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On the Structure of Infinitesimal Automorphisms of the Poisson-Lie Group $SU(2,\mathbb{R})$

Bousselham Ganbouri

Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco
Email: g.busslem@gmail.com

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Abstract

We study the Poisson-Lie structures on the group $SU(2,\mathbb{R})$. We calculate all Poisson-Lie structures on $SU(2,\mathbb{R})$ through the correspondence with Lie bialgebra structures on its Lie algebra $su(2,\mathbb{R})$. We show that all these structures are linearizable in the neighborhood of the unity of the group $SU(2,\mathbb{R})$. Finally, we show that the Lie algebra consisting of all infinitesimal automorphisms is strictly contained in the Lie algebra consisting of Hamiltonian vector fields.

Keywords

Poisson-Lie Structure, Lie Bialgebra, Hamiltonian, Poisson Automorphism, Linearization

1. Introduction

Let G be a Lie group. A Poisson-Lie structure on G is a Poisson structure on G for which the group multiplication is a Poisson map. Then as is usual in [1]-[3], this is equal to giving an antisymmetric contravariant 2-tensor π on G which satisfies Jacobi identity and the relation

$$\pi(xy) = l_{x*}\pi(y) + r_{y*}\pi(x), \quad \forall x, y \in G, \tag{1}$$

where l_{x*} and r_{y*} respectively denote the left and right translations in G by x and y. We note that a Poisson-Lie structure π has rank zero at a neutral element e of G, *i.e.*, $\pi(e) = 0$.

If we choose local coordinates $(x_1, x_2, \cdots, x_n)$ in a neighborhood U of neutral element e of G, the Poisson-Lie structure π reads

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$$\pi(x)=\sum \pi_{ij}(x)\partial x_i\wedge\partial x_j,\qquad x\in U, \tag{2}$$

where π_{ij} are smooth functions vanishing at e and

$$\{x_i,x_j\}(x)=\pi_{ij}(x),\qquad x\in U, \tag{3}$$

where $\{.,.\}$ is the Poisson bracket associated to π. By this Poisson bracket, $C^\infty(G)$ becomes a Lie algebra.

Let $\mathcal{G}$ be a Lie algebra of G. The derivative of π at e defines a skewsymetric co-commutator map $\delta:\mathcal{G}\to\mathcal{G}\wedge\mathcal{G}$ such that:

1) The map δ is a 1-cocycle, *i.e.*,

$$\delta([X,Y])=ad_X\delta(Y)-ad_Y\delta(X),\qquad \forall X,Y\in\mathcal{G}. \tag{4}$$

2) The dual map $\delta^*:\mathcal{G}^*\wedge\mathcal{G}^*\to\mathcal{G}^*$ is a Lie bracket on $\mathcal{G}^*$.

The map δ is said a Lie bialgebra structure associated to π. Conversely, if G is simply connected, any Lie bialgebra structure $\delta:\mathcal{G}\to\mathcal{G}\wedge\mathcal{G}$ on the Lie algebra $\mathcal{G}=:Lie(G)$ can be integrated to define a unique Poisson-Lie structure π on G such that $d_e\pi=\delta$.

The bialgebra structure δ is called a *coboundary* one when there exists an skewsymmetric element r of $\mathcal{G}\wedge\mathcal{G}$ (the classical r-matrix) such that

$$\delta(S)=ad_S r,\qquad \forall S\in\mathcal{G}. \tag{5}$$

Both properties 1) and 2) imply that the element r has to be a constant solution of the modified classical Yang-Baxter equation (mCYBE) [4]-[6]:

$$ad_S[r,r]=0,\qquad S\in\mathcal{G}. \tag{6}$$

Therefore, a constant solution of mCYBE r on a given Lie algebra $\mathcal{G}$ provide a coboundary Poisson-Lie structure π on (connected and simply connected) group G given by

$$\pi(s)=r_{s*}r-l_{s*}r,\qquad \forall s\in G, \tag{7}$$

where l_{s*} and r_{s*} denote respectively the left and right translations in G by s.

Finally, recall that for semisimple Lie algebras, all Lie bialgebra structures are coboundaries, and the corresponding Poisson-Lie structures can be fully solved through the classical r-matrices.

In this work, We shall treat the case of the Poisson-Lie group $SU(2,\mathbb{R})$. We will calculate, firstly, all Poisson-Lie structures through the correspondence with Lie bialgebra; secondly, we will show that these Poisson-Lie structures are linearizable in a neighborhood of the unity e of the group $SU(2,\mathbb{R})$ and, finally, we shall study infinitesimal automorphism of $SU(2,\mathbb{R})$ with a linear Poisson-Lie structure, and show that the Lie algebra $\mathcal{A}$, consisting of all infinitesimal automorphisms is strictly contained in the Lie algebra $\mathcal{H}$ consisting of Hamiltonian vector fields.

2. The Group $SU(2,\mathbb{R})$ and Lie Algebra $su(2,\mathbb{R})$

The special unitary group $SU(2,\mathbb{R})$ is defined by

$$SU(2,\mathbb{R})=\left\{\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}:\alpha,\beta\in\mathbb{C},\alpha\bar\alpha+\beta\bar\beta=1\right\}.$$

Let $\alpha=x+iy$ and $\beta=z+it$. $SU(2,\mathbb{R})$ can be identified with the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ with the unity $e=(1,0,0,0)$.

The Lie algebra $su(2,\mathbb{R})$ of group $SU(2,\mathbb{R})$ is defined by

$$su(2,\mathbb{R})=\left\{S\in\mathbb{C}^{2\times 2}:{}^t\bar S+S=0\text{ and }Tr(S)=0\right\}.$$

Let

$$e_1=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix};\ e_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\ e_3=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

a basis of $su(2,\mathbb{R})$. The Lie bracket on $su(2,\mathbb{R})$ is defined by

$$[e_3,e_1]=2e_2;\quad [e_3,e_2]=-2e_1;\quad [e_1,e_2]=2e_3.$$

Through a straightforward computation, the left invariant fields associated to this basis had this local expression

$$\begin{aligned}X &= -y\partial_x + x\partial_y + t\partial_z - z\partial_t,\\ Y &= -z\partial_x - t\partial_y + x\partial_z + y\partial_t,\\ Z &= -t\partial_x + z\partial_y - y\partial_z + x\partial_t.\end{aligned}$$

3. The Lie Bialgebra Structure on $su(2,\mathbb{R})$ and the Poisson Lie Structure on $SU(2,\mathbb{R})$

3.1. Lie Bialgebra Structures on $su(2,\mathbb{R})$

Recall that the Lie algebra $su(2,\mathbb{R})$ is semisimple. Then, all Lie bialgebra structures on $su(2,\mathbb{R})$ are coboundaries, there exists an skew symmetric element r of $su(2,\mathbb{R})\wedge su(2,\mathbb{R})$ such that the cocommutator δ is given by

$$\delta(S)=ad_S r,\quad \forall S\in su(2,\mathbb{R}).$$

We stress that the element r satisfies the classical Yang-Baxter Equation (CYBE) (6). Through a long but straightforward computation, we show that these solutions are of the form

$$r=k\cdot e_1\wedge e_2,\quad k\in\mathbb{R}_+^*. \tag{8}$$

So any Lie bialgebra structure of $su(2,\mathbb{R})$ can be written as

$$\delta(e_1)=-2ke_3\wedge e_1,\quad \delta(e_2)=2ke_2\wedge e_3,\quad \delta(e_3)=0. \tag{9}$$

3.2. Poisson-Lie Structures on $SU(2,\mathbb{R})$

Since the Lie bialgebra structures δ on $su(2,\mathbb{R})$ are coboundaries, the Poisson-Lie structures on $SU(2,\mathbb{R})$ corresponding to δ are given by

$$\pi(s)=r_{s_*}r-l_{s_*}r,\quad \forall s\in SU(2,\mathbb{R}),$$

where r is the solution of Yang-Baxter equation given by (8) and r_{x_*} and l_{y_*} respectively denote the right and left translations in $SU(2,\mathbb{R})$ by s. Then, using $\alpha=x+iy$, $\beta=z+it$ and $x^2+y^2+z^2+t^2=1$, one gets

$$\pi(x,y,z,t)=2k(xz-yt)Y\wedge Z-2k(xy+zt)Z\wedge X+2k\left(y^2+z^2\right)X\wedge Y. \tag{10}$$

Let

$$\pi_1=2k(xz-yt);\quad \pi_2=-2k(xy+zt);\quad \pi_3=2k\left(y^2+z^2\right), \tag{11}$$

be the components of π in the basis $(Y\wedge X, Z\wedge X, X\wedge Y)$ of the bivector field.

4. Linearization of Poisson-Lie Structures on $SU(2,\mathbb{R})$

By taking back the formula (2), The Taylor series of the functions π_{ij} reads

$$\pi_{ij}(x)=c_{ij}^k x^k+\theta_{ij}^k(x)x^k, \tag{12}$$

where $c_{ij}^k=\frac{\partial\pi_{ij}}{\partial x^k}(e)$ are the structure constants of a Lie algebra $\mathcal{G}^*$, dual of a Lie algebra $\mathcal{G}$, and the θ_{ij}^k are smooth functions vanishing at e.

The term $c_{ij}^k x^k$ of (12) definines a linear Poisson structure, called the linear part of π. The *linearization* *problem* for a structure π around e is the following [7] [8]:

Linearization problem. *Are there new coordinates where the functions* θ_{ij}^k *vanish identically, so that the Poisson structure is linear in these coordinates*?

Let us notice that the Lie bialgebra structure δ associated to π defines a linear Poisson-Lie structure on the additive group $\mathcal{G}$ $\left(\mathcal{G} \simeq \mathbb{R}^n\right)$ that can be expressed as follows

$$\delta(a) = \sum c_{ij}^k a_k \partial_i \wedge \partial_j, \quad a = (a_1, \cdots, a_n) \in \mathbb{R}^n, \tag{13}$$

where $(\partial_1, \cdots, \partial_n)$ is the canonical basis of $\mathbb{R}^n$.

Let us notice that (13) coincides with the linear part of π, so, the linearization problem would be the following:

There is a local Poisson diffeomorphism $\varphi: G \to \mathcal{G}$ *of a neighborhood in* e *of G to a neighborhood of* 0 *in* $\mathcal{G}$?

If $(\varphi_1, \cdots, \varphi_n)$ are the components of φ, then φ is solution of the system of equations

$$\{\varphi_i, \varphi_j\} = \sum c_{ij}^k \varphi_k, \qquad 1 \le i < j \le n. \tag{14}$$

For the Poisson-Lie structure on $SU(2,\mathbb{R})$ given by (10), the system of equations (14) would be

$$\{\varphi_1, \varphi_2\} = 0, \qquad \{\varphi_2, \varphi_3\} = 2k\varphi_2, \qquad \{\varphi_3, \varphi_1\} = -2k\varphi_1. \tag{15}$$

With the identification of the subgroups of the singular points and the symplectic leaves of $SU(2,\mathbb{R})$ and $su(2,\mathbb{R})$, we have:

Proposition 1. The map $\varphi = (\varphi_1, \varphi_2, \varphi_3)$: $\varphi(x,y,z,t) = \left(y, z, \operatorname{Arctan}\frac{t}{x}\right)$ is a diffeomorphism in the neighborhood of $e = (1,0,0,0)$ such that $\varphi(e) = 0$ and

$$\{\varphi_1, \varphi_2\} = 0, \qquad \{\varphi_2, \varphi_3\} = 2k\varphi_2, \qquad \{\varphi_3, \varphi_1\} = -2k\varphi_1.$$

So, the Poisson-Lie structure π on $SU(2,\mathbb{R})$ is linear in the new variables

$$u = y; \quad v = z; \quad w = \operatorname{Arctan}\frac{t}{x}, \tag{16}$$

and will be written

$$\pi(u,v,w) = 2kv \cdot \partial_v \wedge \partial_w - 2ku \cdot \partial_w \wedge \partial_u. \tag{17}$$

The Poisson bracket associated to π reads

$$\{u,v\} = 0, \qquad \{v,w\} = 2kv, \qquad \{w,u\} = -2ku. \tag{18}$$

5. Casimir Functions and Infinitesimal Automorphisms on $SU(2,\mathbb{R})$

Recall that for $f \in C^\infty(SU(2,\mathbb{R}))$, $\{f,.\}$ defines a derivation of $C^\infty(SU(2,\mathbb{R}))$. Hence there corresponds a vector field χ_f, which we call the Hamiltonian vector field. We denote by $\mathcal{H}$ the Lie algebra of Hamiltonian vector fields.

A Casimir function on $SU(2,\mathbb{R})$ is a function C such that $\{C, f\} = 0$ for all function f. On the other words, C is an element of the center of the Lie algebra $C^\infty(SU(2,\mathbb{R}))$. By simple consideration, we know that for each Casimir function C there exists a function ϕ of one variable such that $C(u,v,w) = \phi\left(\frac{u}{v}\right)$.

Each symplectic leaf is the common level manifold of casimir functions. So, these have for equation:

$$\lambda u + \mu v = 0 \quad \left(\lambda, \mu \in \mathbb{R}; (\lambda, \mu) \ne (0,0)\right),$$

and hence are spheres.

By an automorphism of $SU(2,\mathbb{R})$, we mean a smooth vector field ξ on $SU(2,\mathbb{R})$ such that

$$\mathcal{L}_\xi \pi = 0, \tag{19}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along ξ.

If we denote by $\mathcal{A}$ the Lie algebra consisting of all infinitesimal automorphism, it is easy to see that $\mathcal{H}$ is an ideal of $\mathcal{A}$. Let $\xi = f\partial_u + g\partial_v + h\partial_w$ be a vector field of $\mathcal{A}$. Then three function f, g and h must satisfy:

$$\begin{cases} f = u\partial_u f + v\partial_v f + u\partial_w h; \\ g = u\partial_u g + v\partial_v g + v\partial_w h; \\ v\partial_w f = u\partial_w g. \end{cases} \tag{20}$$

Now we shall clarify the gap between $\mathcal{H}$ and $\mathcal{A}$.

We consider the vector field

$$\mathcal{U} = \pi_1[Y,Z] + \pi_2[Z,X] + \pi_3[X,Y], \tag{21}$$

where (π_1, π_2, π_3) are the components of the structure π in the basis $(Y \wedge Z, Z \wedge X, X \wedge Y)$ given by (11).

In the local coordinates (u, v, w) given by (14), this vector field reads

$$\mathcal{U} = -4kv\partial_u + 4ku\partial_v. \tag{22}$$

A simple check shows that the components of $\mathcal{U}$ satisfy the relations (20). So, the vector field $\mathcal{U}$ belongs to $\mathcal{A}$. In other hand, $\mathcal{U}$ is locally Hamiltonian if and only if there exist a smooth function F in a neighborhood of the unity e of the group $SU(2,\mathbb{R})$ such that $\mathcal{U} = \chi_F$, this is translated by the fact that F is a solution of the following system of equations

$$\begin{cases} u\partial_w = -v, \\ v\partial_w = u \\ u\partial_u + v\partial_v = 0. \end{cases} \tag{23}$$

It is easy to see that (23) does not admit solutions. Hence $\mathcal{U}$ does not belong $\mathcal{H}$. Thus we have proved:

Proposition 2. The ideal $\mathcal{H}$ is strictly contained in the Lie algebra $\mathcal{A}$.

In terms of Poisson cohomology [9], recall that the first Poisson cohomology group $H^1_\pi(SU(2,\mathbb{R}))$ is the quotient of the Lie algebra $\mathcal{A}$ by its ideal $\mathcal{H}$. Then, by Proposition 2, we show that the vector field $\mathcal{U}$ defines a non trivial class $[\mathcal{U}] \in H^1_\pi(SU(2,\mathbb{R}))$. On the other hand, this result shows that the classical result due to Conn [10] [11] stating that for a Poisson structure formally linearizable around a singular point any local Poisson automorphism is Hamiltonian, and not just in the C^∞ category.

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