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Robust Resource Management Control for CO_2 Emission and Reduction of Greenhouse Effect: Stochastic Game Approach

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ABSTRACT

With the increasingly severe global warming, investments in clean technology, reforestation and political action have been studied to reduce CO_2 emission. In this study, a nonlinear stochastic model is proposed to describe the dynamics of CO_2 emission with control inputs: clean technology, reforestation and carbon tax, under stochastic uncertainties. For the efficient resources management, a robust tracking control is designed to force resources tracking a desired reference output. The worst-case effect of stochastic parametric fluctuations, external disturbances and uncertain initial conditions on the tracking performance is considered and minimized from the dynamic game theory perspective. This stochastic game problem, in which one player (stochastic uncertainty) maximizes the tracking error and another player (control input) minimizes the tracking error, could be equivalent to a robust minimax tracking problem. To avoid solving the HJI, a fuzzy model is proposed to approximate the nonlinear CO_2 emission model. Then the nonlinear stochastic game problem could be easily solved by fuzzy stochastic game approach via LMI technique.

Keywords: *CO_2 Emission System, Dynamic Game Theory, Greenhouse Effect, LMI, Resource Management Control, Robust Tracking Control, T-S Fuzzy Model*

1. Introduction

In recent years, the world has attracted much attention to environmental issues such as atmospheric pollution, conservation of water reserves and the reduction of tropical forests cover. For example, people feel concern about global warming, caused by greenhouse gases (GHG) such as carbon dioxide (CO_2), methane, nitrous oxide, sulfur hexafluoride, hydrofluorocarbons and perfluorocarbons, which leading to ecological destruction, climatic anomalies and sea level rise [1-2]. However, despite the increasing environmental awareness, global economic success heavily relies on the industrial throughput. People have gained a better life following the expansion of industrial sector and the number of job positions. This has been achieved following the expresses of urban environmental quality, significant increase in pollution, and loss of natural habitats [3]. In order to reduce the emissions of GHG, especially CO_2, without limiting economic growth, substantial investments should target the development of clean technology, expansion of forested areas and some political actions [4-6].

A major problem associated with economic growth is the need for the energy, for which fossil fuel is the primary source. Such economic growth resulted in an increase of atmospheric emission of CO_2, as shown in **Table 1** [7]. From **Figure 1** [8], it is seen that from 1900, the global CO_2 emission increased year by year except in the European Union (EU) that decreased by 2% in the later period (1990-1996), but it is still very high elsewhere. According to United Nations Environment Programme (UNEP) in 2007 [7], this decrease was possible due to many initiatives taken by Germany such as investing in renewable energy, solar power, new technology for car production, reforestation and political actions creating laws requiring 5% reduction of carbon emission. Recently, an indicator called the Ecological Footprint (EF) was concerned by UNEP to relate the 'pressure' exerted by human pollutions on the global ecosystems (**Table 2**) [7]. The EF is expressed in terms of area, and according to the definition provided by WWF [9], it represents "how much productive land and sea is needed to provide the resources such as energy, water and raw materials used everyday. It also calculates

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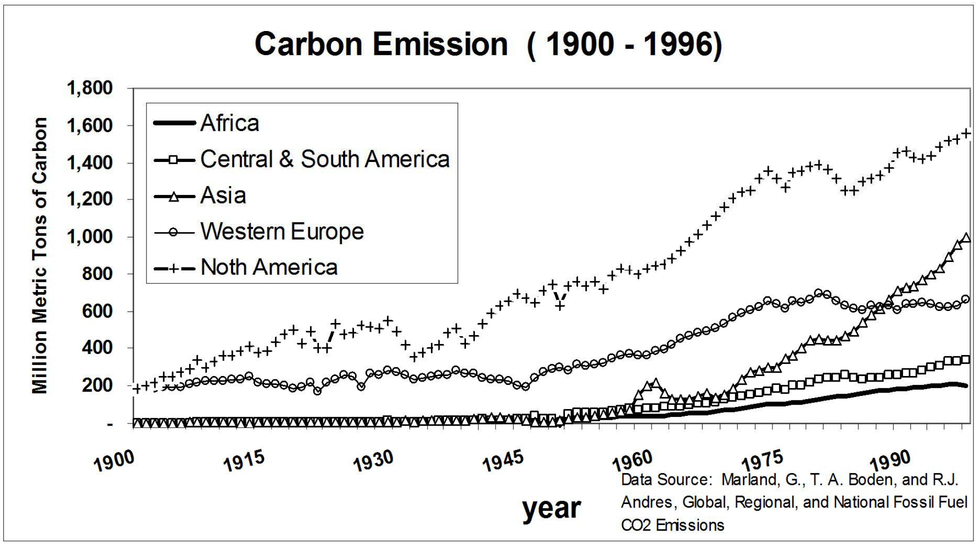

Figure 1. Global carbon emission. The CO_2 emission has an increasing trend in the world, except for an about 2% decreasing in recent period (1990~1996) in EU.

Table 1. Carbon dioxide emission by region (1998).

Region	Million tones carbon/year
Asia and Pacific	2167
Europe	1677
North America	1614
Latin America & Caribbean	365
West Asia	187

Source: UNEP (2007).

Table 2. Ecological footprints.

Region	Hectares/ per capita
Africa	<2
Asia and Pacific	<2
Latin America & Caribbean	Between 2 and 3
West Europe	Between 4 and 5
North America	>11

Source: UNEP (2007).

the emission generated from the oil, coal and gas burnt, and determines how much land is required to absorb the waste". This indicator is very useful in establishing how far the present situation is from the ideal condition in terms of emission of CO_2. The worst EF indicators are found in North America and Western Europe. In order to mitigate the threat of an escalating greenhouse effect, it is necessary to establish a rigorous management process of the available resources to reduce CO_2 emission. These should include direct government incentive to promote pro-environment actions by the private sector and the establishment of stricter pollution regulations. To meet the CO_2 emission limitations and combat global warming, 193 parties (192 states and the EU) have signed and ratified the *Kyoto Protocol* to the United Nations.

Framework Convention on Climate Change (UNFCCC) [10]. The cost estimated for the industrialized countries to implement the *Kyoto Protocol* ranges from 0.1 to 0.2% of their gross domestic product (GDP) [3]. Based on mathematical dynamic models, these costs can be efficiently optimized through control theory methods.

In order to manage the resources commitment to achieve the desired control of CO_2 concentration, mathematical models are required. In previous studies, Nordbous (1991) [11,12] presented a mathematical model to describe the effect of GHG in the economy and to maximize a social welfare function, subject to dynamic constraints for the global temperature and atmospheric concentration of CO_2. He carried out a study considering low, medium and high level of damages as a function of the concentration of CO_2. In another study, Nordhous (1993) [13] used the same mathematical model to evaluate optimal taxation policies to stabilize climate and carbon emissions, *i.e.* enforcing political actions about taxes on the CO_2 emissions from burning coal, petroleum based products and natural gas. Poterba (1993) [14] has discussed the relationship between global warming and GDP growth and considered the influence of certain macroeconomic initiatives on the decrease of the atmospheric emission of CO_2. For example, a consumption-linked carbon tax to reduce CO_2 emissions by 50% would reduce GDP by 4% in North America, 1% in Europe, and by 19% in some oil exporting countries. In the study of

Stollery (1998) [15], an optimal CO_2 emission tax could be initially high, but it would eventually be lowered as emission decline due to energy resource depletion. He also showed that to sustain consumption in the face of both energy resource depletion and economic damage from global warming, it suffices to reinvest the sum of carbon tax revenues and the net energy rents. Caetano *et al*. (2008) [3] follows the ideas in Stollery [15] and offers a quantitative tool for the efficient allocation of resources to reduce the greenhouse effect caused by CO_2 emission. Their approach was developed by a mathematical model to describe the dynamic relation of CO_2 emission with investment in reforestation and clean technology, and propose a method to efficiently manage the available resources by casting an optimal control problem. Also an optimal tracking control of CO_2 emission was addressed to achieve the emission targets proposed in the *Kyoto Protocol* for European countries by numerically solving a Hamiltonian function [16].

In the above methods, ordinary nonlinear differential equations with time-invariant parameters are used to describe the deterministic dynamics among CO_2 emissions, forest area expansion and GDP growth. However, these intrinsic parameters may fluctuate area to area and time to time for different regional development or unpredictable situations, like sub-prime crash that initiated in 2007, which may lead to the necessity of estimating new parameters as time or let the control strategies be limited in some specific area. Further, the external disturbances, due to modeling error and environmental noise, should also be considered in order to mimic the real dynamics of CO_2 emission system. Therefore, the dynamical model of CO_2 emission system should be described by stochastic differential equations (SDEs). A differential equation containing a deterministic part and an additional random fluctuation term is called a stochastic differential equation, which has been frequently used to model diverse phenomena in physics, biology and finance [17]. In this study, a nonlinear stochastic model is proposed to describe the dynamic system with model uncertainties from intrinsic parametric fluctuations, for the CO_2 emission with investments in reforestation, clean technology and political action about carbon tax. In addition to the intrinsic parametric fluctuations, external disturbances from modeling error and environmental noise are also included in the nonlinear stochastic model of CO_2 emission system, thus the generalized dynamic model could be widely applied to different area and time. Then a reference model is developed to generate the desired dynamics of CO_2 emission system. Finally, a robust model reference tracking control is proposed to manage these available resources, so that the nonlinear stochastic CO_2 emission system can track the desired output of the reference model, in spite of parametric fluctuations and external disturbances [18,19]. Since the statistical knowledge of the parametric fluctuation, external disturbance and uncertain initial condition is always unavailable, based on robust H_∞ control theory, the worst-case effect of parametric fluctuations, external disturbances and uncertain initial conditions on the tracking error should be minimized by the control efforts, so that all possible effects on the desired reference tracking, due to these uncertainties, could be attenuated as small as possible.

The parametric fluctuations, external disturbances and uncertain initial conditions are considered as one player to maximize the tracking error, while the control of resource management is considered as another player to minimize the tracking error, from the dynamic game (minimax) theory perspective. This stochastic game problem could be equivalent to a robust minimax tracking problem, to achieve a prescribed reference output, in spite of the worst-case effect of parametric fluctuations, external disturbances and uncertain initial condition. Thus, solving the stochastic game problem for nonlinear stochastic CO_2 emission system will need to solve the Hamilton Jacobi inequality (HJI). At present, there is no analytic or numerical solution for the HJI except simple cases. To avoid solving the HJI for the nonlinear stochastic game problem, a Takagi-Sugeno (T-S) fuzzy model [20] is proposed to interpolate several linearized stochastic systems at different operation points, to approximate the nonlinear dynamics of CO_2 emission system. With the help of fuzzy approximation method to simplify the nonlinear stochastic game problem, it can be easily solved by the proposed fuzzy stochastic game approach via linear matrix inequality (LMI) technique with the help of Robust Control Toolbox in Matlab. Finally, some simulation results are given to confirm the robust minimax tracking performance of the proposed stochastic game approach for reducing the CO_2 emissions and greenhouse effect.

Mathematical Preliminaries:

Before the further analysis of the stochastic CO_2 emission system, some definition and lemma of SDE are given in the following for the convenience of problem description and control design:

Definition 1 (*Ito SDE*) [17]:

For a given stochastic differential equation

$$\dot{x}(t) = a(x(t)) + b(x(t))n(t)$$

where $x(t)$ is a stochastic process; $a(x(t))$ and $b(x(t))$ are functions of $x(t)$; $n(t)$ is the standard white noise with zero mean and unit variance to denote the random fluctuation, which can be considered as the

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derivative of Wiener process (or the Brownian motion). The Ito type stochastic differential equation (*Ito SDE*) of $x(t)$ is represented by

$$dx(t) = a(x(t))dt + b(x(t))dw(t)$$

where $w(t)$ denotes the standard Wiener process, *i.e.* $dw(t) = n(t)dt$.

Lemma 1 (*Ito's formula*) [21,22]:

Let $x(t)$ be an Ito stochastic process in the above equation, if $f:\mathbb{R}\to\mathbb{R}$ is a twice continuous differentiable function of $x(t)$, then $f(x(t))$ is also a stochastic process satisfied with the following dynamic equation

$$\begin{aligned} df(x(t)) &= f'(x(t))dX(t) \\ &\quad + \frac{1}{2}b^T(x(t))f''(x(t))b(x(t))dt \end{aligned}$$

2. Stochastic Model of CO_2 Emission under Parametric Fluctuation and External Disturbance

For the convenience of illustration, some ordinary nonlinear differential equations [3,16] has been proposed to model the dynamics of CO_2 emission. Taking account the political actions mentioned in the previous section, the modified equations by introducing a carbon tax control term are used [23,24]. The more general deterministic model deals only with a few parameters to represent the dynamics of atmospheric CO_2 $\chi(t)$, forest area $z(t)$ and GDP $y(t)$ as follows [3,16]

$$\begin{aligned} \dot{\chi}(t) &= r_1\chi(t)\left(1-\frac{\chi(t)}{s}\right) - \alpha_1 z(t) + (\alpha_2 - u_2)y(t) \\ \dot{z}(t) &= u_1 y(t) - hz(t) \\ \dot{y}(t) &= \gamma y(t) - u_3\chi(t) \end{aligned} \quad (1)$$

The first Equation in (1) is to model the CO_2 emission, in which r_1 is the emission rate, s is the carrying capacity of the atmosphere in terms of CO_2, $\alpha_1 z(t)$ denotes the removal of CO_2 from the atmosphere and is proportional to total forest area $z(t)$, and $\alpha_1(\alpha_2 - u_2)y(t)$ denotes the production of CO_2 due to GDP. It is assumed that CO_2 emission increases with the economic activity term $\alpha_2 y(t)$ and decreases with the clean technology investment term $u_2 y(t)$, which are both proportional to the GDP $y(t)$. The second Equation in (1) is to model the total area of forest, which depends on the reforestation term $u_1 y(t)$ and the forest depletion term $hz(t)$. Consider the impact of economic activity is much larger than natural growth, the reforestation term is assumed be a fraction of GDP, *i.e.* with laws and incentive to promote reforestation, whereas the total area of forest decreasing is mainly due to forest logging or other economic activities. The coefficient u_1 represents the intensity of incentives directed to reforestation and the coefficient h represents the forest depletion rate, which amalgamates a variety of factors such as expansion of cattle ranching, fire, commercial logging, shifted cultivators and colonization, among others. The third Equation in (1) is to model the GDP. Usually the $y(t)$ is assumed to present an exponential growth with rate γ and u_3 is a tax rate which is proportional to the CO_2 emission, thus the CO_2 revenue term $u_3\chi(t)$ including the effects of carbon tax and "virtual tax"—all the effects that are similar to carbon tax, *i.e.* energy cost rise, consumer prices rise, real wages fall and output and employment fall [25] that can also be directly or indirectly controlled by government order. In (1), u_1, u_2 and u_3 are control variables to be specified, so that the state variables $\chi(t)$, $z(t)$ and $y(t)$ can achieve their desired reference outputs.

The above model has some limitations such as 1) the deterministic nature of economic growth (as expressed by GDP), 2) difficulty in limiting the geographic area, as one country, in political sense, can effect a neighboring state, 3) absence of time-varying parameters to adapt the model to changing situation. Further, the model is too simple and some factors may be neglected, *i.e.* there exist some un-modeled dynamics. In order to mimic the stochastic dynamics of CO_2 emission, the parameter fluctuations and external disturbances should be considered in the following stochastic model

$$\begin{aligned} \dot{\chi}(t) &= (r_1 + \Delta r_1)\chi(t)\left(1-\frac{\chi(t)}{s}\right) - (\alpha_1 + \Delta\alpha_1)z(t) \\ &\quad + ((\alpha_2 + \Delta\alpha_2) - u_2(t))y(t) + v_1(t) \\ \dot{z}(t) &= u_1(t)y(t) - (h + \Delta h)z(t) + v_2(t) \\ \dot{y}(t) &= (\gamma + \Delta\gamma)y(t) - u_3(t)\chi(t) + v_3(t) \end{aligned} \quad (2)$$

where Δr_1, $\Delta\alpha_1$, $\Delta\alpha_2$, Δh and $\Delta\gamma$ denote the parametric fluctuations of the coefficients, from one country to one country and from time to time to adapt the model to changing situation. $v_1(t)$, $v_2(t)$ and $v_3(t)$ denote to the external disturbances, due to modeling error and environmental noise.

Suppose the parametric fluctuations can be separated into a deterministic part and a random part as follows:

$$\begin{aligned} &\Delta r_1 = \delta_1 n(t), \Delta\alpha_1 = \delta_2 n(t), \Delta\alpha_2 = \delta_3 n(t), \\ &\Delta h = \delta_4 n(t), \Delta\gamma = \delta_5 n(t) \end{aligned} \quad (3)$$

where δ_i denotes the standard deviation of stochastic parametric fluctuation, and $n(t)$ denotes a standard white noise with unit variance, *i.e.* $\mathrm{var}(\Delta r_1) = \delta_1^2$,

$\operatorname{var}(\Delta\alpha_1)=\delta_2^2$ and so on, *i.e.* the stochastic property of parametric fluctuations is absorbed by a white noise $n(t)$, and the amplitudes of parametric fluctuations are determined by their standard deviations δ_i respectively. Then the stochastic model for dynamics of CO_2 emission could be represented by

$$\begin{bmatrix}\dot{\chi}(t)\\ \dot{z}(t)\\ \dot{y}(t)\end{bmatrix}=\begin{bmatrix}r_1\chi(t)\left(1-\frac{\chi(t)}{s}\right)-\alpha_1 z(t)+\left(\alpha_2-u_2(t)\right)y(t)\\ u_1(t)y(t)-hz(t)\\ \gamma y(t)-u_3(t)\chi(t)\end{bmatrix}+\begin{bmatrix}\delta_1\chi(t)\left(1-\frac{\chi}{s}\right)-\delta_2 z(t)+\delta_3 y(t)\\ -\delta_4 z(t)\\ \delta_5 y(t)\end{bmatrix}n(t)+\begin{bmatrix}v_1(t)\\ v_2(t)\\ v_3(t)\end{bmatrix} \tag{4}$$

For the convenience of analysis and design, the above stochastic CO_2 emission system can be represented by the following Ito stochastic system [17,26]

$$\begin{bmatrix}d\chi(t)\\ dz(t)\\ dy(t)\end{bmatrix}=\begin{bmatrix}r_1\chi(t)\left(1-\frac{\chi(t)}{s}\right)-\alpha_1 z(t)+\alpha_2 y(t)\\ -hz(t)\\ \gamma y(t)\end{bmatrix}dt+\begin{bmatrix}0 & -y(t) & 0\\ y(t) & 0 & 0\\ 0 & 0 & -\chi(t)\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{bmatrix}dt+\begin{bmatrix}v_1(t)\\ v_2(t)\\ v_3(t)\end{bmatrix}dt+\begin{bmatrix}\delta_1\chi(t)\left(1-\frac{\chi(t)}{s}\right)-\delta_2 z(t)+\delta_3 y(t)\\ -\delta_4 z(t)\\ \delta_5 y(t)\end{bmatrix}dw(t) \tag{5}$$

where $w(t)$ with $dw(t)=n(t)dt$ denotes a standard Wiener process or Brownian motion. Actually, the stochastic system for CO_2 emission in (5) can be extended to a more general stochastic CO_2 emission system as follows

$$dx(t)=\left(f(x(t))+g(x(t))u(t)+v(t)\right)dt+h(x(t))dw(t), \quad x(0)=x_0 \tag{6}$$

where $x(t)=\left[x_1(t)\cdots x_n(t)\right]^{\mathrm{T}}$, $u(t)=\left[u_1(t)\cdots u_m(t)\right]^{\mathrm{T}}$, $v(t)=\left[v_1(t)\cdots v_n(t)\right]^{\mathrm{T}}$ denote the state vector, control input vector and external disturbance vector respectively. $f(x(t))\in R^{n\times 1}$ denotes the nonlinear interaction vector among the state variables of the CO_2 emission system. $g(x(t))\in R^{n\times m}$ denotes the control input matrix. $h(x(t))\in R^{n\times 1}$ denotes the noise dependent parameter fluctuation vector. In the more general model of (6), let $x_1(t)=\chi(t)$, $x_2(t)=z(t)$, $x_3(t)=y(t)$ and so on.

Consider a reference model of the stochastic CO_2 emission system in (6) with a desired state output as follows

$$\dot{x}_r(t)=A_r x_r(t)+r(t) \tag{7}$$

where $x_r(t)\in R^{n\times 1}$ is the reference state vector, $A_r\in R^{n\times n}$ is a specific asymptotically stable matrix and $r(t)$ is a desired reference signal. Based on the model reference tracking control, A_r and $r(t)$ are specified beforehand by designer, so that $x_r(t)$ can represent a desired system's state output for the stochastic system of CO_2 emission in (6) to follow. Then, the robust model reference tracking control is to design $u(t)$ to make $x(t)$ in (6) track the desired $x_r(t)$, such that the tracking error $\tilde{x}(t)=x(t)-x_r(t)$ must be as small as possible, in spite of the influence of stochastic parametric fluctuations, external disturbances and the uncertain initial condition $x(0)$ in (6). Since the parametric fluctuations are stochastic, external disturbance $v(t)$ and initial state $x(0)$ are uncertain, and reference signal $r(t)$ could be arbitrarily assigned, the robust model reference tracking control design should be specified, so that the worst-case effect of three uncertainties $v(t)$, $x(0)$ and $r(t)$ on the tracking error $\tilde{x}(t)$ and control effort $u(t)$ should be minimized and below a prescribed value ρ^2, *i.e.* both the minimax tracking and robustness against uncertainties $v(t)$, $x(0)$, and $r(t)$ should be achieved simultaneously as following stochastic game problem

$$\min_{u(t)}\max_{v(t),r(t)}\frac{E\int_0^{t_f}\left(\tilde{x}^T(t)Q\tilde{x}(t)+u^T(t)Ru(t)\right)dt}{E\int_0^{t_f}\left(v^T(t)v(t)+r^T(t)r(t)\right)dt+\tilde{x}^T(0)\tilde{x}(0)}\le\rho^2,\quad\forall\tilde{x}(0) \tag{8}$$

where E denotes the expectation, and the weighting matrices Q and R are assumed diagonal as follow

$$Q=\begin{bmatrix}q_{11} & 0 & \cdots & 0\\ 0 & q_{22} & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & q_{nn}\end{bmatrix},\ R=\begin{bmatrix}r_{11} & 0 & \cdots & 0\\ 0 & r_{22} & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & r_{mm}\end{bmatrix}$$

The diagonal element q_{ii} of Q denotes the punish-

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ment on the corresponding tracking error and the diagonal r_{ii} of R denotes the relative control cost. ρ^2 denotes the upper bound of the stochastic game problem in (8). Since the worst-case effect of $v(t)$, $r(t)$ and $x(0)$ on the tracking error $\tilde{x}(t)$ is minimized by control effort $u(t)$, from the energy point of view, the stochastic game problem in (8) is suitable for a robust minimax tracking problem in which the statistics of $v(t)$, $\tilde{x}(t)$ and $r(t)$ are unknown or uncertain, that are always met in practical control design case, for example, in the stochastic CO_2 emission system.

Remark 1:

If $v(t)$ and $r(t)$ are all deterministic, then the expectation E in the denominator of (8) can be neglected.

Because it is not easy to solve the robust minimax tracking problem in (8) subject to (6) and (7) directly, an upper bound ρ^2 of the minimax tracking problem is proposed to formulate a sub-optimal minimax tracking problem. After that, the sub-optimal minimax tracking problem is solved firstly, then the upper bound ρ^2 is decreased as small as possible to approximate the real robust minimax tracking problem of the stochastic CO_2 emission system.

Since the denominator in (8) is independent of $u(t)$ and is not zero, equation (8) is equivalent to [27]

$$\begin{aligned}&\min_{u(t)}\max_{v(t),r(t)} E\int_0^{t_f}\left(\tilde{x}^T(t)Q\tilde{x}(t)+u^T(t)Ru(t)\right.\\&\qquad\left.-\rho^2v^T(t)v(t)-\rho^2r^T(t)r(t)\right)dt \quad (9)\\&\le\rho^2E\left\{\tilde{x}^T(0)x(0)\right\},\ \forall\tilde{x}(0)\end{aligned}$$

Let us denote

$$\begin{aligned}&\min_{u(t)}\max_{v(t),r(t)} J\left(u(t),v(t),r(t)\right)\\&=\min_{u(t)}\max_{v(t),r(t)} E\int_0^{t_f}\left(\tilde{x}^T(t)Q\tilde{x}(t)+u^T(t)Ru(t)\right. \qquad (10)\\&\qquad\left.-\rho^2v^T(t)v(t)-\rho^2r^T(t)r(t)\right)dt\end{aligned}$$

From the above analysis, the stochastic game problem in (9) or (10) is equivalent to finding the worst-case disturbance $v^*(t)$ and reference signal $r^*(t)$ which maximize $J\left(u(t),v(t),r(t)\right)$, and a minimax tracking control $u^*(t)$ which minimizes $J\left(u(t),v^*(t),r^*(t)\right)$, such that the minimax value $J\left(u^*(t),v^*(t),r^*(t)\right)$ is less than $\rho^2E\left\{\tilde{x}^T(0)\tilde{x}(0)\right\}$, *i.e.*

$$\begin{aligned}&J\left(u^*(t),v^*(t),r^*(t)\right)=\min_{u(t)}J\left(u(t),v^*(t),r^*(t)\right)\\&=\min_{u(t)}\max_{v(t),r(t)}J\left(u(t),v(t),r(t)\right) \qquad (11)\\&\le\rho^2E\left\{\tilde{x}^T(0)\tilde{x}(0)\right\},\forall\tilde{x}(0)\end{aligned}$$

If there exist $u^*(t)$, $v^*(t)$ and $r^*(t)$, such that the robust minimax tracking problem in (11) is solved, then they can satisfy the stochastic game problem in (8) as well. Therefore, the first step of robust minimax tracking control design of stochastic CO_2 emission system is to solve the following minimax tracking problem

$$\min_{u(t)}\max_{v(t),r(t)} J\left(u(t),v(t),r(t)\right) \qquad (12)$$

subject to the CO_2 emission system (6) and the desired reference model (7).

After that, the next step is to check whether the condition $J\left(u^*(t),v^*(t),r^*(t)\right)\le\rho^2E\left\{\tilde{x}^T(0)\tilde{x}(0)\right\}$ is satisfied or not for any $\tilde{x}(0)$.

To solve the minimax tracking problem in (12), it is convenience to transform the problem into an equivalent minimax regulation problem.

Let us denote

$$\begin{aligned}&F\left(\bar{x}(t)\right)=\begin{bmatrix}f\left(x(t)\right)\\A_rx_r(t)\end{bmatrix},\bar{x}(t)=\begin{bmatrix}x(t)\\x_r(t)\end{bmatrix},\bar{v}(t)=\begin{bmatrix}v(t)\\r(t)\end{bmatrix},\\&G\left(\bar{x}(t)\right)=\begin{bmatrix}g\left(x(t)\right)\\0\end{bmatrix},H\left(\bar{x}(t)\right)=\begin{bmatrix}h\left(x(t)\right)\\0\end{bmatrix}\end{aligned} \qquad (13)$$

thus an augmented stochastic system of (6) and (7) is obtained as follows

$$\begin{aligned}&d\bar{x}(t)=\left(F\left(\bar{x}(t)\right)+G\left(\bar{x}(t)\right)u(t)+\bar{C}\bar{v}(t)\right)dt\\&\qquad+H\left(\bar{x}(t)\right)dw(t), \qquad (14)\\&\bar{x}(0)=\bar{x}_0\end{aligned}$$

where $\bar{C}=\begin{bmatrix}I&0\\0&I\end{bmatrix}$.

Then the minimax tracking problem in (12) can be rewriten as the following minimax regulation problem

$$\begin{aligned}&\min_{u(t)}\max_{\bar{v}(t)}J\left(u(t),\bar{v}(t)\right)\\&=\min_{u(t)}\max_{\bar{v}(t)}E\int_0^{t_f}\left(\bar{x}^T(t)\bar{Q}\bar{x}(t)+u^T(t)Ru(t)-\rho^2\bar{v}^T(t)\bar{v}(t)\right)dt,\\&\forall\bar{x}(0)\end{aligned} \qquad (15)$$

subject to (12)

where $\bar{Q}=\begin{bmatrix}Q&-Q\\-Q&Q\end{bmatrix}$.

Then the robust minimax tracking problem in (11) is equivalent to the following constrained minimax regulation problem

$$\min_{u(t)}\max_{\bar{v}(t)} E\int_0^{t_f}\left(\bar{x}^T(t)\bar{Q}\bar{x}(t)+u^T(t)Ru(t)-\rho^2\bar{v}^T(t)\bar{v}(t)\right)dt \le \rho^2 E\left\{\bar{x}^T(0)\bar{I}\bar{x}(0)\right\} \quad (16)$$

subject to (14)

where $\bar{I}=\begin{bmatrix} I & -I \\ -I & I\end{bmatrix}$.

Theorem 1:

The stochastic game problem in (16) for robust tracking control of stochastic CO_2 emission system could be solved by the following minimax tracking control u^* and the worst-case disturbance $\bar{v}^*$

$$u^*(t)=-\frac{1}{2}R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)} \quad (17)$$

$$\bar{v}^*(t)=\frac{1}{2\rho^2}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)} \quad (18)$$

where $V(\bar{x}(t))>0$ is the positive solution of the following HJI

$$\left(\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)}\right)^T F(\bar{x}(t))+\bar{x}^T(t)\bar{Q}\bar{x}(t) -\frac{1}{4}\left(\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)}\right)^T G(\bar{x}(t))R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)} +\frac{1}{4\rho^2}\left(\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)}\right)^T \bar{C}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)} +\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2 V(\bar{x}(t))}{\partial \bar{x}^2(t)}H(\bar{x}(t))<0 \quad (19)$$

with

$$E\left(V(\bar{x}(0))\right)\le\rho^2 E\left\{\bar{x}^T(0)\bar{I}\bar{x}(0)\right\} \quad (20)$$

Proof: See Appendix A.

Since ρ^2 is the upper bound of minimax tracking problem in (8), based on the analysis above, the minimax tracking control $u^*(t)$ and the worst-case disturbance $\bar{v}^*(t)$ still need to minimize the upper bound ρ^2 as follows

$$\rho_0^2=\min_{V(\bar{x}(t))>0}\rho^2 \quad (21)$$

subject to (19) and (20)

After solving a $V(\bar{x})$ and ρ_0^2 from the constrained optimization in (21), the solution $V(\bar{x})$ is substituted into (17) to obtain the robust minimax tracking control $u^*(t)$, for the stochastic CO_2 emission system in (6), to achieve the robust minimax tracking of the desired reference model in (7), in spite of stochastic intrinsic parametric fluctuation and external disturbance.

Remark 2:

If $\rho\to\infty$ in (8), *i.e.* the effect of $v(t)$, $r(t)$ and $\tilde{x}(0)$ on tracking error $\tilde{x}(t)$ and control effort $u(t)$ is neglected in the tracking design procedure. Then the robust minimax tracking problem in (8) is reduced to the following optimal tracking problem [21,28,29]

$$\min_{u(t)} E\int_0^{t_f}\left(\tilde{x}^T(t)Q\tilde{x}(t)+u^T(t)Ru(t)\right)dt \quad (22)$$

In this case, the optimal tracking control $u^*(t)$ is also given by (17), *i.e.* $u^*(t)=-\frac{1}{2}R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)}$. However, $V(\bar{x})$ in (17) should be replaced via solving the following HJI

$$\left(\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)}\right)^T F(\bar{x}(t))+\bar{x}^T(t)\bar{Q}\bar{x}(t) -\frac{1}{4}\left(\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)}\right)^T G(\bar{x}(t))R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial \bar{x}(t)} +\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2 V(\bar{x}(t))}{\partial \bar{x}^2(t)}H(\bar{x}(t))\le 0 \quad (23)$$

which is obtained from (19) but with $\rho\to\infty$. Therefore, if $\rho\to\infty$ in the robust minimax tracking problem, it is equivalent to the optimal tracking problem. Because the effect of external disturbance $\bar{v}(t)$ is neglected by the optimal tracking, its tracking performance will be deteriorated by the external disturbance $\bar{v}(t)$ and uncertain initial condition $\tilde{x}(0)$. Therefore, it is not suitable for robust minimax tracking design.

3. Robust Minimax Tracking Control via Fuzzy Interpolation Method

In general, there is no analytic or numerical solution for the HJI in (19) to solve the constrained optimization problem in (21), for robust minimax tracking control of the stochastic CO_2 emission system in (6). Recently, the T-S fuzzy model has been widely applied to approximate the nonlinear system via interpolating several linearized systems at different operation points [18-20]. By the fuzzy interpolation method, the nonlinear stochastic game problem could be transformed to a fuzzy stochastic game problem so that the HJI in (19) could be replaced by a set of linear matrix inequalities (LMI). In this situation, the nonlinear stochastic game problem in (8) could be easily solved by fuzzy stochastic game approach for the design of robust minimax tracking control of stochastic CO_2

emission system.

Suppose the nonlinear stochastic CO_2 emission system in (6) can be represented by T-S fuzzy model [20]. The T-S model is a piecewise interpolation of several linearized models through membership functions. The fuzzy model is described by fuzzy *if-then* rules and will be employed to deal with the nonlinear stochastic game problem for robust minimax tracking control to achieve a desired CO_2 emission, under stochastic fluctuations, external disturbances and uncertain initial conditions. The *i*-th rule of fuzzy model for nonlinear stochastic system in (6) is of the following form [18,19]

If $z_1(t)$ is F_{i1} and ……and $z_g(t)$ is F_{ig}, *then*

$$dx(t)=\left(A_i x(t)+B_i u(t)+Cv(t)\right)dt+D_i x(t)dw(t),\ i=1\cdots L \quad (24)$$

where F_{ij} is the fuzzy set, A_i, B_i and D_i are linearized system matrices, g is the number of premise variables and $z_1(t)$, … $z_g(t)$ are the premise variables.

The fuzzy system in (24) is inferred as follows [18-20]

$$dx(t)=\frac{\sum_{i=1}^{L}\mu_i(z(t))\left\{\left(A_i x(t)+B_i u(t)+Cv(t)\right)dt+D_i x(t)dw(t)\right\}}{\sum_{i=1}^{L}\mu_i(z(t))}$$
$$=\sum_{i=1}^{L}h_i(z(t))\left\{\left(A_i x(t)+B_i u(t)+Cv(t)\right)dt+D_i x(t)dw(t)\right\} \quad (25)$$

where

$$\mu_i(z(t))=\prod_{j=1}^{g}F_{ij}(z(t)),\ h_i(z(t))=\frac{\mu_i(z(t))}{\sum_{i=1}^{L}\mu_i(z(t))},$$
$$z(t)=\left(z_1(t),\cdots,z_g(t)\right)^T$$

and $F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in F_{ij}. We assume $\mu_i(z(t))\geq 0$ and $\sum_{i=1}^{L}\mu_i(z(t))>0$. Therefore, we get

$$h_i(z(t))\geq 0 \text{ and } \sum_{i=1}^{L}h_i(z(t))=1. \quad (26)$$

The T-S fuzzy model in (25) is to interpolate L linear stochastic systems in (24) to approximate the nonlinear stochastic system in (6) via the fuzzy basis function $h_i(z(t))$. We could specify parameters A_i, B_i and D_i easily in (25), so that $\sum_{i=1}^{L}h_i(z(t))A_i x(t)$, $\sum_{i=1}^{L}h_i(z(t))B_i$ and $\sum_{i=1}^{L}h_i(z(t))D_i x(t)$ can approximate the nonlinear functions $f(x(t))$, $g(x(t))$ and $h(x(t))$ in (6), respectively, by Fuzzy identification method [20]

Remark 3:

Actually, in (25), other interpolation methods such as cubic spline method, can be also employed to interpolate several linear stochastic systems to approximate the nonlinear stochastic system in (6), *i.e*. the smoothing bases $h_i(z(t))$ could be replaced by other interpolation bases of other interpolation methods.

After the nonlinear stochastic system in (6) is approximated by the T-S fuzzy system in (25), the augmented system in (14) can be also approximated by the following fuzzy system

$$d\bar{x}(t)=\sum_{i=1}^{L}h_i(z(t))\left\{\left(\bar{A}_i\bar{x}(t)+\bar{B}_i u(t)+\bar{C}v(t)\right)dt+\bar{D}_i\bar{x}(t)dw(t)\right\} \quad (27)$$

where $\bar{A}_i=\begin{bmatrix}A_i & 0\\ 0 & A_r\end{bmatrix}, \bar{B}_i=\begin{bmatrix}B_i\\ 0\end{bmatrix}, \bar{C}=\begin{bmatrix}c & 0\\ 0 & I\end{bmatrix}, \bar{D}_i=\begin{bmatrix}D_i & 0\\ 0 & 0\end{bmatrix}$.

After the nonlinear augmented stochastic system in (14) is approximated by the T-S fuzzy system in (27), the nonlinear stochastic game problem in (14) and (16) is replaced by solving the fuzzy stochastic game problem in (27) and (16).

Theorem 2:

The minimax tracking control and worst-case disturbance for fuzzy stochastic game problem in (16) subject to (27) are solved respectively as follows

$$u^*(t)=-\sum_{j=1}^{L}h_j(z(t))R^{-1}\bar{B}_j^T P\bar{x}(t),\ v^*(t)=\frac{1}{\rho^2}\bar{C}^T P\bar{x}(t) \quad (28)$$

where P is the positive definite symmetric solution of the following Riccati-like inequalities

$$P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-P\bar{B}_i R^{-1}\bar{B}_j^T P$$
$$+\frac{1}{\rho^2}P^T\bar{C}\bar{C}^T P+D_i^T PD_j<0;\quad i,j=1\cdots L \quad (29)$$
$$P\leq\rho^2\bar{I}$$

Proof: See Appendix B

By fuzzy approximation, obviously, the HJI in (19) is approximated by a set of algebraic inequalities in (29) and the inequality in (20) is also equivalent to the second inequality in (29). Since ρ^2 is the upper bound of minimax tracking problem in (8), the robust minimax tracking problem still needs to minimize ρ^2 as follows

$$\rho_0^2=\min_{P>0}\rho^2 \quad (30)$$

subject to (29)

In order to solve the above constrained optimization problem in (30) by the conventional LMI method, the inequalities in (29) can be rewritten as following relaxed conditions [30]

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$$\begin{aligned}&P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-P\bar{B}_i R^{-1}\bar{B}_i^T P\\&\quad+\frac{1}{\rho^2}P^T\bar{C}\bar{C}^T P+D_i^T PD_i<0; \qquad i=j\\&P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-P\bar{B}_i R^{-1}\bar{B}_j^T P+\frac{1}{\rho^2}P^T\bar{C}\bar{C}^T P\\&\quad+\left(\frac{D_i+D_j}{2}\right)^T P\left(\frac{D_i+D_j}{2}\right)<0;\ i\neq j\\&P\leq\rho^2\bar{I}\end{aligned}\tag{31}$$

Then, we let $W=P^{-1}>0$, and the inequalities in (31) can be equivalent to

$$\begin{aligned}&\bar{A}_i W+W\bar{A}_i^T+W\bar{Q}W-\bar{B}_i R^{-1}\bar{B}_i^T\\&\quad+\frac{1}{\rho^2}\bar{C}\bar{C}^T+WD_i^T W^{-1}D_i W\leq 0;\ i=j\\&\bar{A}_i W+W\bar{A}_i^T+W\bar{Q}W-\bar{B}_i R^{-1}\bar{B}_j^T+\frac{1}{\rho^2}\bar{C}\bar{C}^T\\&\quad+W\left(\frac{D_i+D_j}{2}\right)^T W^{-1}\left(\frac{D_i+D_j}{2}\right)W\leq 0;\ i\neq j\\&\rho^2 W>\bar{I}\end{aligned}$$

or

$$\begin{aligned}&\bar{A}_i W+W\bar{A}_i^T+W\bar{\bar{Q}}^T\bar{\bar{Q}}W-\bar{B}_i R^{-1}\bar{B}_i^T\\&\quad+\frac{1}{\rho^2}\bar{C}\bar{C}^T+WD_i^T W^{-1}D_i W\leq 0;\ i=j\\&\bar{A}_i W+W\bar{A}_i^T+W\bar{\bar{Q}}^T\bar{\bar{Q}}W-\bar{B}_i R^{-1}\bar{B}_j^T+\frac{1}{\rho^2}\bar{C}\bar{C}^T\\&\quad+W\left(\frac{D_i+D_j}{2}\right)^T W^{-1}\left(\frac{D_i+D_j}{2}\right)W\leq 0;\ i\neq j\\&\rho^2 W>\bar{I}\end{aligned}$$

where $\bar{\bar{Q}}=\left[Q^{\frac{1}{2}}\ -Q^{\frac{1}{2}}\right]$.

By the schur complement [27]. The constrained optimization problem in (30) is equivalent to the following LMI-constrained optimization problem

$$\rho_0^2=\min_{W>0}\rho^2\tag{32}$$

subject to

$$\begin{aligned}&\begin{bmatrix}\bar{A}_i W+W\bar{A}_i^T-\bar{B}_i R^{-1}\bar{B}_i^T+\frac{1}{\rho^2}\bar{C}\bar{C}^T & WD_i^T & W\bar{\bar{Q}}^T\\ D_i W & -W & 0\\ \bar{\bar{Q}}W & 0 & -I\end{bmatrix}<0;\ i=j\\&\begin{bmatrix}\bar{A}_i W+W\bar{A}_i^T-\bar{B}_i R^{-1}\bar{B}_j^T+\frac{1}{\rho^2}\bar{C}\bar{C}^T & W\left(\frac{D_i+D_j}{2}\right)^T & W\bar{\bar{Q}}^T\\ \left(\frac{D_i+D_j}{2}\right)W & -W & 0\\ \bar{\bar{Q}}W & 0 & -I\end{bmatrix}<0; i\neq j\\&\rho^2 W>\bar{I}\end{aligned}\tag{33}$$

Remark 4:

1) The fuzzy basis function $h_i(z(t))$ in (25) can be replaced by other interpolation function, for example, cubic spline function.

2) By fuzzy approximation, the HJI in (19) of nonlinear stochastic game problem for the robust minimax tracking of nonlinear stochastic CO_2 emission system is replaced by a set of inequalities in (29), which can be easily solved by LMI-constrained optimization in (33).

3) The constrained optimization to solve ρ_0 and W in (32), (33), can be easily solved by decreasing ρ^2 until there exists no $W>0$ solution in (32), (33).

4) After solving W and then $P=W^{-1}$ from the constrained optimization problem in (32), (33), ρ_0 can be solved by Robust Control Toolbox in Matlab efficiently.

5) If the conventional optimal tracking control in (22) is considered, *i.e*. the effect of disturbance $v(t)$ is not considered in the control design problem, then the optimal tracking control problem is equivalent to letting $\rho^2=\infty$ in (8). The optimal fuzzy tracking control design $u^*(t)=-\sum_{j=1}^{L}h_j(z(t))R^{-1}\bar{B}_j^T P\bar{x}(t)$ can be solved by a common positive definite symmetric matrix P from the inequalities in (29) with $\rho^2=\infty$, *i.e*. solving a common positive definite symmetric matrix $P>0$ from the following inequalities [27]

$$\begin{aligned}&P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-PB_i R^{-1}B_j P+D_i^T PD_j<0;\\&i,j=1\cdots L\end{aligned}\tag{34}$$

or the following relaxed conditions [30]

$$\begin{aligned}&P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-P\bar{B}_i R^{-1}\bar{B}_i^T P+D_i^T PD_i<0;\ i=j\\&P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-P\bar{B}_i R^{-1}\bar{B}_j^T P\\&\quad+\left(\frac{D_i+D_j}{2}\right)^T P\left(\frac{D_i+D_j}{2}\right)<0; i\neq j\end{aligned}$$

In order to solve the optimal tracking problem by LMI technique, the optimal tracking control is equivalent to solving a common $W=P^{-1}$ from the following inequalities,

$$\bar{A}_i W + W\bar{A}_i^T + W\bar{\bar{Q}}^T I\bar{\bar{Q}}W - \bar{B}_i R^{-1}\bar{B}_i^T + WD_i^T W^{-1} D_i W < 0;\ i = j$$

$$\bar{A}_i W + W\bar{A}_i^T + W\bar{\bar{Q}}^T I\bar{\bar{Q}}W - \bar{B}_i R^{-1}\bar{B}_j^T + W\left(\frac{D_i + D_j}{2}\right)^T W^{-1}\left(\frac{D_i + D_j}{2}\right)W < 0;\ i \neq j$$

or following LMIs,

$$\begin{bmatrix} \bar{A}_i W + W\bar{A}_i^T - \bar{B}_i R^{-1}\bar{B}_i^T & WD_i^T & W\bar{\bar{Q}}^T \\ D_i W & -W & 0 \\ \bar{\bar{Q}}W & 0 & -I \end{bmatrix} < 0;\ i = j$$

$$\begin{bmatrix} \bar{A}_i W + W\bar{A}_i^T - \bar{B}_i R^{-1}\bar{B}_j^T & W\left(\frac{D_i + D_j}{2}\right)^T & W\bar{\bar{Q}}^T \\ \left(\frac{D_i + D_j}{2}\right)W & -W & 0 \\ \bar{\bar{Q}}W & 0 & -I \end{bmatrix} < 0;\ i \neq j \quad (36)$$

i.e., if $W = P^{-1}$ is solved from (36), then the optimal fuzzy tracking control can be obtained as

$$u^*(t) = -\sum_{j=1}^{L} h_j(z(t)) R^{-1}\bar{B}_j^T P\bar{x}(t)$$

According to the analysis above, the robust minimax tracking control of CO_2 emission system via fuzzy interpolation method is summarized as follows.

Design Procedure:

Step 1. Give a desired reference model in (7) for the stochastic CO_2 emission system in (6).

Step 2. Select membership functions and construct fuzzy plant rules in (24) and (25).

Step 3. Give the weighting matrices Q and R of minimax tracking problem in (8).

Step 4. Solve the LMI-constrained optimization in (32), (33) to obtain W (thus $P = W^{-1}$) and ρ_0^2.

Step 5. Construct the robust minimax tracking control $u^*(t)$ in (28).

Remark 5:

The software package such as Robust Control Toolbox in Matlab can be employed to solve the LMI-constrained optimization problem in (32), (33) easily.

4. Computational Simulation

Consider the stochastic CO_2 emission system in (5). The values of system parameters are given in **Table 3** to fit the actual CO_2 emission in Western Europe [3]. In order to emphasize the influence of disturbances on the CO_2 emission system, the bounded standard deviations are assumed that $\delta_1 = r_1$, $\delta_2 = \alpha_1$, $\delta_3 = \alpha_2$, $\delta_4 = h$, $\delta_5 = \gamma$, *i.e.* the standard deviations of parametric fluctuations are equal to the original system parameters; $w(t)$ is a standard Wiener of presses with unit variance. The environmental disturbances $v_1(t) \sim v_3(t)$ are unknown but bounded signals. For the conventional of simulation, $v_1(t) \sim v_3(t)$ are assumed to be zero mean white noise with variances equal to 10^3, 10^2 and 10^4, respectively. To simulate the dynamic CO_2 emission in Western Europe, the initial values in 1960 are given as $\chi(t) = 398$ million tones of CO_2 emission [7], $z(t) = 43$ million m^3 of conifer forest area [31,32] and $y(t) = 2787$ billion international dollars of GDP [33-35]. The control efforts were assumed to be invariant from 1960 to 2010, *i.e.* $u_1 = 0.00012$, $u_2 = 0.0008$, $u_3 = 0$, to fit the actual data [3] (**Figure 2**). But these constant control efforts would limit the system behavior too rigid for actual performance demand, which could not guarantee the control ability of CO_2 emission under disturbances (**Figure 3**). In order to attenuate the effect of stochactic disturbance on CO_2 emission system and make a flexible control design for actual demand immediately, the robust minimax tracking control method will be applied after 2010.

Table 3. Model parameters for Western Europe [3].

Parameters	Values
r_1	0.15
s	700
h	0.0001
u_1	0.00012
u_2	0.0008
u_3	0
γ	0.035
α_1	0.0006
α_2	0.00005

For the robust minimax tracking control purpose, the reference model design requests a prescribed trajectory behavior for CO_2 emission system. Thus, the system matrix A_r and reference signal $r(t)$ should be specified, based on some standards in prior, to determine the transient response and steady state of the reference model, so that the desired reference signal can perform as a guideline for the tracking control system, for example, if the real parts of eigenvalues of A_r are more negative, the tracking control system follow a trajectory prescribed by $r(t)$ sooner. In Europe, consider the historical data starting at 1960, it is reasonable to assume an average growth rate of GDP around 3.5%, and the present growth rate of GDP is around 4% for Europe. Moreover, the change in total forest cover from 1990 to 2000 was positive due to reforestation, but corresponding to only 0.3%

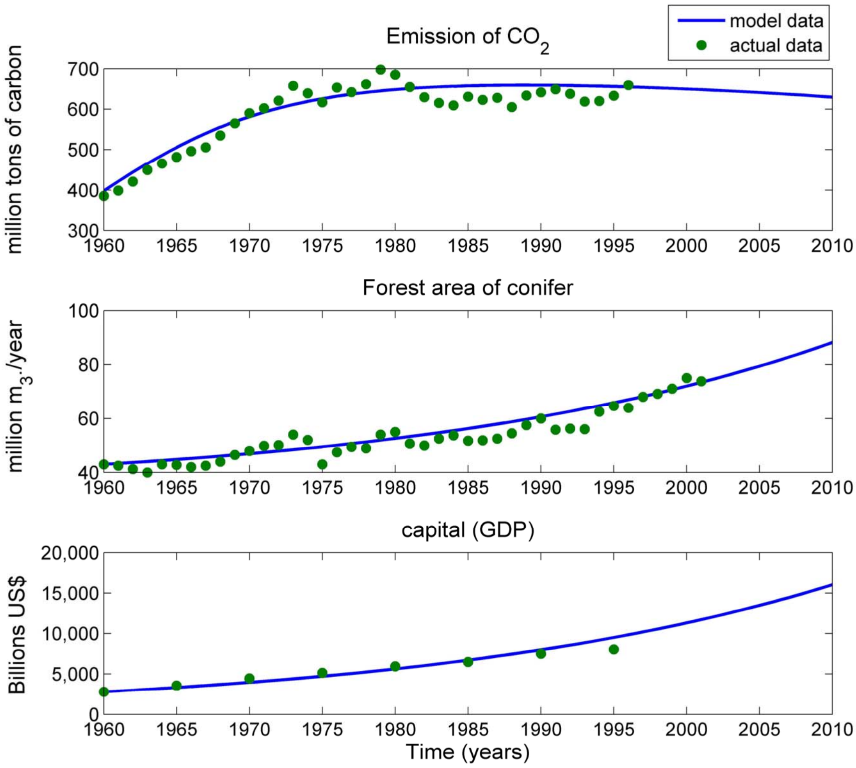

Figure 2. Simulation and comparison between model and actual data. To fit the actual data, the invariant control efforts: reforestation u_1, clean technology u_2 and CO_2 tax u_3, are assumed to be 0.012%, 0.08% and 0 respectively [3].

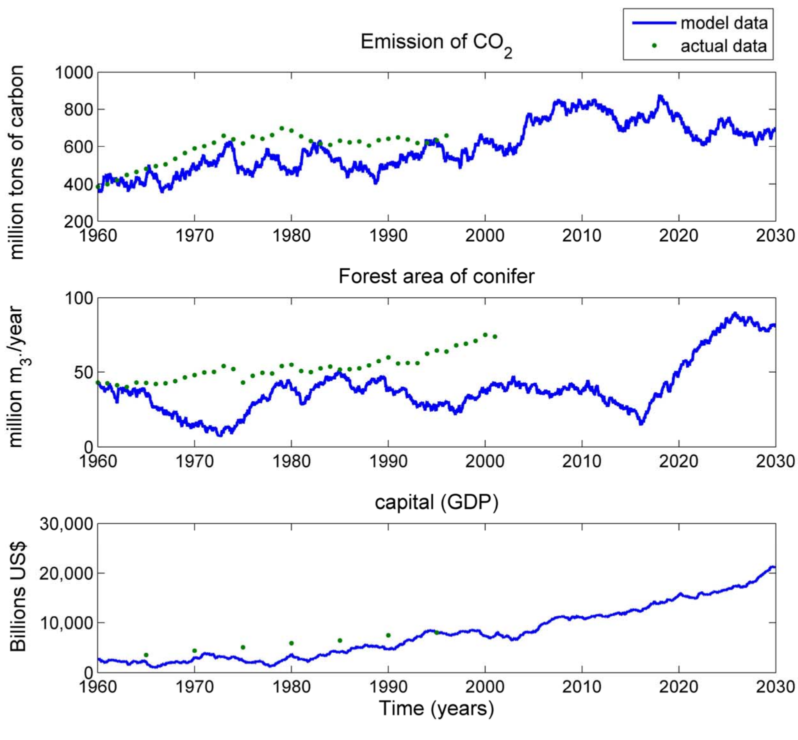

Figure 3. The CO_2 emission system with invariant control efforts under stochastic disturbance. It is seen that the control ability would not be guaranteed under parametric fluctuations and environmental noises.

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per year [3]. Thus, for the purpose of robust resource management control for CO_2 emission and reduction of greenhouse effect, the reference model is set via

$$A_r = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \quad r(t) = \begin{bmatrix} 398 \\ 19.31 \\ 3514.14 \end{bmatrix}$$

and the initial state value in 2010 as $x_r(0)=x(0)$ to simulate the desired progressive process of clean technology improvement, forest expansion and GDP increase after 2010. Therefore, based on the reference model, the CO_2 concentration could be decreased to the value in 1960, and GDP could reach a desired steady state that is prescribed without limiting the growth of GDP (**Figure 4**), *i.e*. the GDP growth can not be less than the original GDP growth rate 4% in Europe. And the expansion rate of forested area can also be higher than 0.3% until reach an appropriate value.

To avoid solving the HJI in **Theorem 1**, the T-S fuzzy model is employed to approximate the nonlinear stochastic system described in above section. For the convenience of control design, each state is taken with 3 operation points respectively, and triangle type membership functions are taken for the 27 Rules (**Figure 5**). In order to accomplish the robust minimax tracking performance of the desired reference signal, in spite of the worst influence of stochastic parametric fluctuation, environmental noise, and minimize the control efforts, a set of weighting matrices Q and R are tuned up as follows

$$Q = \begin{bmatrix} 10^{-4} & 0 & 0 \\ 0 & 10^{-4} & 0 \\ 0 & 0 & 10^{-4} \end{bmatrix}; R = \begin{bmatrix} 10^{4} & 0 & 0 \\ 0 & 10^{4} & 0 \\ 0 & 0 & 10^{4} \end{bmatrix}$$

i.e. with a heavy penalty on the control effort and a light penalty on the tracking error in (8).

After that the LMI-constraind optimization problem in (32) and (33) for the robust minimax tracking control can be solved by using Matlab Robust Control Toolbox. Finally, a minimum $\rho_0 = 1.39$ and the associated common positive definite symmetric matrix P can be obtained as follows

$$P = \begin{bmatrix} 0.0009 & 0.0007 & 0.0000 & -0.0009 & -0.0007 & -0.0000 \\ 0.0007 & 0.0015 & 0.0000 & -0.0007 & -0.0015 & -0.0000 \\ 0.0000 & 0.0000 & 0.0161 & -0.0000 & -0.0000 & -0.0161 \\ -0.0009 & -0.0007 & -0.0000 & 0.0022 & 0.0010 & 0.0000 \\ -0.0007 & -0.0015 & -0.0000 & 0.0010 & 0.0030 & 0.0000 \\ -0.0000 & -0.0000 & -0.0161 & 0.0000 & 0.0000 & 0.0252 \end{bmatrix}$$

Thus the robust minimax tracking control is designed according to these imperative parameters and matrix, *i.e.* $u^*(t) = -\sum_{j=1}^{27} h_j(z(t)) R^{-1} \bar{B}_j^T P \bar{x}(t)$ to track the desired reference signal to the end (**Figure 6**). In **Figure 7**, it shows the responses of the controlled CO_2 emission system with the robust minimax tracking control. As the CO_2 emission target is approached, both investments in reforestation and clean technology tend to decrease, and a positive carbon tax revenue could be achieved in the end. From the simulation results, it is seen that the effect of intrinsic parametric fluctuations and external disturbances on the reference model tracking of CO_2 emission system can be overcome efficiently by the proposed robust minimax tracking control design. Thus, the tracking performance of the robust minimax tracking control via T–S fuzzy interpolation is quite satisfactory.

5. Discussion

From the computer simulation, it is shown that the CO_2 emission system with invariant control efforts can fit the actual data perfectly from 1960 to 2010, but could not guarantee its performance under intrinsic or external disturbances (**Figure 3**). To achieve actual demands, *i.e.* the system can track an appropriate reference model as soon as possible without limiting GDP growth, forest area increase and guarantee CO_2 emission decrease under disturbances or modeling errors, the robust model reference tracking control is proposed from a dynamic game theory perspective, and then can be efficiently solved by fuzzy stochastic game approach.

By employing the robust minimax tracking controls $u^*(t)$ (**Figure 6**) instead of using the invariant controls from 2010 to 2030, the robust minimax tracking performance is guaranteed under an upper bound ρ_0, no matter what stochastic property of noise $v(t)$ and what value of the uncertain initial condition $x(0)$. Within the controlled period (2010~2030) (**Figure 7**), the government and companies following the robust minimax tracking control $u^*(t)$ can reduce CO_2 emission to a desired value without limiting forest area and GDP increasing by

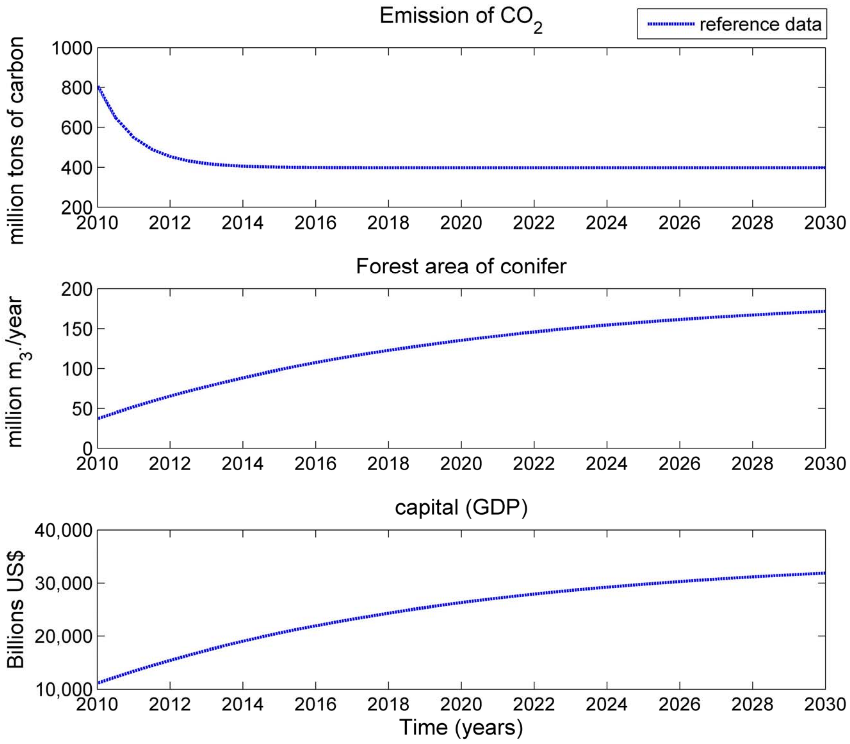

Figure 4. The desired trajectory of reference model for CO_2 emission system, from 2010 to 2030. The tracking control goal is to decrease CO_2 to the value in 1960, *i.e.* $\chi(t) = 398$ million tones and make sure the increasing rate of forest area and GDP are both higher than 4% in each year.

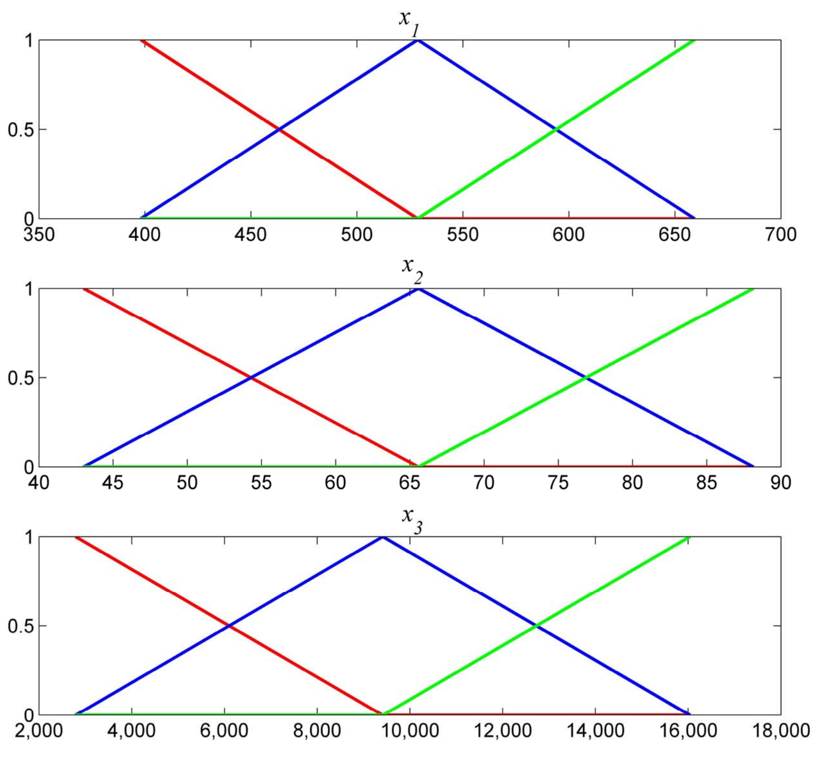

Figure 5. Membership functions for the three states x_1, x_2 and x_3 in the CO_2 emission system.

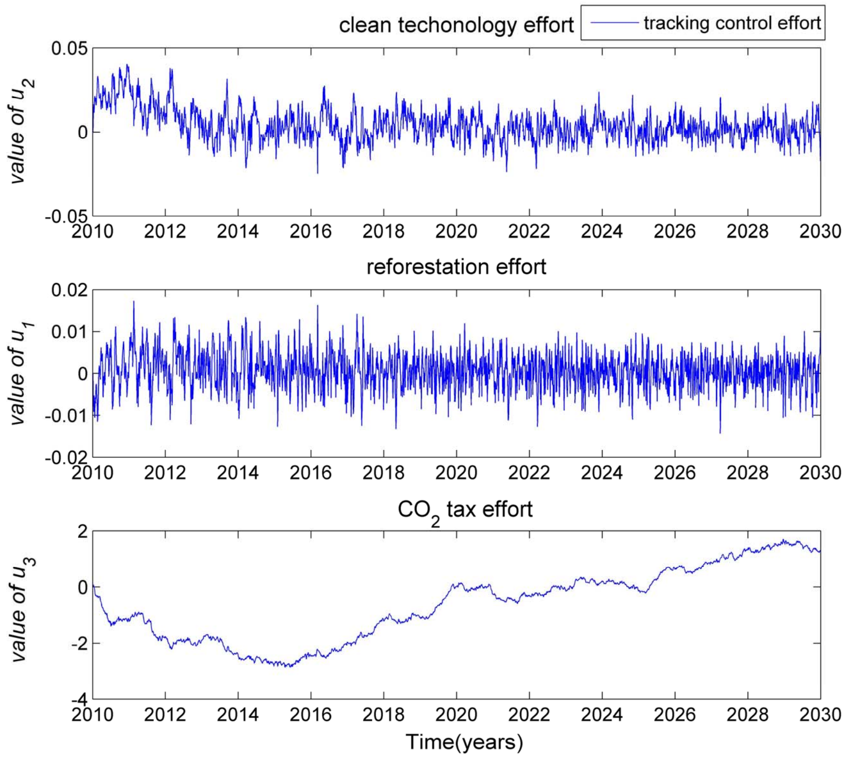

Figure 6. The robust minimax tracking control in the simulation example.

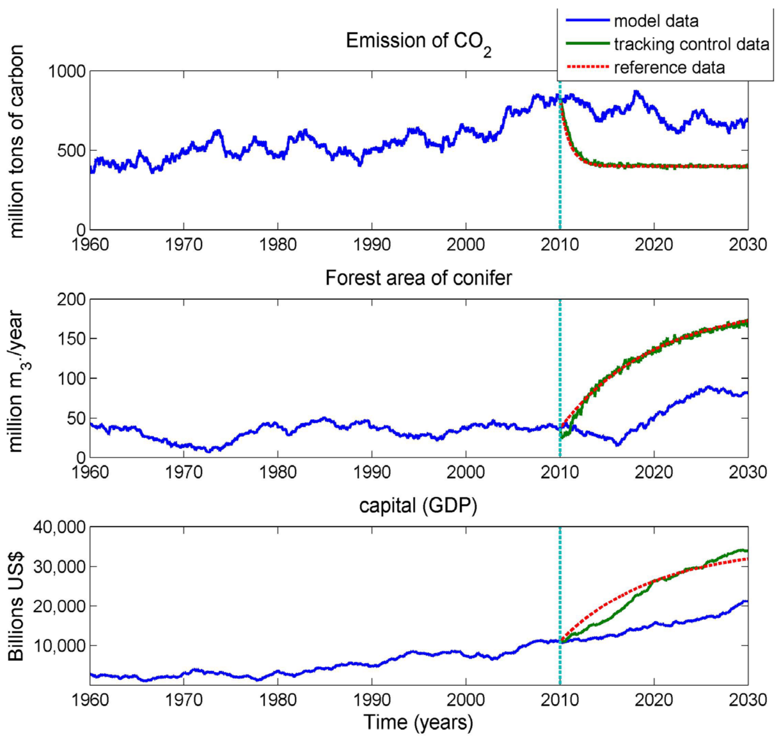

Figure 7. The tracking performance of CO_2 emission system to a desired reference model by the robust minimax tracking control, under the influence of parametric fluctuations and environmental noises.

managing expenditure in clean technology $u_2(t)$, reforestation $u_1(t)$ and CO_2 revenue $u_3(t)$.

From **Figure 6**, the control efforts of reforestation $u_1(t)$, and investment in innovating or improving clean technology $u_2(t)$, in simulation increase acutely during the early stage and then converge to the stable values, which clearly points out the urgent need of a concerted effort of reforestation and clean technology to change the CO_2 emission dynamics into a desired trajectory. From the perspective of desired economic development, at the beginning, companies produced more pollutions (which means they emit more CO_2) may also have more economic contributions to GDP growth [36] and then GDP deteriorates because of the cost of global warming and greenhouse effect until a balance. The control effort of CO_2 revenue $u_3(t)$ in our simulation also interprets this situation reasonably by an acute descent to negative and then climb to positive gently, which means to ensure the unlimited GDP growth and following a desired reference model, the government should provide a financial subsidy to improve industrial throughput in early years, even it creates more pollution, until the scale enterprises can bear the loss of carbon tax.

By tuning the weighting matrices of error punishment Q and control cost R, more situation in reality can be taken account, because the cost of control inputs may not be invariant when the robust minimax tracking control starts. In this study, it is shown that the tracking error is punished by a low Q and a high control cost R, which means to guarantee the robust minimax tracking control performance, the control strategy can endure more tracking error by using less control, thus making the control method efficiently and viably.

If the CO_2 emission model is free of external disturbance, *i.e.* $c=0$, the robust tracking control performance has a lower suboptimal upper bound $\rho_0=0.41$. It implies that if the measurements of states are more accurate, and the controlled system could track the desired reference trajectory more sophistically, which means the control design for the CO_2 emission system is more precisely.

Although international cooperation from tradable quotas and permits can reduce CO_2 emission efficiently, uncertainties about compliance costs have caused countries to withdraw from negotiations. Without tuning any system parameters, these time-invariant control efforts could make the CO_2 emission system too rigid to respond for an immediate need or lead the CO_2 emission system toward an uncontrollable circumstance under disturbance, which may finally lose its control ability for actual dynamics of CO_2 emission system. Optimal control method without take account of the effect of intrinsic and external disturbances in the design procedure could even not guarantee the control performance. If the more flexible CO_2 emission targets can be made to incorporate optimum choices of investments with minimum impact on the GDP growth, *i.e.* taking account of the stochastic disturbances with respect to minimax tracking control problem, then climate agreements for reducing greenhouse effect may become more attractive and efficient [25,37].

In this study, the fuzzy interpolation technique is employed to approximate the nonlinear CO_2 emission system, so that LMIs technique is used to efficiently solve the nonlinear minimax optimization problem in our robust minimax tracking design procedure. Since the proposed robust minimax tracking control design can efficiently control the CO_2 emission in real time to protect environment from the global warming and reduce greenhouse effect, in the future, the applications of robust minimax tracking control design for environmental resource conservation and pollution control under stochastic disturbance would be potential in ecological and economic field.

6. Conclusions

If current GHG concentrations remain constant, the world would be committed to several centuries of increasing global mean temperatures and sea level rise. Slowing such climate change requires overcoming inertia in political, technological, and geophysical systems. To efficiently manage the resources commitment for decreasing the atmospheric CO_2, mathematical methods have been proposed to help people make decision. However, how to ensure the desired CO_2 emission performance under stochastic disturbances is still important and infancy. In this study, based on robust control theory and dynamic game theory, a nonlinear stochastic game problem is equivalent to a nonlinear robust minimax tracking problem, for controlling the CO_2 emission system to achieve a desired time response under the influence of parametric fluctuations, environmental noises and unknown initial conditions.

To solve the nonlinear HJI-constrained problem for the robust minimax tracking control design is generally difficult. Instead of solving the HJI-constrained problem, a fuzzy stochastic game approach is proposed to transform this nonlinear robust minimax tracking control problem into a set of equivalent linear robust minimax problems. Such transformation allows us to solve an equivalent LMI-constrained problem for this robust minimax tracking control design in an easier way with the help of Robust Control Toolbox in Matlab.

This robust minimax tracking method not only considers the parametric fluctuation and environmental noise

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but also guarantees the tracking performance in a suboptimal condition. And the unknown initial condition of the system also be considered as a random factor, thus this method can be used to control the CO_2 emission system tracking around any feasible reference model whenever the control of this system starts. Although this theoretical method rests on the conservative suboptimal method, this fact doesn't frustrate its potential as a government policy guideline and the power of prediction in public decision-making. Once these obstacles have been surmounted, *i.e*. more rapid response by real time monitor via e-government implementation, this method would be powerful to control and manage the economic and ecological resource. What is more is that for its convenient and efficient control design for nonlinear systems with parametric fluctuation and stochastic uncertainties, this dynamic game approach can be applied in other fields with similar demands.

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Appendix A: Proof of Theorem 1

Equation Section (Next)

For the augmented stochastic system in (14), let us denote a Lyapunov energy function $V(\bar{x}(t))>0$ for $\bar{x}(0)\neq 0$ with $V(0)=0$. Then the regulation problem in (15) is equivalent to the following minimax problem

$$\min_{u(t)}\max_{\bar{v}(t)} J(u(t),\bar{v}(t))=\min_{u(t)}\max_{\bar{v}(t)} E\left\{V(\bar{x}(0))-V(\bar{x}(t_f))+\int_0^{t_f}\left(\bar{x}^T(t)\bar{Q}\bar{x}(t)+u^T(t)Ru(t)-\rho^2\bar{v}^T(t)\bar{v}(t)+\frac{dV(\bar{x}(t))}{dt}\right)dt\right\},\forall\bar{x}(0) \quad \text{(A1)}$$

By Ito formula in **Lemma 1** [21,22], we get

$$\begin{aligned}\frac{\mathrm{d}V(\bar{x}(t))}{\mathrm{d}t}&=\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t}+\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2V(\bar{x}(t))}{\partial\bar{x}^2(t)}H(\bar{x}(t))\\&=\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T\left(\left(F(\bar{x}(t))+G(\bar{x}(t))u(t)+\bar{C}\bar{v}(t)\right)+H(\bar{x}(t))\frac{\mathrm{d}w(t)}{\mathrm{d}t}\right)+\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2V(\bar{x}(t))}{\partial\bar{x}^2(t)}H(\bar{x}(t))\end{aligned} \quad \text{(A2)}$$

Substituting (A2) to (A1) and by the fact that $E\left\{\frac{\mathrm{d}w(t)}{\mathrm{d}t}\right\}=0$, we get

$$\begin{aligned}&\min_{u(t)}\max_{\bar{v}(t)} J(u(t),\bar{v}(t))\\&=\min_{u(t)}\max_{\bar{v}(t)} E\Bigg\{V(\bar{x}(0))-V(\bar{x}(t_f))\\&\quad+\int_0^{t_f}\Bigg(\bar{x}^T(t)\bar{Q}\bar{x}(t)+u^T(t)Ru(t)-\rho^2\bar{v}^T(t)\bar{v}(t)+\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T F(\bar{x}(t))+\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T G(\bar{x}(t))u(t)\\&\quad+\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T\bar{C}\bar{v}(t)+\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2V(\bar{x}(t))}{\partial\bar{x}^2(t)}H(\bar{x}(t))\Bigg)\mathrm{d}t\Bigg\},\forall\bar{x}(0)\\&=\min_{u(t)}\max_{\bar{v}(t)} E\Bigg\{V(\bar{x}(0))-V(\bar{x}(t_f))\\&\quad+\int_0^{t_f}\Bigg(\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T F(\bar{x}(t))+\bar{x}^T(t)\bar{Q}\bar{x}(t)-\frac{1}{4}\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T G(\bar{x}(t))R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}+\frac{1}{4\rho^2}\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T\bar{C}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\\&\quad+\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2V(\bar{x}(t))}{\partial\bar{x}^2(t)}H(\bar{x}(t))+\left(u(t)+\frac{1}{2}R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T R\left(u(t)+\frac{1}{2}R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)\\&\quad-\left(\rho\bar{v}(t)-\frac{1}{2\rho}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T\left(\rho\bar{v}(t)-\frac{1}{2\rho}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)\Bigg)\mathrm{d}t\Bigg\},\forall\bar{x}(0)\end{aligned}$$

Then the minimax solution is given as follows

$$\begin{aligned}J(u^*(t),\bar{v}^*(t))&=E\Bigg\{V(\bar{x}(0))-V(\bar{x}(t_f))\\&\quad+\int_0^{t_f}\Bigg(\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T F(\bar{x}(t))+\bar{x}^T(t)\bar{Q}\bar{x}(t)-\frac{1}{4}\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T G(\bar{x}(t))R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\\&\quad+\frac{1}{4\rho^2}\left(\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}\right)^T\bar{C}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}+\frac{1}{2}H^T(\bar{x}(t))\frac{\partial^2V(\bar{x}(t))}{\partial\bar{x}^2(t)}H(\bar{x}(t))\Bigg\},\forall\bar{x}(0)\end{aligned}$$

with

$$u^*(t)=-\frac{1}{2}R^{-1}G^T(\bar{x}(t))\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)},\quad \bar{v}^*(t)=\frac{1}{2\rho^2}\bar{C}^T\frac{\partial V(\bar{x}(t))}{\partial\bar{x}(t)}.$$

If Equation (19) holds, then

$$J\left(u^*(t),\bar{v}^*(t)\right)\le E\left\{V\left(\bar{x}(0)\right)-V\left(\bar{x}\left(t_f\right)\right)\right\}.$$

From the inequality in (16), the minimax solution should be less than $\rho^2E\left\{\bar{x}^T(0)\bar{I}\bar{x}(0)\right\}$.

After that the inequality in (20) is obtained as follows

$$J\left(u^*(t),\bar{v}^*(t)\right)\le E\left\{V\left(\bar{x}(0)\right)-V\left(\bar{x}\left(t_f\right)\right)\right\}\le E\left\{V\left(\bar{x}(0)\right)\right\}\le\rho^2E\left\{\bar{x}^T(0)\bar{I}\bar{x}(0)\right\},\forall\bar{x}(0).$$

Q.E.D.

Appendis B: Proof of Theorem 2

Equation Section (Next)

For the fuzzy system in (27), let us denote a Lyapunov function $V\left(\bar{x}(t)\right)=\bar{x}^T(t)P\bar{x}(t)>0$, for $\bar{x}(0)\ne0$ with $V(0)=0$. Then the minimax regulation problem in (16) is equivalent to the following

$$\min_{u(t)}\max_{\bar{v}(t)}J\left(u(t),\bar{v}(t)\right)=\min_{u(t)}\max_{\bar{v}(t)}E\left\{\bar{x}^T(0)P\bar{x}(0)-\bar{x}^T\left(t_f\right)P\bar{x}\left(t_f\right)+\int_0^{t_f}\left(\bar{x}^T(t)\bar{Q}\bar{x}(t)+u^T(t)Ru(t)-\rho^2\bar{v}^T(t)\bar{v}(t)+\frac{\mathrm{d}V\left(\bar{x}(t)\right)}{\mathrm{d}t}\right)\mathrm{d}t\right\},\forall\bar{x}(0) \tag{B1}$$

By Ito formula in **Lemma 1** [21,22], we get

$$\frac{\mathrm{d}V\left(\bar{x}(t)\right)}{\mathrm{d}t}=2\bar{x}^T(t)P\sum_{i=1}^{L}h_i\left(z(t)\right)\left(\bar{A}_i\bar{x}(t)+\bar{B}_iu(t)+\bar{C}\bar{v}(t)+\bar{D}_i\frac{dw(t)}{dt}\right)+\sum_{i=1}^{L}\sum_{j=1}^{L}h_i\left(z(t)\right)h_j\left(z(t)\right)\bar{x}^T(t)\bar{D}_i^TP\bar{D}_j\bar{x}(t) \tag{B2}$$

Substituting (B2) to (B1) and by the fact that $E\left\{\frac{\mathrm{d}w(t)}{\mathrm{d}t}\right\}=0$, we get

$$\begin{aligned}
&\min_{u(t)}\max_{\bar{v}(t)}E\Big\{\bar{x}^T(0)P\bar{x}(0)-\bar{x}^T\left(t_f\right)P\bar{x}\left(t_f\right)\\
&\quad+\int_0^{t_f}\Big(\bar{x}^T(t)\bar{Q}\bar{x}(t)+u^T(t)Ru(t)-\rho^2\bar{v}^T(t)\bar{v}(t)+2\bar{x}^T(t)P\cdot\sum_{i=1}^{L}h_i\left(z(t)\right)\left(\bar{A}_i\bar{x}(t)+\bar{B}_iu(t)+\bar{C}\bar{v}(t)\right)\\
&\quad+\sum_{i=1}^{L}\sum_{j=1}^{L}h_i\left(z(t)\right)h_j\left(z(t)\right)\bar{x}^T(t)\bar{D}_i^TP\bar{D}_j\bar{x}(t)\Big)\mathrm{d}t\Big\}\\
&=\min_{u(t)}\max_{\bar{v}(t)}E\Big\{\bar{x}^T(0)P\bar{x}(0)-\bar{x}^T\left(t_f\right)P\bar{x}\left(t_f\right)\\
&\quad+\int_0^{t_f}\Big(\bar{x}^T(t)\bar{Q}\bar{x}(t)+\sum_{i=1}^{L}h_i\left(z(t)\right)\left(\bar{x}^T(t)P\bar{A}_i\bar{x}(t)+\bar{x}^T(t)\bar{A}_i^TP\bar{x}(t)\right)+\frac{1}{\rho^2}\bar{x}^T(t)P\bar{C}\bar{C}^TP\bar{x}(t)\\
&\quad-\sum_{i=1}^{L}\sum_{j=1}^{L}h_i\left(z(t)\right)h_j\left(z(t)\right)\bar{x}^T(t)P\bar{B}_iR^{-1}\bar{B}_j^TP\bar{x}(t)+\sum_{i=1}^{L}\sum_{j=1}^{L}h_i\left(z(t)\right)h_j\left(z(t)\right)\bar{x}^T(t)\bar{D}_i^TP\bar{D}_j\bar{x}(t)\\
&\quad+\left(Ru(t)+\sum_{i=1}^{L}h_i\left(z(t)\right)\bar{B}_i^TP\bar{x}(t)\right)^TR^{-1}\left(Ru(t)+\sum_{i=1}^{L}h_j\left(z(t)\right)\bar{B}_j^TP\bar{x}(t)\right)-\left(\rho\bar{v}(t)-\frac{1}{\rho}\bar{C}^TP\bar{x}(t)\right)^T\left(\rho\bar{v}(t)-\frac{1}{\rho}\bar{C}^TP\bar{x}(t)\right)\Big)\mathrm{d}t\Big\}\\
&=\min_{u(t)}\max_{\bar{v}(t)}E\Big\{\bar{x}^T(0)P\bar{x}(0)-\bar{x}^T\left(t_f\right)P\bar{x}\left(t_f\right)\\
&\quad+\int_0^{t_f}\Big(\sum_{i=1}^{L}\sum_{j=1}^{L}h_i\left(z(t)\right)h_j\left(z(t)\right)\bar{x}^T(t)\left(P\bar{A}_i+\bar{A}_i^TP+\bar{Q}-P\bar{B}_iR^{-1}\bar{B}_j^TP+\frac{1}{\rho^2}P\bar{C}\bar{C}^TP+\bar{D}_i^TP\bar{D}_j\right)\bar{x}(t)\\
&\quad+\left(Ru(t)+\sum_{i=1}^{L}h_i\left(z(t)\right)\bar{B}_i^TP\bar{x}(t)\right)^TR^{-1}\left(Ru(t)+\sum_{i=1}^{L}h_i\left(z(t)\right)\bar{B}_i^TP\bar{x}(t)\right)-\left(\rho\bar{v}(t)-\frac{1}{\rho}\bar{C}^TP\bar{x}(t)\right)^T\left(\rho\bar{v}(t)-\frac{1}{\rho}\bar{C}^TP\bar{x}(t)\right)\Big)\mathrm{d}t\Big\}
\end{aligned}$$

The minimax solution is given as follows

$$J\left(u^*(t),\bar{v}^*(t)\right)=E\Big\{\bar{x}^T(0)P\bar{x}(0)-\bar{x}^T\left(t_f\right)P\bar{x}\left(t_f\right)$$
$$+\int_0^{t_f}\left(\sum_{i=1}^{L}\sum_{j=1}^{L}h_i\left(z(t)\right)h_j\left(z(t)\right)\bar{x}^T(t)\left(P\bar{A}_i+\bar{A}_i^T P+\bar{Q}-P\bar{B}_i R^{-1}\bar{B}_j^T P+\frac{1}{\rho^2}P\bar{C}\bar{C}^T P+\bar{D}_i^T P\bar{D}_j\right)\bar{x}(t)\right)\mathrm{d}t\Big\}$$

with

$$u^*(t)=-\sum_{j=1}^{L}h_j\left(z(t)\right)R^{-1}\bar{B}_j^T P\bar{x}(t),$$

$$\bar{v}^*(t)=\frac{1}{\rho^2}\bar{C}^T P\bar{x}(t).$$

In order to simplify the above equation, suppose the inequality in (29) hold, then

$$\begin{aligned}&\min_{u(t)}\max_{\bar{v}(t)}J\left(u(t),\bar{v}(t)\right)=J\left(u^*(t),\bar{v}^*(t)\right)\\&\le E\left\{\bar{x}^T(0)P\bar{x}(0)-\bar{x}^T\left(t_f\right)P\bar{x}\left(t_f\right)\right\}\\&\le E\bar{x}^T(0)P\bar{x}(0)\le\rho^2 E\left(\bar{x}^T(0)\bar{I}\bar{x}(0)\right)\end{aligned}$$

i.e. $P\le\rho^2\bar{I}$.

Q.E.D.

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