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No Degeneracy of the Ground State for the Impact Parameter Model

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Abstract

A time dependent Hamiltonian associated to the impact parameter model for the scattering of a light particle and two heavy ones is considered. Existence and non degeneracy of the ground state is shown.

Keywords: Impact Parameter Model, Non Degeneracy of the Ground State

1. Introduction

In [1,2], the impact parameter model for the scattering of two heavy particles and a light one is studied, where it is assumed that the heavy particles are infinitely massive and that their motion along a classical trajectory is not affected by the light particle. Also, rigorous proof from first principles of the validity of Massey's criterion is given [1,3].

The above mentioned results were proved for a simple Hamiltonian, by means of an adiabatic argumentation. Now we study a more complicated one than in [1], where a precise knowledge of the discrete spectrum of the corresponding Hamiltonian was needed.

A physical ground state is a state of minimal energy, and therefore it has a relevant role in quantum theories. See for instance [4-17].

In this work we prove non degeneracy of the ground state for the Hamiltonian

$$H(t) = -\frac{1}{2}\Delta - \lambda_1 V_1 - \mu_1 V_2 - \lambda_2 V_{1,\rho} - \mu_2 V_{2,\rho}, \qquad (1)$$

defined as an operator in the Hilbert space $L^2(\mathbb{R}^n)$ of all complex valued Lebesgue measurable square integrable functions on $\mathbb{R}^n$, with domain $H^2(\mathbb{R}^n)$, the Sobolev space of order two [18]. Δ is the Laplace operator [11].

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

with derivatives in the distribution sense, and, λ_1, λ_2, μ_1, μ_2 are positive constants. Also, for $k = 1,2$, we will take the potentials V_k of rank one:

$$V_k\varphi = (g_k, \varphi) g_k, \quad \forall \varphi \in L^2(\mathbb{R}^n), \qquad (2)$$

with g_1, g_2 fixed elements in $L^2(\mathbb{R}^n)$. Here $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbb{R}^n)$, antilinear on the factor on the left. Moreover,

$$V_{\kappa,\rho}\varphi = (g_{\kappa,\rho}, \varphi) g_{\kappa,\rho}, \quad g_{\kappa,\rho}(x) := g_\kappa(x - \rho(t)), \qquad (3)$$

$\rho(t)$ being a continuous function on $\mathbb{R}$ with values in $\mathbb{R}^n$ satisfying $\rho(0) = 0 \in \mathbb{R}^n$ and

$$\lim_{|t|\to\infty} |\rho(t)| = \infty.$$

We denote by ^ the Fourier transform [19], as an unitary operator in $L^2(\mathbb{R}^n)$:

$$\hat{g}(\boldsymbol{p}) = \lim_{K\to\infty} \int_{|x|\le K} \mathrm{e}^{-ipx} g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \quad g \in L^2(\mathbb{R}^n),$$

where the limit is taken in the L^2-norm.

2. Main Theorem

From Weyl's theorem [16], one knows that for each $t \in \mathbb{R}$, $H(t)$ is a self-adjoint operator with discrete spectrum in $(-\infty, 0)$. The eigenvector corresponding to the infimum of the spectrum of $H(t)$ is called the ground state for $H(t)$. The following theorem was proved in [20].

Theorem 2.1. *For* $i=1,2$, *let* $g_i \in L^2(\mathbb{R}^n)$ *and* $\hat{g}_1$ *nonnegative functions obeying* $|\boldsymbol{p}|\hat{g}_1 \in L^2(\mathbb{R})$. *Moreover, we suppose the constants* λ_i, μ_i *in Equation* (1) *satisfy*

$$\lambda_1 > \mu_1 + \mu_2 > \mu_1 > \lambda_2 > \mu_2 > 0.$$

such that $0 < E_0(2) < E_1$ *and* $0 < E_{\lambda_2} < E_{\mu_1}$. *Here* $-E_1, -E_0(2), -E_{\lambda_2}$, *and* $-E_{\mu_1}$ *are the ground state eigenvalues associated to*

$$-\frac{1}{2}\Delta - \lambda_1 V_1,\ -\frac{1}{2}\Delta - (\mu_1+\mu_2)V_2,$$
$$-\frac{1}{2}\Delta - \lambda_2 V_1,\ -\frac{1}{2}\Delta - \mu_1 V_2,$$

respectively. Then the following statements are valid:

1) *The eigenvalue* $-E_0$, *corresponding to the ground state for the operator*

$$H(0) = -\frac{1}{2}\Delta - (\lambda_1+\lambda_2)V_1 - (\mu_1+\mu_2)V_2,$$

and the eigenvalue $-E_\infty$, *corresponding to the ground state for the operator*

$$H(\pm\infty) = -\frac{1}{2}\Delta - \lambda_1 V_1 - \mu_1 V_2,$$

are strictly negative and the inequality $-E_0 < -E_\infty$ *holds.*

2) *The eigenvale* $-E(t)$, *corresponding to the ground state for* $H(t)$ *for all* $t\in\mathbb{R}$ *lies in the interval* $[-E_0, -E_\infty)$.

3) *In the interval* $(-E_\infty, -E_1]$ *there are no eigenvalues of* $H(t)$ *for every* $t \in \mathbb{R}$.

We mention that for a given function $0 \neq g \in L^2(\mathbb{R}^n)$, one can find a sufficiently large positive constant α_0 such that the operator

$$-\frac{1}{2}\Delta - \alpha(g,\cdot)g \qquad (4)$$

has a (unique) negative eigenvalue $-E_\alpha$ for $\alpha \ge \alpha_0$. In fact, $-E$ is a negative eigenvalue iff [1]

$$\frac{1}{\alpha} = \left\| \frac{\hat{g}}{\left(\frac{p^2}{2}+E\right)^{1/2}} \right\|^2, \qquad (4)$$

where we denote $p^2 := |\boldsymbol{p}|^2$. Note also that for a given *g* the right hand side of (5) is a monotone decreasing function of *E*. Therefore, given functions g_i in $L^2(\mathbb{R}^n)$ one can find constants $\lambda_i, \mu_i\ (i=1,2)$ large enough for the hypotheses of the theorem to hold.

We will prove in this manuscript that under the hypotheses of theorem 2.1, for $t\in\mathbb{R}$ the ground state of $H(t)$ is not degenerate.

Let $-E(t)$ be the ground state eigenvalue of the time dependent operator given by Equation (1). We define $\Theta(p) := \frac{p^2}{2} + E(t)$ and

$$a_{ii} = \frac{1}{\lambda_i} - \left\|\frac{\hat{g}_1}{\sqrt{\Theta}}\right\|^2;\ d_{ii} = \frac{1}{\mu_i} - \left\|\frac{\hat{g}_2}{\sqrt{\Theta}}\right\|^2 \ \text{ for } i=1,2. \qquad (6)$$

Moreover,

$$\begin{aligned} a_{12} &= -\left(\hat{g}_1, \Theta^{-1}\hat{g}_{1,\rho}\right),\\ b_{11} &= -\left(\hat{g}_{\mathbf{1}}, \Theta^{-1}\hat{g}_2\right),\\ b_{12} &= -\left(\hat{g}_2, \Theta^{-1}\hat{g}_{1,\rho}\right),\\ b_{21} &= -\left(\hat{g}_{\mathbf{2},\rho}, \Theta^{-1}\hat{g}_1\right),\\ d_{12} &= -\left(\hat{g}_{\mathbf{2}}, \Theta^{-1}\hat{g}_{2,\rho}\right) \end{aligned} \qquad (7)$$

Lemma 2.1. *Let* $-E(t)$ *be the ground state eigenvalue of the time dependent operator* $H(t)$ *given by Equation* (1). *Then, the matrix equation*

$$M = \begin{pmatrix} \boldsymbol{x} \\ \\ \boldsymbol{y} \end{pmatrix} = 0 \in \mathbb{R}^4,$$
$$M = \begin{pmatrix} A & B^T \\ B & D \end{pmatrix} := \begin{Bmatrix} a_{11} & a_{12} & b_{11} & b_{21} \\ a_{12} & a_{22} & b_{12} & b_{11} \\ b_{11} & b_{12} & d_{11} & d_{12} \\ b_{21} & b_{11} & d_{12} & d_{22} \end{Bmatrix}, \qquad (8)$$

has a nontrivial solution. Furthermore

$$\det(D) \equiv \det\left[\begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}\right] > 0 \quad (\forall t \in \mathbb{R}).$$

Proof: Let $\psi(t)$ the eigenvector for $H(t)$ with respective eigenvalue $-E(t)$, then the Fourier transform of $\psi(t)$ is given by

$$\begin{aligned} \hat{\psi}(t) &= \lambda_1(g_1,\psi)\frac{\hat{g}_1}{\Theta(\boldsymbol{p})} + \lambda_2(g_{1,\rho},\psi)\frac{\hat{g}_{1,\rho}}{\Theta(\boldsymbol{p})} \\ &\quad + \mu_1(g_2,\psi)\frac{\hat{g}_2}{\Theta(\boldsymbol{p})} + \mu_2(g_{2,\rho},\psi)\frac{\hat{g}_{2,\rho}}{\Theta(\boldsymbol{p})}, \end{aligned} \qquad (9)$$

where $\Theta(\boldsymbol{p}) := \frac{p^2}{2} + E(t)$. The Plancherel theorem implies that $(u,v) = (\hat{u},\hat{v})\,\forall u,v \in L^2(\mathbb{R}^n)$. Taking inner products in (9) with $\hat{g}_1$ and $\hat{g}_{1,\rho}$ for $i=1,2$, we get

$$
\begin{aligned}
\left(\hat{g}_1,\hat{\psi}\right) &= \lambda_1\left(\hat{g}_1,\hat{\psi}\right)\left\|\frac{\hat{g}_1}{\sqrt{\Theta}}\right\|^2 + \lambda_2\left(\hat{g}_{1,\rho},\hat{\psi}\right)\left(\hat{g}_1,\Theta^{-1}\hat{g}_{1,\rho}\right) + \mu_1\left(\hat{g}_2,\hat{\psi}\right)\left(\hat{g}_1,\Theta^{-1}\hat{g}_2\right) + \mu_2\left(\hat{g}_{2,\rho},\hat{\psi}\right)\left(\hat{g}_1,\Theta^{-1}\hat{g}_{2,\rho}\right),\\
\left(\hat{g}_{1,\rho},\hat{\psi}\right) &= \lambda_1\left(\hat{g}_1,\hat{\psi}\right)\left(\hat{g}_{1,\rho},\Theta^{-1}\hat{g}_1\right) + \lambda_2\left(\hat{g}_{1,\rho}\hat{\psi}\right)\left\|\frac{\hat{g}_{1,\rho}}{\sqrt{\Theta}}\right\|^2 + \mu_1\left(\hat{g}_2,\hat{\psi}\right)\left(\hat{g}_{1,\rho},\Theta^{-1}\hat{g}_2\right) + \mu_2\left(\hat{g}_{2,\rho},\hat{\psi}\right)\left(\hat{g}_{1,\rho},\Theta^{-1}\hat{g}_{2,\rho}\right),\\
\left(\hat{g}_2,\hat{\psi}\right) &= \lambda_1\left(\hat{g}_1,\hat{\psi}\right)\left(\hat{g}_2,\Theta^{-1}\hat{g}_1\right) + \lambda_2\left(\hat{g}_{1,\rho},\hat{\psi}\right)\left(\hat{g}_2,\Theta^{-1}\hat{g}_{1,\rho}\right) + \mu_1\left(\hat{g}_2,\hat{\psi}\right)\left\|\frac{\hat{g}_2}{\sqrt{\Theta}}\right\|^2 + \mu_2\left(\hat{g}_{2,\rho},\hat{\psi}\right)\left(\hat{g}_2,\Theta^{-1}\hat{g}_{2,\rho}\right),\\
\left(\hat{g}_{2,\rho},\hat{\psi}\right) &= \lambda_1\left(\hat{g}_1,\hat{\psi}\right)\left(\hat{g}_{2,\rho},\Theta^{-1}\hat{g}_1\right) + \lambda_2\left(\hat{g}_{1,\rho},\hat{\psi}\right)\left(\hat{g}_{2,\rho},\Theta^{-1}\hat{g}_{1,\rho}\right) + \mu_1\left(\hat{g}_2,\hat{\psi}\right)\left(\hat{g}_{2,\rho},\Theta^{-1}\hat{g}_2\right) + \mu_2\left(\hat{g}_{2,\rho},\hat{\psi}\right),\left\|\frac{\hat{g}_{2,\rho}}{\sqrt{\Theta}}\right\|^2.
\end{aligned}
\tag{10}
$$

This system of equations is represented in matrix form precisely by Equation (8), where

$$
\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda_1\left(\hat{g}_1,\hat{\psi}\right) \\ \lambda_2\left(\hat{g}_{1,\rho},\hat{\psi}\right) \end{pmatrix};\qquad
\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} \mu_1\left(\hat{g}_2,\hat{\psi}\right) \\ \mu_2\left(\hat{g}_{2,\rho},\hat{\psi}\right) \end{pmatrix}
\tag{11}
$$

From Theorem 2.1 we deduce the existence of a non-trivial solution to Equation (8).

Now we fix $E > 0$. For every $t \in \mathbb{R}$ let us consider the function,

$$
R_E(t) = \left(\frac{1}{\mu_1} - \left\|\frac{\hat{g}_2}{\left(\frac{p^2}{2}+E\right)^{1/2}}\right\|^2\right)\left(\frac{1}{\mu_2} - \left\|\frac{\hat{g}_2}{\left(\frac{p^2}{2}+E\right)^{1/2}}\right\|^2\right) - \left(\hat{g}_2, \frac{\hat{g}_{2,\rho}}{\frac{p^2}{2}+E}\right)^2
$$

and observe that for $E_0(2) < E$,

$$
\begin{aligned}
R_E(t) &= \frac{1}{\mu_1\mu_2} - \frac{\mu_1+\mu_2}{\mu_1\mu_2}\left\|\frac{\hat{g}_2}{\left(\frac{p^2}{2}+E\right)^{1/2}}\right\|^2 + \left\|\frac{\hat{g}_2}{\left(\frac{p^2}{2}+E\right)^{1/2}}\right\|^4 - \left(\hat{g}_2,\frac{\hat{g}_{2,\rho}}{\frac{p^2}{2}+E}\right)^2 \geq \frac{1}{\mu_1\mu_2} - \frac{\mu_1+\mu_2}{\mu_1\mu_2}\left\|\frac{\hat{g}_2}{\left(\frac{p^2}{2}+E\right)^{1/2}}\right\|^2\\
&= \frac{\mu_1+\mu_2}{\mu_1\mu_2}\left(\frac{1}{\mu_1+\mu_2} - \left\|\frac{\hat{g}_2}{\left(\frac{p^2}{2}+E\right)^{1/2}}\right\|^2\right) > 0.
\end{aligned}
\tag{12}
$$

The last inequality being true because of the remark following Equation (5). Also, we have used the Schwarz inequality and the Fourier transform property $\hat{g}_{2,\rho}(\boldsymbol{p}) = e^{-i\rho(t)\cdot\boldsymbol{p}}\hat{g}_2(\boldsymbol{p})$. When $-E \equiv -E(t)$ is the eigenvalue for $H(t)$, then the determinant of matrix D in Equation (8) satisfies, $\det(D) = R(E(t))$. Theorem 2.1 states that $-E(t) \in [-E_0, -E_\infty)$ and $E_\infty > E_1 > E_0(2)$. Then, (12) gives $\det(D) > 0$. □

The main result will be proved by showing that the dimension of the eigenspace associated to the ground sate remains constant over time.

Lema 2.2. *The ground state for the operator*

$$
H(0) = -\frac{1}{2}\Delta - (\lambda_1+\lambda_2)V_1 - (\mu_1+\mu_2)V_2
$$

is not degenerate.

Proof: Lemma (2.1) assures that D^{-1} exists. Equation (8) implies,

$$
\boldsymbol{y} = -D^{-1}B\boldsymbol{x},\quad \left(A - B^T D^{-1} B\right)\boldsymbol{x} = \mathbf{0}. \tag{13}
$$

We take $C := A - B^T D^{-1} B,$ so that,

$$
C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix},
$$

where

$$c_{11} = a_{11} - \frac{\left(b^2_{11}d_{22} - 2b_{11}b_{21}d_{12} + b^2_{21}d_{11}\right)}{\det D},$$

$$c_{12} = a_{12} - \frac{\left(b_{11}b_{12}d_{22} + b_{11}b_{21}d_{11} - b^2_{11}d_{12} - b_{12}b_{21}d_{12}\right)}{\det D}, \quad (14)$$

$$c_{22} = a_{22} - \frac{\left(b^2_{11}d_{11} - 2b_{11}b_{12}d_{12} + b^2_{21}d_{22}\right)}{\det D}.$$

From Theorem 2.1, we know that there exists a non-trivial solution to system (8). Thus $\det C = 0$. Accordingly,

$$C = \begin{pmatrix} c_{11} & c_{12} \\ kc_{11} & kc_{12} \end{pmatrix}, \quad (15)$$

for some constant $k = k(t)$. Moreover, for $t = 0$ the matrix $C = C(t)$ is not null. In fact, for this value of *t*, the following terms simplify

$$a_{12} = -\left\| \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_1 \right\|^2,$$

$$d_{12} = -\left\| \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right\|^2,$$

$$b_{21} = b_{12} = b_{11} = -\left(\hat{g}_1, \left(\frac{p^2}{2} + E_0\right)^{-1} \hat{g}_2 \right).$$

It follows that,

$$\begin{aligned} c_{12} = a_{12} - \frac{b^2_{11}\left(d_{11} + d_{22}\right) - 2d_{12}b_{11}}{\det D} = &-\left\| \frac{\hat{g}_1}{\left(\frac{p^2}{2} + E_0\right)^{1/2}} \right\|^2 \\ &- \frac{\left(\hat{g}_1, \frac{\hat{g}_2}{\frac{p^2}{2} + E_0} \right)^2 \left(\frac{\mu_1 + \mu_2}{\mu_1\mu_2} - 2\left\| \frac{\hat{g}_2}{\left(\frac{p^2}{2} + E_0\right)^{1/2}} \right\|^2 \right)}{\det D} \\ &- \frac{2\left\| \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right\|^2 \left(\hat{g}_1, \left(\frac{p^2}{2} + E_0\right)^{-1} \hat{g}_2 \right)^2}{\det D} \\ &< 0, \end{aligned} \quad (16)$$

where we use equation (5), the hypothesis $E_0(2) < E_1$ and statement (3) of theorem 2.1. Therefore,

$$\left\| \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right\|^2 < \frac{\mu_1 + \mu_2}{2\mu_1\mu_2}.$$

Equations (11)-(16) imply,

$$\mu_1\left(\hat{g}_2, \hat{\psi}\right) = \left(\frac{b_{21}d_{12} - b_{11}d_{22}}{\det D} + \frac{c_{11}\left(b_{12}d_{22} - b_{22}d_{12}\right)}{c_{12}\det D} \right) x_1$$

$$\mu_2\left(\hat{g}_{2,\rho}, \hat{\psi}\right) = \left(\frac{b_{11}d_{12} - b_{21}d_{11}}{\det D} + \frac{c_{11}\left(b_{22}d_{11} - b_{12}d_{12}\right)}{c_{12}\det D} \right) x_1 \quad (17)$$

Substitution of these equalities in Equation (9) gives,

$$\hat{\psi}(0) = x_1 \left(\frac{\hat{g}_1}{\frac{p^2}{2} + E_0} + k_1 \frac{\hat{g}_{1,\rho}}{\frac{p^2}{2} + E_0} + k_2 \frac{\hat{g}_2}{\frac{p^2}{2} + E_0} + k_3 \frac{\hat{g}_{2,\rho}}{\frac{p^2}{2} + E_0} \right). \quad (18)$$

Here,

$$k_1 = -\frac{c_{11}}{c_{12}}$$

$$k_2 = \frac{\left(c_{12}b_{21}d_{12} + c_{11}b_{12}d_{22}\right) - \left(c_{12}b_{11}d_{22} + c_{11}b_{22}d_{12}\right)}{c_{12}\det D} \quad (19)$$

$$k_3 = \frac{\left(c_{12}b_{11}d_{12} + c_{11}b_{22}d_{11}\right) - \left(c_{12}b_{21}d_{11} + c_{11}b_{12}d_{12}\right)}{c_{12}\det D}.$$

This determines the vector $\hat{\psi}(0)$ up to a multiplicative constant, and from the Plancherel theorem, also the eigenspace associated to the ground state for $H(0)$, proving the statement of the lemma. □

Theorem 2.2. *Let* $H(t)$ *be defined by Equation* (1) *and suppose the hypotheses of theorem* 2.1 *hold true. Moreover, we take the curve* $\rho: \mathbb{R} \to \mathbb{R}^n$ *so that* $\rho(t) = a + vt, \ \forall |t| \geq M$, *for some positive constant M and fixed vectors* $a, v \in \mathbb{R}^n$. *Then the dimension of the spectral projection onto the interval* $[-E_0, -E_\infty]$, *associated with the selfadjoint operator* $H(t)$, *is equal to one for each* $t \in \mathbb{R}$.

Proof: The resolvent $R_i(A)$ of a self-adjoint operator *A* at $i \in \mathbb{C}$ is defined by $(iI - A)^{-1}$ with *I* denoting the identity operator on $L^2(\mathbb{R}^n)$. We take $H_2 = H(t_2)$, $H_1 = H(t_1)$, for two distinct values t_1 and t_2 and calculate the difference $R_i(H_2) - R_i(H_1)$.

$$\begin{aligned} R_i(H_2) - R_i(H_1) &= R_i(H_2)(H_2 - H_1)R_i(H_1) \\ &= \lambda_2 R_i(H_2)\left(V_{1,\rho_1} - V_{1,\rho_2}\right)R_i(H_1) \\ &\quad + \mu_2 R_i(H_2)\left(V_{2,\rho_1} - V_{2,\rho_2}\right)R_i(H_1) \end{aligned} \quad (20)$$

Here V_{1,ρ_1} is given as in Equation (3) with $g_{i,\rho}(\boldsymbol{x}) = g_i(\boldsymbol{x} - \rho(t))$ replaced with $g_{1,\rho_1}(\boldsymbol{x}) = g_1(\boldsymbol{x} - \rho(t_1))$. Also $V_{1,\rho_2}, V_{2,\rho_1}$, and V_{2,ρ_2} being defined similarly. It follows from Equation (1) and standard arguments that

$$\left\| R_i(H_2) - R_i(H_1) \right\| \le \Upsilon |t_2 - t_1|,$$

where Υ is a constant uniform in $t_1, t_2 \in \mathbb{R}/[-M, M]$ depending on $\left\| |\boldsymbol{p}| g_\ell \right\|$ and $\|g_\ell\|$, $\ell = 1, 2$. This implies that $R_i(H(t))$ is uniformly continuous on $\mathbb{R}$ with respect to the norm topology. Let $P_S(B)$ denote the spectral projection of a self-adjoint operator B corresponding to the Borel set $S \subseteq \mathbb{R}$. By functional calculus, we get

$$P_{[-E_0, -E\infty]}(H(t_2)) \to P_{[-E_0, -E\infty]}(H(t_1)) \quad \text{as} \quad t_2 \to t_1,$$

in the operator norm. Therefore, by standard arguments

$$\dim P_{[-E_0, -E\infty]}(H(t_2)) = \dim P_{[-E_0, -E\infty]}(H(t_1)),$$

For t_2 close enough to t_1. It follows from lemma 2.2 that

$$\dim P_{[-E_0, -E_\infty]}(H(t)) = 1 \quad (\forall t \in \mathbb{R}).$$

Remark: We mention that the hypothesis for the curve $\rho(t)$ can be relaxed to the condition that $\rho(t)$ is asymptotic to a straight line.

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