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Sequential Test of Fuzzy Hypotheses

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Abstract

In testing statistical hypotheses, as in other statistical problems, we may be confronted with fuzzy concepts. This paper deals with the problem of testing hypotheses, when the hypotheses are fuzzy and the data are crisp. We first give new definitions for notion of mass (density) probability function with fuzzy parameter, probability of type I and type II errors and then state and prove the sequential probability ratio test, on the basis of these new errors, for testing fuzzy hypotheses. Numerical examples are also provided to illustrate the approach.

1. Introduction

Statistical analysis, in traditional form, is based on crispness of data, random variable, point estimation, hypotheses, parameter and so on. As there are many different situations in which the above mentioned concepts are imprecise. On the other hand, the theory of fuzzy sets is a well known tool for formulation and analysis of imprecise and subjective concepts. Therefore the sequential probability ratio test with fuzzy hypotheses can be important. The problem of statistical inference in fuzzy environments are developed in different approaches.

Delgado *et al*. [1] consider the problem of fuzzy hypotheses testing with crisp data. Arnold [2,3] presents an approach to test fuzzily formulated hypotheses, in which he considered fuzzy constraints on the type I and II errors. Holena [4] considers a fuzzy generalization of a sophisticated approach to exploratory data analysis, the general unary hypotheses automaton. Holena [5] presents a principally different approach and motivates by the observational logic and its success in automated knowledge discovery. Neyman-pearson lemma for fuzzy hypotheses testing and Neyman-pearson lemma for fuzzy hypotheses testing with vague data is given by Taheri *et al*. and Torabi *et al*. [6,7]. Filzmoser and Viertl [8] present an approach for statistical testing at the basis of fuzzy values by introducing the fuzzy *p*-value. Some methods of statistical inference with fuzzy data, are reviewed by Viertl [9]. Buckley [10,11] studies the problems of statistical inference in fuzzy environment.

Thompson and Geyer [12] proposed the Fuzzy *p*-values in latent variable problems. Taheri and Arefi [13] exhibit an approach for testing fuzzy hypotheses based on fuzzy test statistics. Parchami *et al*. [14] consider the problem of testing hypotheses, when the hypotheses are fuzzy and the data are crisp. they first introduce the notion of fuzzy p-value, by applying the extension principle and then present an approach for testing fuzzy hypotheses by comparing a fuzzy *p*-value and a fuzzy significance level, based on a comparison of two fuzzy sets.

In present work, we first define a new approach for obtaining the probability (density) function, when the random variable is crisp and the parameter of interest is imprecise (fuzzy). Also, the type I and type II errors are introduced based on fuzzy hypotheses. Then, the sequential probability ratio test (SPRT) is defined and extended based on such hypotheses.

We organize the matter in the following way:

In section 2 we describe some basic concepts of fuzzy hypotheses, density (Mass) probability function with fuzzy parameter and necessary definitions. In section 3 we come up sequential probability ratio test based on fuzzy hypotheses. In section 4 the previous definitions and the sequential probability ratio test will be illustrated by examples.

2. Preliminaries

In this section we describe fuzzy hypotheses, density (Mass) probability function with fuzzy parameter and

necessary definitions.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, a random variable (RV) X is a measurable function from $(\Omega, \mathcal{F}, P)$ to $(\mathcal{X}, \mathcal{B}, P_X)$, where P_X is the probability measure induced by X and is called the distribution of the RV X, *i.e.*,

$$P_X(A) = P(X \in A) = \int_{X \in A} \mathrm{d}P \quad \forall A \in \mathcal{B}.$$

If P_X is dominated by a σ-finite measure υ, *i.e.* $P_X \ll \upsilon$ then by the Radon-Nikodym theorem (Billingsley, [15]), we have

$$P_X(A) = \int_{X \in A} f(x|\theta) \mathrm{d}\upsilon(x) \quad \forall A \in \mathcal{B},$$

where $f(x|\theta)$ is the Radon-Nikodym derivative of P_X with respect to υ and is called the probability density function of X with respect to υ. In a statistical context, the measure υ is usually a "counting measure" or a "Lebesgue measure", hence $P_X(A)$ is $\sum_{x \in A} P(X = x|\theta)$ or $\int_A f(x|\theta) \mathrm{d}x$, respectively.

2.1. Canonical Fuzzy Numbers

Let $\Theta = \{\theta \mid f(x|\theta) > 0\}$ be the "support" or "sample space" of Θ, then a fuzzy subset $\tilde{\theta}$ of S_X is defined by its membership function $\mu_{\tilde{\theta}} : \Theta \to [0.1]$. We denote by $\tilde{\theta}_\alpha = \{\theta : \mu_{\tilde{\theta}}(x) \geq \alpha\}$ the α-cut set of $\tilde{\theta}$ and $\tilde{\theta}_0$ is the closure of the set $\{\theta : \mu_{\tilde{\theta}}(x) > 0\}$, and

1) $\tilde{\theta}$ is called a normal fuzzy set if there exists $\theta \in \Theta$ such that $\mu_{\tilde{\theta}}(x) = 1$;

2) $\tilde{\theta}$ is called a convex fuzzy set if $\mu_{\tilde{\theta}}(\lambda x + (1 - \lambda)y) \geq \min(\mu_{\tilde{\theta}}(x), \mu_{\tilde{\theta}}(y))$ for all $\lambda \in [0,1]$;

3) $\tilde{\theta}$ is called a fuzzy number if $\tilde{\theta}$ is a normal convex fuzzy set and its α-cut sets, are bounded $\forall \alpha \neq 0$;

4) $\tilde{\theta}$ is called a closed fuzzy number if $\tilde{\theta}$ is a fuzzy number and its membership function $\mu_{\tilde{\theta}}$ is upper semicontinuous;

5) $\tilde{\theta}$ is called a bounded fuzzy number if $\tilde{\theta}$ is a fuzzy number and the support of its membership function $\mu_{\tilde{\theta}}$ is compact.

If $\tilde{\theta}$ is a closed and bounded fuzzy number with $\theta_\alpha^L = \inf\{\theta : \theta \in \theta_\alpha\}$ and $\theta_\alpha^U = \sup\{\theta : \theta \in \theta_\alpha\}$ and its membership function be strictly increasing on the interval $\left[\theta_\alpha^L, \theta_1^L\right]$ and strictly decreasing on the interval $\left[\theta_1^U, \theta_\alpha^U\right]$, then $\tilde{\theta}$ is called a canonical fuzzy number (Klir and Yuan, [16]).

The fuzzy canonical numbers (such as triangular or trapezoidal fuzzy numbers) are very realistic in fuzzy set theory, so we use this numbers for our goal.

2.2. Fuzzy Hypotheses

We define some models, as fuzzy sets of real numbers, for modeling the extended versions of the simple, the one-sided, and the two-sided ordinary (crisp) hypotheses to the fuzzy ones.

Testing statistical hypothesis is a main branch of statistical inference. Typically, a statistical hypothesis is an assertion about the probability distribution of one or more random variable(s). Traditionally, all statisticians assume the hypothesis for which we wish provide a test are well-defined. This limitation, sometimes, force the statistician to make decision procedure in an unrealistic manner. This is because in realistic problems, we may come across non-precise (fuzzy) hypothesis. For example, suppose that θ is the proportion of a population which have a disease. We take a random sample of elements and study the sample for having some idea about θ. In crisp hypothesis testing, one uses the hypotheses of the form: $H_0 : \theta = 0.2$ versus $H_1 : \theta \neq 0.2$ or $H_0 : \theta \leq 0.2$ versus $H_0 : \theta > 0.2$, and so on. However, we would sometimes like to test more realistic hypotheses. In this example, more realistic expressions about θ would be considered as: "small", "very small", "large", "approximately 0.2", "essentially larger" and so on. Therefore, more realistic formulation of the hypotheses might be $H_0 : \theta$ is small, versus $H_1 : \theta$ is not small. We call such expressions as fuzzy hypotheses.

We define some models, as fuzzy sets of real numbers, for modeling the extended versions of the simple, the one-sided, and the two-sided crisp hypotheses to the fuzzy ones (Akbari and Rezaei, [17]).

Definition 2.1 *Let θ_0 be a real number and known.*

1) Any hypothesis of the form $\left(H : \theta .3cm is .3cm approximately .3cm \theta_0\right)$ is called to be a fuzzy simple hypothesis.

2) Any hypothesis of the form $\left(H : \theta .3cm is .15cm not .3cm approximately .3cm \theta_0\right)$ is called to be a fuzzy two-sided hypothesis.

3) Any hypothesis of the form $\left(H : \theta .3cm is .15cm essentially .5cm larger .15cm than .15cm .13cm \theta_0\right)$ is called to be a fuzzy right one-sided hypothesis.

4) Any hypothesis of the form $\left(H : \theta .3cm is .15cm essentially .15cm smaller .15cm than .15cm .3cm \theta_0\right)$ is called to be a fuzzy left one-sided hypothesis.

We denote the above definitions by

$$(a).3cm \{H_0 : \theta .3cm is .3cm approximately$$
$$.3cm \theta_0 .65cm (\theta .3cm is .3cm H_0(\theta))\}$$

$$(b).3cm \{H_0 : \theta .3cm is .3cm essentially.$$
$$3cm larger .3cm than .3cm \theta_0 1.4cm\}$$

$(c).3cm\{H_0 : \theta.3cm is.3cm essentially.3cm smaller$

$.3cm than.3cm\theta_0 1.6cm\}$

$(d).3cm\{H_0 : \theta.3cm is.3cm approximately.3cm\theta_0 .65cm\}$

2.3. Density (Mass) Probability Function

Let X is a RV and let $S_X = \{x \in R : f(x|\theta) > 0\}$ be the "support" or "sample" space of X and

$$f\left(x\middle|\tilde{\theta}\right) = \frac{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) f(x|\theta) \mathrm{d}\theta \mathrm{d}\alpha}{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \mathrm{d}\theta \mathrm{d}\alpha}$$

where $H(\theta)$ is the membership function of canonical fuzzy hypothesis and $\tilde{\theta}_\alpha$ is its α-cuts.

We call the **new** density $f\left(x\middle|\tilde{\theta}\right)$ as the fuzzy probability density (mass) function (FPDF) of X (Akbari and Rezaei [18]). We note that, $f\left(x\middle|\tilde{\theta}\right) \geq 0$ and

$$\int_{x \in S_X} f\left(x\middle|\tilde{\theta}\right) \mathrm{d}x = \int_{x \in S_X} \frac{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) f(x|\theta) \mathrm{d}\theta \mathrm{d}\alpha}{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \mathrm{d}\theta \mathrm{d}\alpha}$$

$$= \frac{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \int_{x \in S_X} f(x|\theta) \mathrm{d}x \mathrm{d}\theta \mathrm{d}\alpha}{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \mathrm{d}\theta \mathrm{d}\alpha}$$

$$= 1$$

(substitute the summation by integral in discrete cases).

Let $g(x) : R \to R$ be arbitrary function in x. Then we define

$$E_{\tilde{\theta}}(g(X)) = \int_{x \in S_X} g(x) f\left(x\middle|\tilde{\theta}\right) \mathrm{d}x$$

$$= \frac{\int_{x \in S_X} \int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} g(x) H(\theta) f(x|\theta) \mathrm{d}\theta \mathrm{d}\alpha \mathrm{d}x}{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \mathrm{d}\theta \mathrm{d}\alpha}$$

$$= \frac{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \int_{x \in S_X} g(x) f(x|\theta) \mathrm{d}x \mathrm{d}\theta \mathrm{d}\alpha}{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \mathrm{d}\theta \mathrm{d}\alpha}$$

$$= \frac{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) E_{\tilde{\theta}}(g(X)) \mathrm{d}\theta \mathrm{d}\alpha}{\int_0^1 \int_{\theta \in \tilde{\theta}_\alpha} H(\theta) \mathrm{d}\theta \mathrm{d}\alpha}$$

Let $X = (X_1, X_2, \cdots, X_n)$ be a random sample, with observed value $x = (x_1, x_2, \cdots, x_n)$, where X_i has the FPDF $f\left(x\middle|\tilde{\theta}\right)$ with unknown $\theta \in \Theta$. For testing

$$\begin{cases} H_0 : \theta \text{ is } H_0(\theta) \text{ or } \theta \text{ is } \tilde{\theta}_0 \\ H_1 : \theta \text{ is } H_1(\theta) \text{ or } \theta \text{ is } \tilde{\theta}_1 \end{cases}$$

we state the following definitions:

Definition 2.2 *Let* $\psi(X)$ *be a test function. The probability of type I error of* $\psi(X)$ *is*

$$\alpha_\psi = E_{\tilde{\theta}_0}\left[\psi(X)\right],$$

and the probability of type II error of $\psi(X)$ is

$$\beta_\psi = 1 - E_{\tilde{\theta}_1}\left[\psi(X)\right] = E_{\tilde{\theta}_1}\left[1 - \psi(X)\right].$$

Definition 2.3 *A teat* ψ *is said to be a test of level* α *if* $\alpha_\psi \leq \alpha$, *where* $\alpha \in [0,1]$.
we call α_ψ the size of ψ.

3. Sequential Probability Ratio Test

Consider testing a null fuzzy hypothesis against a alternative fuzzy hypothesis. In other words, suppose a sample can be drawn from one of two FPDFs and it is desired to test that the sample came from one distribution against the possibility that is came from the other. If $X_1, X_2, \cdots$ denotes the random variables, we want to test $H_0 : X_i \sim f\left(.\middle|\tilde{\theta}_0\right)$ versus $H_0 : X_i \sim f\left(.\middle|\tilde{\theta}_1\right)$. The simple likelihood-ratio test was of the following form:

reject H_0 *if* $\lambda = \dfrac{L_0}{L1} \leq k$ *for some constant* $k > 0$.

The sequential test that we propose to consider employs the likelihood-ratios sequentially. Define

$$\lambda_m = \lambda_m(x) = \frac{L_0(x)}{L_1(x)} = \frac{L_0(x_1, x_2, \cdots, x_m)}{L_1(x_1, x_2, \cdots, x_m)}$$

$$= \frac{L_0(m)}{L_1(m)} = \frac{\prod_{i=1}^m f\left(x_i\middle|\tilde{\theta}_0\right)}{\prod_{i=1}^m f\left(x_i\middle|\tilde{\theta}_1\right)}$$

for $m = 1, 2, \cdots$, and compute sequentially $\lambda_1, \lambda_2, \cdots$. for fixed k_0 and k_1 satisfying $0 < k_0 < k_1$, adopt the following procedure: take observation x_1 and compute λ_1; if $\lambda_1 \leq k_0$, reject H_0; if $\lambda_1 \geq k_1$, accept H_0; and if $k_0 < \lambda_1 < k_1$, take observation x_2, and compute λ_2. If $\lambda_2 \leq k_0$, reject H_0; if $\lambda_2 \geq k_1$, accept H_0; and if $k_0 < \lambda_2 < k_1$, take observation x_3, and etc. The idea is to continue sampling as long as $k_0 < \lambda_j < k_1$ and stop as soon as $\lambda_j \leq k_0$ or $\lambda_j \geq k_1$, rejecting H_0 if $\lambda_m \leq k_0$ and accepting H_0 if $\lambda_m \geq k_1$. The critical region of the described sequential test can be define as $C = \cup_{n=1}^\infty C_n$, where

$$C_n = \Big\{x : k_0 < \lambda(x_1, x_2, \cdots, x_j) < k_1,$$

$$j = 1, 2, \cdots, n-1, \lambda(x_1, x_2, \cdots, x_n) \leq k_0\Big\}$$

Similarly, the acceptance region can be defined as $A = \cup_{n=1}^\infty A_n$, where

$$A_n = \left\{ x : k_0 < \lambda\left(x_1, x_2, \cdots, x_j\right) < k_1, \right.$$
$$\left. j = 1, 2, \ldots, n-1, \lambda\left(x_1, x_2, \cdots, x_n\right) \ge k_1 \right\}$$

When we considered the simple likelihood-ratio test for fixed sample size n, we determined k so that the test would have preassigned size α. We know want to determine k_0 and k_1 so that the sequential probability ratio test will have preassigned α and β for its respective sizes of type I and type II errors. Note that

$$\alpha = P\left(\text{reject } H_0 \mid H_0 \text{ is true}\right) = \sum_{n=1}^{\infty} \int_{C_n} L_0(n)$$

and

$$\beta = P\left(\text{accept } H_0 \mid H_0 \text{ is false}\right) = \sum_{n=1}^{\infty} \int_{A_n} L_1(n),$$

where, as before, $\int_{C_n} L_0(n)$ is a shortened notation for

$$\int \cdots \int_{C_n} \left[\prod_{i=1}^{n} f\left(x_i \mid \tilde{\theta}_0\right) dx_i \right].$$

For fixed α and β, the above equations are two equations in the two unknown k_0 and k_1. A solution of these two equations would give the sequential probability ratio test having the desired preassigned error sizes α and β. As might be anticipated, the actual determination of k_0 and k_1 from above equations can be a major computational project.

We note that the sample size of a sequential probability ratio test is a random variable. The procedure says to continue sampling until λ_n first falls outside the interval (k_0, k_1). The actual sample size then depend on which x_i s observed; it is a function of the random variables $X_1, X_2, \cdots$ and consequently is itself a RV. Denote it by N. Ideally, we would like to know the distribution of N or at least the expectation of N. One way of assessing the performance of the sequential probability ratio test would be to evaluate the expected sample size that is required under each hypothesis. The following lemma, given without proof (Lehmann, [20]), state that the sequential probability ratio test with crisp hypotheses is an optimal test if performance is measured using expected sample size. We can similarly prove this lemma with fuzzy hypotheses based on introduced FDPF.

Lemma 3.1 *The sequential probability ratio test with error sizes α and β minimizes both* $E\left(N \mid H_0 \text{ is true}\right)$ *and* $E\left(N \mid H_1 \text{ is true}\right)$ *among all tests which satisfy the following:*

$P\left(H_0 \text{ is rejected} \mid H_0 \text{ is true}\right) \le \alpha$,

$P\left(H_0 \text{ is rejected} \mid H_0 \text{ is true}\right) \le \beta$, *and the expected sample size is finite.*

We noted above that the determination of k_0 and k_1

that defines that particular sequential probability ratio test which has error sizes α and β is in general computationally quite difficult. The following lemma (with simple proof) gives an approximation to k_0 and k_1.

Lemma 3.2 *Let k_0 and k_1 be defined so that the sequential probability ratio test corresponding to k_0 and k_1 has error sizes α and β; then k_0 and k_1 can be approximated by, say k_0' and k_1', where*

$$k_0' = \frac{\alpha}{1-\beta} \quad \text{and} \quad k_1' = \frac{1-\alpha}{\beta}.$$

Lemma 3.3 *Let α' and β' be the error sizes of the sequential probability ratio test defined by k_0' and k_1' given in before lemma. Then $\alpha' + \beta' \le \alpha + \beta$.*

Naturally, one would prefer to use that sequential probability ratio test having the desired preassigned error sizes α and β; however, since it is difficult to to find the k_0 and k_1 corresponding to such a sequential probability ratio test, instead one can use that sequential probability ratio test defined by α' and β' of before equation and be assured that the the sum of the error sizes α' and β' is less than or equal to the sum of the desired error sizes α and β.

The procedure used in performing a sequential probability ratio test is to continue sampling as long as $k_0 < \lambda_m < k_1$ and stop sampling as soon as $\lambda_m \le k_0$ or $< \lambda_m \ge k_1$. If $z_i = \ln\left[\dfrac{f\left(x_i \mid \tilde{\theta}_0\right)}{f\left(x_i \mid \tilde{\theta}_1\right)} \right]$, an equivalent test is given by the following: continue sampling as long as $\ln k_0 < \sum_{i=1}^{n} z_i < \ln k_1$, and stop sampling as soon as $\sum_{i=1}^{n} z_i \le \ln k_0$ or $\sum_{i=1}^{n} z_i \ge \ln k_1$. As before, let N be a RV denoting the sample size of the sequential probability ratio test, and let $z_i = \ln\left[\dfrac{f\left(x_i \mid \tilde{\theta}_0\right)}{f\left(x_i \mid \tilde{\theta}_1\right)} \right]$.

If the sequential probability ratio test leads to rejection of H_0, then the RV $\sum_{i=1}^{N} Z_i \ge \ln k_0$, but $\sum_{i=1}^{N} Z_i$ is close to $\ln k_0$ since $\sum_{i=1}^{N} Z_i$ first became less than or equal to $\ln k_0$ at the N th observation; hence $E\left[\sum_{i=a}^{N} Z_i\right] \approx \ln k_0$. Similarly $E\left[\sum_{i=1}^{N} Z_i\right] \approx \ln k_1$; hence $E\left[\sum_{i=1}^{N} Z_i\right] \approx \xi \ln k_0 + (1-\xi) \ln k_1$, where $\xi = P\left(H_0 \text{ is rejected}\right)$. Using Wald's equation (Casella and Berger, [19])

$$E(N) = \frac{E\left[\displaystyle\sum_{i=1}^{N} Z_i\right]}{E(Z_i)} \approx \frac{\xi \ln k_0 + (1-\xi) \ln k_1}{E(Z_i)}$$

we obtain

$$E\left(N\middle|H_0\ is\ true\right) \approx \frac{\alpha \ln k_0 + (1-\alpha)\ln k_1}{E\left(Z_i N\middle|H_0\ is\ true\right)}$$

$$\approx \frac{\alpha \ln\left(\frac{\alpha}{1-\beta}\right) + (1-\alpha)\ln\left(\frac{1-\alpha}{\beta}\right)}{E\left(Z_i\middle|H_0\ is\ true\right)}$$

and

$$E\left(N\middle|H_0\ is\ flase\right) \approx \frac{(1-\beta)\ln k_0 + \beta \ln k_1}{E\left(Z_i\middle|H_0\ is\ flase\right)}$$

$$\approx \frac{(1-\beta)\ln\left(\frac{\alpha}{1-\beta}\right) + \beta \ln\left(\frac{1-\alpha}{\beta}\right)}{E\left(Z_i\middle|H_0\ is\ true\right)}.$$

4. Numerical Examples

In this section, we illustrate the proposed approach for some distributions and use the ability of package "Maple 6" [21] for this examples.

Example 4.1 (*Taheri and Behboodian*, [6]) *Let* X *be a continues r.v. with PDF*

$$f\left(x\middle|\theta\right) = 2\theta x + 2(1-\theta)(1-x) \quad 0 < x < 1 \quad 0 < \theta < 1.$$

we want to test

$$\begin{cases} H_0 : \theta\ is\ approximatly\ \dfrac{1}{3} \\ H_1 : \theta\ is\ approximatly\ \dfrac{1}{2} \end{cases}$$

where the membership functions H_0 and H_1 are defined in the following way:

$$H_0(\theta) = \begin{cases} 3\theta & 0 \le \theta < \dfrac{1}{3} \\ 2 - 3\theta & \dfrac{1}{3} \le \theta < \dfrac{2}{3} \\ 0 & otherwise \end{cases}$$

$$H_1(\theta) = \begin{cases} 2\theta & 0 \le \theta < \dfrac{1}{2} \\ 2 - 2\theta & \dfrac{1}{2} \le \theta < 1 \\ 0 & otherwise \end{cases}$$

We can interpret $\tilde{\theta}_0$ and $\tilde{\theta}_1$ as the value of " *near to* $\dfrac{1}{3}$ " and " *near to* $\dfrac{1}{2}$ ".

Let $\alpha = 0.05$ and $\beta = 0.01$. We obtain

$\ln(k_0') = -2.986$, $\ln(k_1') = 4.554$. Hence, $E\left(N\middle|H_0\ is\ true\right) = 38.556$, and we must take $n = 39$, whereas $E\left(N\middle|H_1\ is\ true\right) = 25.432$, thus we take $n = 25$.

Example 4.2 *Let* $X = (X_1, X_2, \cdots, X_n)$ *be a random sample where* $X_i \sim (\theta, 1)$ *population, i.e.,*

$$f\left(x\middle|\theta\right) = \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{1}{2}(x-\theta)^2\right\} \qquad x, \theta \in R$$

and $H_i(\theta)$'s are our fuzzy hypotheses with membership functions given by:

$$H_0(\theta) = \begin{cases} \theta - 11 & 11 \le \theta < 12 \\ 13 - \theta & 12 \le \theta < 13 \\ 0 & otherwise \end{cases}$$

$$H_1(\theta) = \begin{cases} \theta - 9 & 9 \le \theta < 10 \\ 11 - \theta & 10 \le \theta < 11 \\ 0 & otherwise \end{cases}$$

We can interpret H_0 and H_1 as the value of " *near to* 12 " and " *near to* 10 ".

Let $\alpha = \beta = 0.05$. Hence, $E\left(N\middle|H_0\ is\ true\right) = 85.155$, and we must take $n = 85$, whereas $E\left(N\middle|H_1\ is\ true\right) = 87.63$, thus we take $n = 88$.

Example 4.3 *Let* X *be a random sample where* $X_i \sim E(\theta, 1)$ *population, i.e.,*

$$f\left(x\middle|\theta\right) = \exp\left(-(x-\theta)\right) \qquad x > \theta, \theta > 0,$$

and $H_i(\theta)$'s are our triangular fuzzy parameters with membership functions

$$H_i(\theta) = \begin{cases} \dfrac{\theta - a_i + a}{a} & a_i - a \le \theta \le a_i \\ \dfrac{a_i + b - \theta}{b} & a_i \le \theta \le a_i + b, \end{cases}$$

for $a, b \ge 0$.

We can interpret the canonical parameters as having values that are "near to a_i".

Let $a_0 = 8$, $a_1 = 4$, $\alpha = 0.05$ and $\beta = 0.1$. Hence, $E\left(N\middle|H_0\ is\ true\right) = 21.325$, and we must take $n = 21$, whereas $E\left(N\middle|H_1\ is\ true\right) = 28.22$, thus we take $n = 28$.

Example 4.4 *Let* X *be a RV from the* $U \sim (0, \theta)$ *population, i.e.,*

$$f\left(x\middle|\theta\right) = \frac{1}{\theta} \qquad 0 \le x \le \theta, \quad \theta > 0,$$

and $H_i(\theta)$'s are our trapezoidal fuzzy parameters with membership functions given by:

$$H_i(\theta) = \begin{cases} \dfrac{2(\theta - x_i + a)}{a} & x_i - a \le \theta \le x_i - \dfrac{a}{2} \\ 1 & x_i - \dfrac{a}{2} \le \theta \le x_i + \dfrac{b}{2} \\ \dfrac{2(x_i + b - \theta)}{b} & x_i + \dfrac{b}{2} \le \theta \le X_i + b. \end{cases}$$

for $a, b \ge 0$.

Let $a = b = 1$, $x_0 = 5$ and $x_1 = 3$. If $\alpha = \beta = 0.05$, then $E(N | H_0 \text{ is true}) = 8.75$, and we must take $n = 9$, whereas $E(N | H_1 \text{ is true}) = 9.11$, thus we take $n = 9$.

5. Conclusions

In this paper, an new approach for sequential test of fuzzy hypotheses based on fuzzy hypotheses for one-sample and two-sample when the available data are crisp, is presented. As for this paper, it sound the introduced method is very simple and applicable in the statistics and other sciences.

Extension of the proposed method to test the variance, correlation and parameters of linear models (regression models), design of experiment is a potential area for the future work. Furthermore, we can construct sequential test of fuzzy hypotheses based on intuitionistic fuzzy hypotheses or fuzzy data for the parameters of interest.

6. References

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