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Gauss' Problem, Negative Pell's Equation and Odd Graphs

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Abstract

In this paper we present some results connected with still open problem of Gauss, negative Pell's equation and some type graphs. In particular we prove in the Theorem 1 that all real quadratic fields $K = Q\left(\sqrt{d}\right)$, generated by Fermat's numbers with $d = F_{m+1} = 2^{2^{m+1}} + 1,\ m \geq 2$ have not unique factorization. Theorem 2 give a connection of the Gauss problem with primitive Pythagorean triples. Moreover, in final part of our paper we indicate on some connections of the Gauss problem with odd graphs investigated by Cremona and Odoni in the papper [1].

Keywords: Fermat Numbers, Class-Number Gauss' Problem, Odd Graphs

1. Introduction

Let Z_s be the set of all square-free positive integers. Moreover, let $h(d)$ denote the class-number of real quadratic field $K = Q\left(\sqrt{d}\right), d \in Z_s$. It is well-known (Cf. [2]) that the condition $h(d) = 1$ is equivalent to unique factorization in the ring R_K of the algebraic integers of the field K. The difficult and still open problem posed by Gauss concern of the existence infinitely many $d \in Z_s$ such that $h(d) = 1$. It is known that if $h(d) = 1$ in the field $K = Q\left(\sqrt{d}\right)$ then $d = p,\ 2q,\ q \cdot r$, where p, q and r are primes such that $q \equiv r \equiv 3(mod 4)$. This result has been proved by Hasse in the paper [3].Another proof of this result has been presented by Szymiczek [4]. In the paper [5] we give some arithmetic description of the set Z_s and as consequence we also obtained this result.Many others interesting and important results concerning factorization problem have been given in the papers [1,6-7,9-14]. In this paper we prove of the following two theorems:

Theorem 1: *Let* $d = F_{m+1} = 2^{2^{m+1}} + 1, m \geq 2$ *be the Fermat's number and let* $h(d) = h(F_{m+1})$ *be the class-number of the real quadratic field* $K = Q\left(\sqrt{d}\right)$. *Then we have*

$$M_m \,\middle|\, h(F_{m+1}) \qquad (*)$$

where $M_m = 2^m - 1$ *is the Mersenne number.*

Theorem 2: *Let* $d \in Z_s$ *and* $\omega(d) \geq 2$, *where is the number of all distinct prime divisors of* d. *If there is primitive Pythagorean triple* $\langle \alpha, \beta, \gamma \rangle$ *and positive relatively prime integers a, b such that*

$$d = a^2 + b^2, \quad |\alpha \cdot a - \beta \cdot b| = 1 \qquad (**)$$

then

$$h(d) > 1 \qquad (***)$$

2. Basic Lemmas

Lemma 1: (H. W. Lu, [13]). *Let* $d = 4k^{2n} + 1$ *where* k, n *are positive integers such that* $k > 1$ *and* $n > 1$. *Then we have*

$$h(d) \equiv 0(\mathrm{mod} n) \qquad (2.1)$$

where $h(d)$ *denote of the class-number of the field* $K = Q\left(\sqrt{d}\right)$.

Lemma 2: (A. Grytczuk, F. Luca, M. Wójtowicz, [15]). *The negative Pell equation*

$$x^2 - dy^2 = -1 \qquad (2.2)$$

has a solution in positive integers x, y if and only if there is a primitive Pythagorean triple $\langle \alpha, \beta, \gamma \rangle$ *and positive relatively prime integers a, b such that*

$$d = a^2 + b^2, \ |\alpha \cdot a - \beta \cdot b| = 1 \qquad (2.3)$$

Lemma 3: (A. Grytczuk, J. Grytczuk [5]). *Let* Z_s *be the set of all square-free integers and let* p, p_j *for* $j = 1,2,...,k$ *be primes. Moreover, let*

$$A = \{d \in Z_s; d = p \cdot p_1 \cdot ... \cdot p_k k \geq 1, p \equiv 1(\text{mod}4)\}$$

$$B = \{d \in Z_s; d = 2p, p \equiv 1(\text{mod}4)\}$$

$$C = \{d \in Z_s; d = 2p_1 \cdot p_2 \cdot ... \cdot p_k, k \geq 2, p_j \equiv 3(\text{mod}4), j = 1,2,...,k\}$$

$$D = \{d \in Z_s; d = p_1 \cdot p_2 \cdot ... \cdot p_k, k \geq 3, p_j \equiv 3(\text{mod}4), j = 1,2,...,k\}$$

and $h(d)$ be the class-number of the real quadratic number field $K = Q(\sqrt{d})$.

If $d \in A \cup B \cup C \cup D$, *then* $h(d) > 1$.

3. Proof of the Theorem 1

For the proof of (*) in the Theorem 1 we use Lemma 1. First we note that $k > 1$ is positive integer, hence we put $k = 2$ and let $n = 2^m - 1 > 1$, so $m \geq 2$. From this fact we obtain

$$d = 4k^{2n} + 1 = 2^2 \cdot 2^{2(2^m-1)} + 1 = 2^{2^{m+1}} + 1 = F_{m+1} \quad (3.1)$$

By (2.1) of Lemma 1 it follows that

$$n \mid h(d) \quad (3.2)$$

Since $n = 2^m - 1 = M_m$ is *m-th* Mersenne number then from (3.1) and (3.2) follows that (*) is true .

The proof of the Theorem 1 is complete. □

From the Theorem 1 and some property of Fermat numbers follows the following Corollary:

Corollary1: *For each positive integer* $k \geq 1$ *we have*

$$F_0 \cdot F_1 \cdot ... \cdot F_{k-1} \mid h\left(F_{2^k+1}\right) \quad (3.3)$$

where $F_j = 2^{2^j} + 1$ *are Fermat numbers.*

Proof. We use well-known (see, Cf. [16]) the following identity:

$$F_k = F_0 \cdot F_1 \cdot ... \cdot F_{k-1} + 2; \quad k \geq 1 \quad (3.4)$$

Since $F_k = 2^{2^k} + 1$ then from (3.4) we obtain

$$2^{2^k} - 1 = F_0 \cdot F_1 \cdot ... \cdot F_{k-1} \quad (3.5)$$

Since $2^{2^k} - 1 = M_{2^k}$, thus by (3.5) and (*) it follows the divisibility (3.3) and the proof of Corollary 1 is finished. □

4. Proof of the Theorem 2

Since $d \in Z_s$ and $\omega(d) \geq 2$ then by the assumption of the Theorem 2 and Lemma 2 it follows that negative Pell's equation (2.2) has a solution in positive relatively prime integers *x*, *y*. From the assumption that $\omega(d) \geq 2$ it follows that there is a prime p such that $p \mid d$. By (2.2) it follows that

$$x^2 + 1 = dy^2 \quad (4.1)$$

From the well-known properties of divisibility relation and (4.1) we get

$$p \mid x^2 + 1 \quad (4.2)$$

By the relation (4.2) it follows that

$$x^2 \equiv -1(modp) \quad (4.3)$$

From (4.3) we see that -1 is a quadratic residue and consequently we have

$$\left(\frac{-1}{p}\right) = +1 \quad (4.4)$$

On the other hand from the property of Legendre's symbol (Cf. [17], p. 342) we have

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} \quad (4.5)$$

By (4.4) and (4.5) it follows that the prime number p is the form: $p = 4k + 1$. Since $\omega(d) \geq 2$ then we see that $d \in A$ or $d \in B$ and by Lemma 3 it follows that $h(d) > 1$.

The proof of the Theorem 2 is complete. □

5. Connections Negative Pell's Equation with Gauss' Problem and Graphs Theory

Let *P* denote the set of all primes and let

$$D^* = \{1 < d \in Z_s, if\ p \in P \text{ and } p \mid d \text{ then } p \equiv 1(mod4)\} \quad (5.1)$$

Let $D_n \subset D^*$, $n = \omega(d)$; $p,q \in P$ and $p,q \in D_1$. Then the following relation R has been defined by Cremona and Odoni in the paper [1]:

$$\langle p,q \rangle \in R \subset D_1 \times D_1 \Leftrightarrow p \neq q \text{ and } p^3 \neq x^2 \left(modq^3\right)$$

for some $x \in Z$ (5.2)

Let $d \in D_n$ and $d = p_1 \cdot p_2 \cdot ... \cdot p_n$; $p_1 < p_2 < ... < p_n$, $p_j \in P$, $1 \leq j \leq n$.

Now, we form graph $H(d)$ with vertex set $S_n = \{1,2,\cdots,n\}$ in such way that vertices *i*, *j* are adjacent if and only if $\langle p_i, p_j \rangle \in R$ for $i \neq j$. (5.3)

A graph G with vertex set S_n is called as **odd graph** when he has of the following property:

Whenever S_n is partitioned into disjoin union $X \cup Y$ of two non-empty sets X,Y then either exists

$x \in X$ joined in G to an odd number of vertices in Y or there exists $y \in Y$ joined in G to an odd number of vertices in X.

Cremona-Odoni Theorem: [1] *If* $d \in D_n$ *and* $H(d)$ *is an odd graph then* d *is negative Pellian.*

Remark: If the Diophantine equation $x^2 - dy^2 = -1$ has a solution in positive integers x, y then the number d is called as negative Pellian.

From this Remark, Theorem 2 and The Cremona-Odoni theorem it follows the following Corollary:

Corollary 2: If $d \in D_n, n \geq 2$ and $H(d)$ is an odd graph, then $h(d) > 1$.

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