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Compactness, Contractibility and Fixed Point Properties of the Pareto Sets in Multi-Objective Programming

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Abstract

This paper presents the Pareto solutions in continuous multi-objective mathematical programming. We discuss the role of some assumptions on the objective functions and feasible domain, the relationship between them, and compactness, contractibility and fixed point properties of the Pareto sets. The authors have tried to remove the concavity assumptions on the objective functions which are usually used in multi-objective maximization problems. The results are based on constructing a retraction from the feasible domain onto the Pareto-optimal set.

Keywords: Multi-Objective Programming, Pareto-Optimal, Pareto-Front, Compact, Contractible, Fixed Point, Retraction

1. Introduction

During the last four decades, the topological properties of the Pareto solutions in multi-objective optimization problems have attracted much attention from researchers, see [1-8] for more details. The aim of this paper is to present some new facts on compactness, contractibility and fixed point properties of the Pareto-optimal and the Pareto-front sets, shortly Pareto sets, in a multi-objective maximization problem. The authors have tried to remove the concavity assumptions of the objective functions which are usually used in this optimization problem.

The standard form of the multi-objective maximization problem is to find a variable $x(x_1, x_2, \cdots, x_m) \in R^m$, $m \geq 1$, so as to maximize $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$ subject to $x \in X$, where the feasible domain X is nonempty, $J_n = \{1, 2, \cdots, n\}$ is the index set, $n \geq 2$, $f_i : X \to R$ is a given continuous function for all $i \in J_n$.

Since the objective functions $\{f_i\}_{i=1}^n$ may conflict with each other, it is usually difficult to obtain a global maximum for all objective functions at the same time. Therefore, the target of the maximization problem is to achieve a set of solutions that are Pareto-optimal. Historically, the first reference to address such situations of conflicting objectives is usually attributed to Vilfredo Pareto (1848-1923).

Definition 1. a) A point $x \in X$ is called a Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $f_k(y) > f_k(x)$ for some $k \in J_n$. The set of Pareto-optimal solutions of X is denoted by $PO(X, f)$ and is called a Pareto-optimal set. Its image $f(PO(X, f)) = PF(X, f)$ is called a Pareto-front set.

b) A point $x \in X$ is called a strictly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $x \neq y$. The set of strictly Pareto-optimal solutions of X is denoted by $SPO(X, f)$ and is called a strictly Pareto-optimal set. □

The strictly Pareto-optimal solutions are the multi-objective analogue of unique optimal solutions in scalar optimization.

In this paper, let the feasible domain X be compact. It is well-known that $PO(X, f)$ and $PF(X, f)$ are nonempty, $SPO(X, f) \subset PO(X, f)$ and $PF(X, f) \subset \partial f(X)$ [4].

One of the most important problems in multi-criteria maximization is the investigation of the topological properties of the Pareto-optimal and Pareto-front sets. Information about these properties of the Pareto sets is very important for computational algorithms generating Pareto solutions [8,9]. Consideration of topological properties of Pareto solutions sets is started by [5], see also [4,6,9,10].

We focus our attention on the compactness, contractibility and fixed point properties of the Pareto sets. Compactness of these sets is studied in [4,6,11]. Contractibility of Pareto sets is considered in [12-14]. Fixed point properties of Pareto-optimal and Pareto-front sets have been addressed in [7,15].

This paper is organized as follows: In Section 2, we describe some definitions and notions from topology and optimization theory. In Section 3, we study compactness, contractibility and fixed point properties of the Pareto-optimal and Pareto-front sets.

2. Definitions and Notions

Recall some topological definitions.

Definition 2. a) The set $Y \subset X$ is a retract of X if and only if there exists a continuous function $r: X \to Y$ such that $r(x) = x$ for all $x \in Y$. The function r is called a retraction.

b) The set $Y \subset X$ is a deformation retract of X if and only if there exist a retraction $r: X \to Y$ and a homotopy $H: X \times [0;1] \to X$ such that $H(x,0) = x$ and $H(x,1) = r(x)$ for all $x \in X$.

c) The set Y is contractible (contractible to a point) if and only if there exists a point $a \in Y$ such that $\{a\}$ is a deformation retract of Y. □

Definition 3. The topological space Y is said to have the fixed point property if and only if every continuous function $h: Y \to Y$ from this set into itself has a fixed point, *i.e*. there is a point $x \in Y$ such that $x = h(x)$.

Remark 1. Let X and Y be two topological spaces. A homotopy between two continuous functions $f, g: X \to Y$ is defined to be a continuous function $H: X \times [0;1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Note that we can consider the homotopy H as a continuously deformation of f to g [16]. □

Remark 2. From a more formal viewpoint, a retraction is a function $r: X \to Y$ such that $r \circ r(x) = r(x)$ for all $x \in X$, since this equation says exactly that r is the identity on its image. Retractions are the topological analogs of projection operators in other parts of mathematics. Clearly, every deformation retract is a retract, but in generally the converse does not hold [16]. □

Remark 3. It is known that convexity implies contractibility, but the converse does not hold in general. Contractibility of sets is preserved under retraction. This means that if set X is contractible and Y is a retract of X, then set Y is contractible too. □

Let us consider a multifunction $\varphi: Y \Rightarrow Y$. Let it be upper semi-continuous with a nonempty, compact and convex image, shortly we say that φ is *cusco*.

Definition 4. The topological space Y is said to have the Kakutani fixed point property if and only if every *cusco* $\varphi: Y \Rightarrow Y$ has a fixed point, *i.e*. there is a point $x \in Y$ such that $x \in \varphi(x)$. □

Remark 4. A property is called a topological property if and only if an arbitrary topological space X has this property, then Y has this property too, where Y is homeomorphic to X. Compactness, contractibility and the fixed point properties (the fixed point property and the Kakutani fixed point property) are topological properties. □

Of course, the topological properties of the Pareto-optimal set relate to the topological properties of the Pareto-front set, respectively.

Remark 5. The fixed point and the Kakutani fixed point properties of sets are preserved under retraction [16]. This means that the following statements are true: if set X has the fixed point property and Y is a retract of X, then set Y has the fixed point property too; if set X has the Kakutani fixed point property and Y is a retract of X, then set Y has the Kakutani fixed point property too. □

Remark 6. The Kakutani fixed point property is very closely related to the fixed point property. If $S \subset R^n$ has the Kakutani fixed point property, then since any continuous function from S into itself can be viewed as a *cusco* it follows that set S will also have the fixed point property. □

Remark 7. Let $S \subset R^n$ be compact. It can be shown that set S having the Kakutani fixed point property is equivalent to S having the fixed point property. Remark 6 has shown that if S has the Kakutani fixed point property, then S has the fixed point property. Now, let $\varphi: S \Rightarrow S$ be *cusco*, S have the fixed point property and $gph(\varphi) = \{(x,y) \in S \times S \mid y \in \varphi(x)\}$. From Cellina's Theorem it follows that there is an approximate continuous selection h of ϕ [17,18]. That is, for each $k \in N$ there exists a continuous function $h_k: S \to S$ such that $d((x, h_k(x)), gph(\varphi)) < \frac{1}{k}$ for all $x \in S$. From the assumption that S has the fixed point property, it follows that each function h_k has a fixed point $x_k \in S$. As a result we get a sequence $\{x_k\}_{k=1}^{\infty} \subset S$ such that $d\left(\left(x_k, x_k\right), gph(\varphi)\right) < \frac{1}{k}$, *i.e.* the point $\left(x_k, x_k\right)$ approaches the set $gph(\phi)$. The set S is compact, implying that there exists a convergent subsequence $\left\{x'_{m(k)}\right\}_{m(k)=1}^{\infty} \subset \left\{x_k\right\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x'_{m(k)} = x_0 \in S$. We also see that $d\left(\left(x'_{m(k)}, x'_{m(k)}\right), gph(\varphi)\right) < \frac{1}{m(k)}$. But φ is *cusco*, then $gph(\varphi)$ is closed. Taking the limit as $k \to \infty$ we have $m(k) \to \infty$ and obtain

$\lim_{k\to\infty}\left(x'_{m(k)}, x'_{m(k)}\right) = \left(x_0, x_0\right) \in gph(\varphi)$. This means that $x_0 \in S$ is a fixed point for φ, see also [19]. Finally, we find that S has the Kakutani fixed point property. □

We use R^m and R^n as generic finite-dimensional vector spaces. In addition, we also introduce the following notations: for every two vectors $x, y \in R^n$,

$x\left(x_1, x_2, \cdots, x_n\right) = y\left(y_1, y_2, \cdots, y_n\right)$ means $x_i = y_i$ for all $i \in J_n$, $x\left(x_1, x_2, \cdots, x_n\right) \geqq y\left(y_1, y_2, \cdots, y_n\right)$ means $x_i \ge y_i$ for all $i \in J_n$ (weakly componentwise order), $x\left(x_1, x_2, \cdots, x_n\right) > y\left(y_1, y_2, \cdots, y_n\right)$ means $x_i > y_i$ for all $i \in J_n$ (strictly componentwise order), and

$x\left(x_1, x_2, \cdots, x_n\right) \geq y\left(y_1, y_2, \cdots, y_n\right)$ means $x_i \ge y_i$ for all $i \in J_n$ and $x_k > y_k$ for some $k \in J_n$ (componentwise order).

3. Main Result

As usual, the key idea is to transfer our multi-objective optimization problem to mono-objective optimization problem by defining a unique objective function.

We begin with the following definition: define a multifunction $\psi : X \Rightarrow X$ by $\psi(x) = \left\{y \in X \mid f(y) \geqq f(x)\right\}$ for all $x \in X$; define a function $s : X \to R$ by $s(x) = \sum_{j=1}^{n} f_j(x)$ for all $x \in X$.

Choose $x \in X$ and consider an optimization problem with a single objective function as follows: maximize $s(y)$ subject to $y \in \psi(x)$. By letting x vary over all of X we can identify different Pareto-optimal solutions. This optimization technique will allow us to find the whole Pareto-optimal set.

In this paper, we will discuss the role of the following assumptions:

Assumption 1. X is a nonempty, compact and convex set.

Assumption 2. $\left|\text{Arg max}\left(s, \psi(x)\right)\right| = 1$ for all $x \in X$.

Assumption 3. If d is a metric in R^m, $\left\{x_i\right\}_{i=0}^{\infty} \subset X$ and $\lim_{k\to\infty} d\left(x_k, x_0\right) = 0$, then $\lim_{k\to\infty} d\left(y_0, \psi\left(x_k\right)\right) = 0$ for all $y_0 \in \psi\left(x_0\right)$.

Assumption 4. f is bijective.

Our aim is to obtain a set of conditions which guarantee the compactness, contractibility and fixed point properties of the Pareto sets.

Remark 8. Let $Cl(X)$ be the set of all nonempty compact subsets of X. A sequence of sets $\left\{A_k\right\}_{k=1}^{\infty} \subset Cl(X)$ is said to Wijsman converge to $A \in Cl(X)$ if and only if for each $x \in X$, $\lim_{k\to\infty} d\left(x, A_k\right) = d(x, A)$. See also Assumption 3.

These definitions and assumptions allow us to present the main theorem of this paper.

<u>Theorem 1.</u> If Assumptions 1, 2, 3 and 4 hold, then:

a) $PO(X, f) = SPO(X, f)$.

b) $PO(X, f)$ and $PF(X, f)$ are compact.

c) $PO(X, f)$ and $PF(X, f)$ are contractible.

d) $PO(X, f)$ and $PF(X, f)$ have the fixed point properties.

In order to give the proof of Theorem 1, we will prove some lemmas.

Lemma 1. If $x \in X$, $x \in PO(X, f)$ is equivalent to $\{x\} = \psi(x)$.

Proof. Let $x \in PO(X, f)$ and assume that $\{x\} \neq \psi(x)$. From the conditions $x \in \psi(x)$ and $\{x\} \neq \psi(x)$, it follows that there exists $y \in \psi(x) \setminus \{x\}$ such that $f(y) \geqq f(x)$. As a result we get that $s(y) \ge s(x)$, but $x \in PO(X, f)$ implies $s(y) = s(x)$ and $f(y) = f(x)$. Since by assumption f is bijective, we deduce $x = y$; which contradicts the condition $y \in \psi(x) \setminus \{x\}$. Thus, we obtain $\{x\} = \psi(x)$.

Conversely, let $\{x\} = \psi(x)$ and assume that $x \notin PO(X, f)$. From the assumption $x \notin PO(X, f)$, it follows that there exists $y \in X \setminus \{x\}$ such that $f(y) \geq f(x)$. Thus we deduce that $y \in \psi(x)$ and $x \neq y$, which contradicts the condition $\{x\} = \psi(x)$. Therefore, we obtain $x \in PO(X, f)$.

The lemma is proven.

Lemma 2. If $x \in X$, then $\text{Arg max}\left(s, \psi(x)\right) \subset PO(X, f)$.

Proof. Let us choose $y \in \text{Arg max}(s, \psi(x))$ and assume that $y \notin PO(X, f)$. From the assumption $y \notin PO(X, f)$ it follows that there exists $z \in X \setminus \{y\}$ such that $f(z) \geq f(y)$. As a result we derive $z \in \psi(x)$ and $s(z) > s(y)$. This leads to a contradiction; therefore, we obtain $y \in PO(X, f)$.

The lemma is proven.

Now, using $\text{Arg max}(s, \psi(x)) \subset PO(X, f)$ (Lemma 2) and $|\text{Arg max}(s, \psi(x))| = 1$ (Assumption 3), we are in a position to construct a function $r : X \to PO(X, f)$ such that $r(x) = \text{Arg max}(s, \psi(x))$ for all $x \in X$, see also Definition 1 and Lemma 1.

Remark 9. Applying Lemmas 1 and 2 it follows that: if $x \in PO(X, f)$, then $x = r(x)$; if $x \in X \setminus PO(X, f)$, then $x \neq r(x)$. This means that $r(X) = PO(X, f)$. □

Lemma 3. ψ is continuous on X.

Proof. First, we will prove that if $\{x_k\}_{k=1}^{\infty} \subset X$ and $\{y_k\}_{k=1}^{\infty} \subset X$ are a pair of sequences such that $\lim_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \psi(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^{\infty}$ whose limit belongs to $\psi(x_0)$.

The assumption $y_k \in \psi(x_k)$ for all $k \in N$ implies $f(y_k) \geqq f(x_k)$ for all $k \in N$. From the condition $\{y_k\}_{k=1}^{\infty} \subset X$ it follows that there exists a convergent subsequence $\{q_k\}_{k=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} q_k = y_0 \in X$. Therefore, there exists a convergent subsequence $\{p_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that $q_k \in \psi(p_k)$ and $\lim_{k\to\infty} p_k = x_0$. Thus, we find that $f(q_k) \geqq f(p_k)$ for all $k \in N$. Taking the limit as $k \to \infty$ we obtain $f(y_0) \geq f(x_0)$. This implies $y_0 \in \psi(x_0)$.

This means that ψ is upper semi-continuous on X [20].

Second, we will prove that if $\{x_k\}_{k=1}^{\infty} \subset X$ is a sequence convergent to $x_0 \in X$ and $y_0 \in \psi(x_0)$, then there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi(x_k)$ for all $k \in N$ and $\lim_{k\to\infty} y_k = y_0$.

As usual, the distance between the point $y_0 \in X$ and the set $\psi(x_k) \subset X$ is denoted by $d_k = \inf\{dis(y_0, x) \mid x \in \psi(x_k)\}$. Clearly, $\psi(x_k)$ is nonempty and compact; therefore, if $y_0 \notin \psi(x_k)$, then there exists $\hat{y} \in \psi(x_k)$ such that $d_k = d(\hat{y}, y_0)$. There are two cases as follows: if $y_0 \in \psi(x_k)$, then $d_k = 0$ and let $y_k = y_0$; if $y_0 \notin \psi(x_k)$, then $d_k > 0$ and let $y_k = \hat{y}$. Thus, we get a sequence $\{d_k\}_{k=1}^{\infty} \subset R_+$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi(x_k)$ for all $k \in N$ and $d_k = d(y_0, y_k)$. Assumption 3 and $\lim_{k\to\infty} x_k = x_0$ imply that the sequence $\{d_k\}_{k=1}^{\infty}$ is convergent and $\lim_{k\to\infty} d_k = 0$. Finally, we obtain $\lim_{k\to\infty} y_k = y_0$.

This means that ψ is lower semi-continuous on X [20].

In summary, ψ is continuous on X.

The lemma is proven.

Lemma 4 [21, Theorem 9.14 – The Maximum Theorem]. Let $S \subset R^n$, $\Theta \subset R^m$, $h : S \times \Theta \to R$ a continuous function, and $D : \Theta \rightrightarrows S$ be a compact-valued and continuous multifunction. Then, the function $h^* : \Theta \to R$ defined by is continuous on Θ, and the multifunction $D^* : \Theta \rightrightarrows S$ defined by $D^*(\theta) = \{x \in D(\theta) \mid h(x, \theta) = h^*(\theta)\}$ is compact-valued and upper semi-continuous on Θ.

Lemma 5. r is continuous on X.

Proof. As was mentioned before, the multifunction ψ is compact-valued and continuous on X. Applying Lemma 6 for $S = \Theta = X$ and $D = \psi$, we deduce that r is an upper semi-continuous multifunction on X. Obviously, an upper semi-continuous multifunction is continuous when viewed as a function. This shows that r is continuous on X.

The lemma is proven.

Lemma 6 [21, Theorem 9.31 – Schauder's Fixed Point Theorem]. Let $h : S \to S$ be a continuous function from a nonempty, compact and convex set $S \subset R^n$ into itself, then h has a fixed point.

Lemma 7 [21, Theorem 9.26 - Kakutani's Fixed Point Theorem]. Let $S \subset R^n$ be a nonempty, compact and convex set and the multifunction $\varphi : S \rightrightarrows S$ be *cusco*, then φ has a fixed point.

Lemma 8. $PO(X, f)$ is homeomorphic to $PF(X, f)$.

Proof. It is well-known that every continuous image of a compact set is compact. In fact, the set X is compact and the function r is continuous on X. Hence, the set

$PO(X,f)=r(X)$ is compact. Recalling that the function $f:X\to R^n$ is continuous, we deduce that the restriction $h:PO(X,f)\to PF(X,f)$ of f is continuous too. We know that the function h is bijective. Consider the inverse function $h^{-1}:PF(X,f)\to PO(X,f)$ of h. As we proved above, the set $PO(X,f)$ is compact; therefore, h^{-1} is continuous too [22]. As a result we find that the function h is homeomorphism.

The lemma is proven.

Proof of Theorem 1. Applying Lemma 1 we get that $PO(X,f)=SPO(X,f)$.

From Lemma 5 it follows that there exists a continuous function $r:X\to PO(X,f)$ such that $r(X)=PO(X,f)$ and $r(x)=\text{Arg max}(s,\psi(x))$ for all $x\in X$. This means that $PO(X,f)$ is a retract of X.

We know that X is nonempty, compact and convex (Assumption 1), *i.e*. it is contractible, has the fixed point and the Kakutani fixed point properties, see Lemmas 6 and 7. This remark allows us to deduce that $PO(X,f)$ is compact and contractible, has the fixed point and the Kakutani fixed point properties.

According to Lemma 8 we obtain that the Pareto-front set $PF(X,f)$ is compact and contractible, has the fixed point and the Kakutani fixed point properties too.

The theorem is proven.

Remark 10. It is important to note that the Pareto-optimal and Pareto-front sets are not convex in general. □

Remark 11. As we have shown in Lemma 6, if an arbitrary set is nonempty, compact and convex, then it has the fixed point property. In general, it is not difficult to verify that the Pareto-optimal and Pareto-front sets are nonconvex, but they are compact and contractible. Thus among nonconvex sets, compactness and contractibility do not have direct relationship with the fixed point property. There are examples of compact and contractible sets which do not have the fixed point property. It is not known what types on nonconvex sets have this property.

4. Conclusions

We have shown the compactness, contractibility and fixed point properties of the Pareto-optimal and Pareto-front sets in multi-objective mathematical programming when the feasible domain is compact and convex. The two important facts are that the proof did not use the concavity assumptions on the objective functions which are usually used in this optimization problem, and that the Pareto-optimal and Pareto-front sets are not compact and convex in general.

The authors see three directions for future research related to this article: one would look for general conditions on the objective functions without the assumption of their concavity; one would analyze specific types of concave or quasi-concave objective functions; and one would study the relationship between the first two.

5. References

[1] J. Cohon, "Multi-Objective Programming and Planning," Academic Press, Cambridge, 1978.

[2] Y. Collette and P. Siarry, "Multi-Objective Optimization," Springer, Berlin, 2003.

[3] J. Jahn, "Vector Optimization: Theory, Applications, and Extensions," Springer, Berlin, 2004.

[4] D. Luc, "Theory of Vector Optimization," Springer, Berlin, 1989.

[5] B. Peleg, "Topological Properties of the Efficient Point Set," *Proceedings of the American Mathematical Society*, Vol. 35, No. 2, 1972, pp. 531-536. doi:10.1090/S0002-9939-1972-0303916-2

[6] Z. Slavov and C. Evans, "On the Structure of the Efficient Set," *Mathematics and Education in Mathematics*, Vol. 33, 2004, pp. 247-250.

[7] Z. Slavov, "The Fixed Point Property in Convex Multi-Objective Optimization Problem," *Acta Universitatis Apulensis*, Vol. 15, 2008, pp. 405-414.

[8] R. Steuer, "Multiple Criteria Optimization: Theory, Computation and Application," John Wiley and Sons, Hoboken, 1986.

[9] M. Ehrgott, "Multi-Criteria Optimization," Springer, Berlin, 2005.

[10] S. Boyd and L. Vandenberghe, "Convex Optimization," Cambridge University Press, Cambridge, 2004.

[11] Z. Slavov, "Compactness of the Pareto Sets in Multi-Objective Optimization with Quasi-Concave Functions," *Mathematics and Education in Mathematics*, Vol. 35, 2006, pp. 298-301.

[12] J. Benoist, "Contractibility of the Efficient Set in Strictly Quasi-Concave Vector Maximization," *Journal of Optimization Theory and Applications*, Vol. 110, No. 2, 2001, pp. 325-336. doi:10.1023/A:1017527329601

[13] N. Huy and N. Yen, "Contractibility of the Solution Sets in Strictly Quasi-Concave Vector Maximization on Noncompact Domains," *Journal of Optimization Theory and Applications*, Vol. 124, No. 3, 2005, pp. 615-635. doi:10.1007/s10957-004-1177-9

[14] Z. Slavov, "Contractibility of Pareto Solutions Sets in Concave Vector Maximization," *Mathematics and Education in Mathematics*, Vol. 36, 2007, pp. 299-304.

[15] Z. Slavov, "On the Engineering Multi-Objective Maximization and Properties of the Pareto-Optimal Set," *International e-Journal of Engineering Mathematics*: *Theory and Application*, Vol. 7, 2009, pp. 32-46.

[16] A. Hatcher, "Algebraic Topology," Cambridge Univer-

sity Press, Cambridge, 2002.

[17] J. Borwein and A. Lewis, “Convex Analysis and Nonlinear Optimization: Theory and Examples,” Springer, Berlin, 2000.

[18] A. Cellina, “Fixed Point of Noncontinuous Mapping,” *Atti della Accademia Nazionale dei Lincei Serie Ottava Rendiconti*, Vol. 49, 1970, pp. 30-33.

[19] J. Franklin, “Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed Point Theorems,” Springer, Berlin, 1980.

[20] A. Mukherjea and K. Pothoven, “Real and Functional Analysis,” Plenum Press, New York, 1978.

[21] R. Sundaran, “A First Course in Optimization Theory,” Cambridge University Press, London, 1996.

[22] A. Wilansky, “Topology for Analysis,” Dover Publications, New York, 1998.

