.

$$\begin{aligned} u_2(x,t) &= u_1(x,t) \\ &-\int_0^t \left(\frac{\partial u_1(x,\tau)}{\partial \tau} - \frac{\partial^3 u_1(x,\tau)}{\partial x^2 \partial \tau} - 3u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial \tau} \right. \\ &-3\frac{\partial u_1(x,\tau)}{\partial x}\int_0^x \frac{\partial u_1(\delta,\tau)}{\partial \tau}\mathrm{d}\delta + \frac{\partial u_1(x,\tau)}{\partial x} \\ &\left. + \frac{\partial^3 u_1(x,\tau)}{\partial x^3} + 6u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} \right)\mathrm{d}\tau \end{aligned} \tag{34}$$

In **Figure 2** we can see the 3-D result of second extended model of shallow water wave equation by VIM.

5. Acknowledgements

In this paper, He's variational iteration method has been successfully applied to find the solution of two extended model equations for shallow water. The obtained results were showed graphically it is proved that He's variational iteration method is a powerful method for solving these equations. In our work; we used the Maple Package to calculate the functions obtained from the He's variational iteration method.

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