

Reducibility of Periodic Quasi-Periodic Systems

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Received 28 September 2013; revised 28 October 2013; accepted 5 November 2013

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Abstract

In this work, the reducibility of quasi-periodic systems with strong parametric excitation is studied. We first applied a special case of Lyapunov-Perron (L-P) transformation for time periodic system known as the Lyapunov-Floquet (L-F) transformation to generate a dynamically equivalent system. Then, we used the quasi-periodic near-identity transformation to reduce this dynamically equivalent system to a constant coefficient system by solving homological equations via harmonic balance. In this process, we obtained the reducibility/resonance conditions that needed to be satisfied to convert a quasi-periodic system into a constant one. Assuming the reducibility is possible, we obtain the L-P transformation that can transform original quasi-periodic system into a system with constant coefficients. Two examples are presented that show the application of this approach.

Keywords

L-P Transformation; Quasi-Periodic System; Reducibility

1. Introduction

A matrix function $A(t)$ with a square matrix of dimension n is termed quasi-periodic with k incommensurable frequencies $\omega_1, \dots, \omega_k$ [1] [2]. A quasi-periodic function $f(t)$ can be showed in the form

$$f(t) = F(\omega_1 t, \dots, \omega_n t) \quad (1)$$

where a continuous function is $F(x_1, \dots, x_n)$ of period 2π in x_1, \dots, x_n . In addition, we can always assume that $\omega_1, \dots, \omega_n$ are independent [2]. As Moser [3] stated, the class of all almost periodic functions is not separable while $C^0(\omega_1, \dots, \omega_n)$ is. The integral of a quasi-periodic function is not quasi-periodic even if the mean value of $f(t)$ is zero [3].

Let's consider the linear equation

$$\dot{x} = A(t)x \quad (2)$$

where $x \in \mathbb{R}^n$ and $A(t)$ is a matrix depending quasi-periodically on time. The quasi-periodicity of A enables it possible to raise the Equation (2) to a system of linear equations [4] [5] on $\mathbb{R}^n \times \mathbb{T}^d$ basically writing

$$\dot{x} = \tilde{A}(\theta)x, \dot{\theta} = \omega, \quad (3)$$

where the Equation (2) is acquired when the initial value for θ is zero.

Now let's consider the matrix equation for (3), so that the lifted system [2] becomes

$$\dot{X} = \tilde{A}(\theta)X, \dot{\theta} = \omega, \quad (4)$$

Equation (2) is said to be reducible whenever there is a linear time-varying change of variables

$$x = Z(t)y, \quad (5)$$

called Lyapunov-Perron (L-P) transformation [2], which transforms the system into an equation like $\dot{y} = By$, where B is a constant matrix. As Fink [6] mentioned, if $k = 1$, then all systems are reducible by the Floquet theory. This view is also applicable for common linear systems [7] like

$$\dot{x}(t) = A(t)x(t), \quad (6)$$

and it means that, whenever a system like that is L-P reducible to a constant coefficients system like

$$\dot{x}(t) = Ax(t), \quad (7)$$

This will result in many properties of the original system “such as the growth of the solutions or their boundness” being the same as those of the reduced system with constant coefficients [2].

The primary objective of this work is to develop a practical approach for reducibility of quasi-periodic system with strong parametric periodic excitation. It is noted that in the past, the researchers have studied quasi-periodic system where the order of quasi-periodicity is less than the order of the linear term [8] [9] as given by Equation (8)

$$\ddot{x}(t) + (\delta + \varepsilon(\cos \omega_1 t + \cos \omega_2 t)) \quad (8)$$

In this work, we relax this requirement. We assume the order of the most dominant “strong” periodic excitation to be of the same order as the constant term given by Equation (9) and use the L-F transformation and quasi-periodic near identity transformation to

$$\ddot{x}(t) + (\delta + (\cos \omega_1 t + \varepsilon \cos \omega_2 t)) \quad (9)$$

reduce the system to a constant form. This research will present a practical approach to achieve reducibility of quasi-periodic system and will lay the groundwork for future efforts in optimizing such a process.

1.1. Floquet Theory Overview

Floquet theory is very useful for finding the response or stability of linear time-periodic equations [4] [10] [11]. Consider the linear periodic system

$$\dot{x} = A(t)x \quad (10)$$

$\Phi(t)$ denotes the State Transition Matrix (STM) (fundamental solution matrix) that contains n linearly independent solutions of Equation (10) with the initial conditions $\Phi(0) = I$ as I is an identity square matrix of dimension n .

As such the following conditions hold:

- 1) $\Phi(t+T) = \Phi(T)\Phi(t), 0 \leq t \leq T$ and, consequently
- 2) $\Phi(t+jT) = \Phi(T)^j \Phi(t), 0 \leq t \leq T, j = 2, 3, \dots$
- 3) $x(t) = \Phi(t)x(0), t \geq 0$

These conditions suggest that, if the solution is known for the first period, it can be designed for all time t . The matrix $\Phi(T)$ is called the Floquet transition matrix (FTM). The next condition considers the stability of Equation (10). Let $\zeta_i (i=1, \dots, n)$ denote the eigenvalues of $\Phi(T)$. System given by Equation (10) is asymptotically stable if all ζ_i lie inside the unit circle of the complex plane. If one or more of the eigenvalues of the FTM has magnitude greater than one, the system is unstable. The Floquet multipliers are the eigenvalues ζ_i .

According to the Lyapunov-Floquet (L-F) theorem, STM (the fundamental matrix) $\Phi(t)$ of equation (10) can be written as a product of two matrices as:

$$\Phi(t) = Q(t)e^{Rt} \quad (11)$$

where $Q(t)$ is T -periodic and R is a constant matrix, both, in general, are complex. There also exists factorization of the same form, where R is a real constant matrix and $Q(t)$ is real $2T$ -periodic. $Q(t)$ is called the (or $2T$ -periodic) L-F transformation matrix [12]. For the details on computation of the L-F transformation and its applications, we refer the reader to references [13]-[15].

1.2. Quasi-Periodic System Reducibility

In the paper by Wooden and Sinha [15], it is mentioned that an essential class of dynamical systems may be showed by a set of nonlinear differential equations with periodic/quasi-periodic coefficients multiplying the nonlinearity. They analyzed the system where the linear term was periodic but the nonlinear terms were quasi-periodic as given by equation

$$\dot{x} = [A_0 + A_1(t)]x + \varepsilon f(x, t) \quad (12)$$

where A_0 is a matrix of constant coefficients, $A_1(t)$ is matrix with time periodic coefficients and $f(x, t)$ is a vector with quasi-periodic coefficients. The matrices have dimensions $n \times n$ and the vector is $n \times 1$ dimensional. The authors used the L-F transformation and nonlinear quasi-periodic transformations to study Equation (12) and obtained resonance conditions.

In the past, Arnold [10] demonstrated normal forms of quasi-periodic nonlinear systems with time-invariant linear part. Bogoljubov *et al.* [11] presented the reducibility of quasi-periodic systems to approximate time-invariant forms using a small parameter strategy. Even though small parameters/perturbation type techniques have been successfully used to study stability and reducibility of quasiperiodic systems [9], these techniques are limited by *small parameters* multiplying the nonlinear and/or time-varying conditions. Belhaq *et al.* [16] considered a homogeneous Mathieu equation with quasi-periodic linear coefficients and a constant nonlinear coefficient. The small parameter strategy of multiple scales was used twice to the system to acquire an approximate time-invariant system. Researchers have used small parameter assumption to plot stability charts for quasiperiodic systems [17] [18].

The system studied in this research is different that the systems studied earlier. Here, we consider the system of the form given by Equation (13)

$$\dot{x} = [A_0 + A_1(t) + \varepsilon A_2(t)]x \quad (13)$$

where $A_1(t)$ is the time periodic matrix $A_1(t) = A_1(t + T_1)$ and $A_2(t)$ matrix has the coefficients with incommensurate frequencies and A_0 is a constant matrix of appropriate dimensions. We present technique to reduce Equation (13) to a constant coefficient system without any limitation on the magnitude of $A_1(t)$ *i.e.* small parameter assumption. This paper is organized as follows. In section two, the reducibility formulation is presented and the resonance/reducibility conditions are derived. In section three, application of this approach is presented with two examples. Section four has discussion and conclusions.

2. Reducibility Problem Formulation

Consider the time periodic part of Equation (13),

$$\dot{x} = [A_0 + A_1(t)]x \quad (14)$$

The state transition matrix (STM) of Equation (11) can be factored as

$$\Phi_1 = Q_1(t)e^{R_1 t} \quad (15)$$

where $Q(t)$ is typically $2T_1$, periodic such that $Q(t) = Q(t + 2T_1)$ and R is a real-valued $n \times n$ constant matrix. Applying the L-F transformation $x = Q_1(t)z$ to Equation (13) gives

$$\dot{z} = [R + \varepsilon Q_1^{-1}(t)A_2(t)Q_1(t)]z \quad (16)$$

Application of modal transformation $z = M\bar{z}$ to Equation (16) converts the constant part of the system in the Jordan form, where M is eigenvector of R

$$\dot{\bar{z}} = [J + \varepsilon M^{-1}Q_1^{-1}(t)A_2(t)Q_1(t)M]\bar{z} \quad (17)$$

where J is the Jordan Form of R and $M^{-1}Q_1^{-1}(t)A_2(t)Q_1(t)M$ is quasi-periodic matrix with ω_1 and ω_2 . We use following near-identity transformation to Equation (17)

$$\bar{z} = y + \varepsilon Q(t)y \quad (18)$$

where the unknown nonlinear function $Q(t)$ is quasi-periodic. Substituting Equation (18) in Equation (16), we obtain

$$[J + \varepsilon M^{-1}Q_1^{-1}(t)A_2(t)Q_1(t)M]\{y + \varepsilon Q(t)y\} = \dot{y} + \varepsilon \dot{Q}(t)y + \varepsilon Q(t)\dot{y} \quad (19)$$

Assuming $M^{-1}Q_1^{-1}(t)A_2(t)Q_1(t)M = G(t)$, and expanding Equation (19) and simplification, we obtain

$$\begin{aligned} \dot{y} = & Jy + \varepsilon JQ(t)y + \varepsilon G(t)y + \varepsilon^2 G(t)Q(t)y - \varepsilon \dot{Q}(t)y - \varepsilon Q(t) \cdot J - \varepsilon^2 Q(t) \cdot JQ(t)y \\ & - \varepsilon^2 Q(t) \cdot G(t)y - \varepsilon^3 Q(t) \cdot G(t)Q(t)y + \varepsilon^2 Q(t) \cdot \dot{Q}(t)y \end{aligned} \quad (20)$$

Now, if

$$\begin{aligned} \varepsilon JQ(t) + \varepsilon G(t) + \varepsilon^2 G(t)Q(t) - \varepsilon \dot{Q}(t) - \varepsilon Q(t) \cdot J - \varepsilon^2 Q(t) \cdot JQ(t) \\ - \varepsilon^2 Q(t) \cdot G(t) - \varepsilon^3 Q(t) \cdot G(t)Q(t) + \varepsilon^2 Q(t) \cdot \dot{Q}(t) = 0 \end{aligned} \quad (21)$$

Then, Equation (20) reduces to $\dot{y} = Jy$. Thus, quasi-periodic system given by Equation (16) will be reduced to a constant system.

Equation (21) is similar to the homological equation obtained in the normal form reduction. By collecting the coefficient of ε , we get the reducibility equation

$$JQ(t) + G(t) - Q(t)J - \dot{Q}(t) = 0 \quad (22)$$

For illustration, assume all the matrices in Equation (22) are 2×2 of the form

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} + \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} - \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} - \begin{bmatrix} \dot{Q}_{11} & \dot{Q}_{12} \\ \dot{Q}_{21} & \dot{Q}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (23)$$

Equation (23) can be expanded in the scalar form as

$$\begin{aligned} \lambda_1 Q_{11}(t) + G_{11}(t) - Q_{11}(t)\lambda_1 - \dot{Q}_{11}(t) &= 0 \quad (a) \\ \lambda_1 Q_{12}(t) + G_{12}(t) - Q_{12}(t)\lambda_2 - \dot{Q}_{12}(t) &= 0 \quad (b) \\ \lambda_2 Q_{21}(t) + G_{21}(t) - Q_{21}(t)\lambda_1 - \dot{Q}_{21}(t) &= 0 \quad (c) \\ \lambda_2 Q_{22}(t) + G_{22}(t) - Q_{22}(t)\lambda_2 - \dot{Q}_{22}(t) &= 0 \quad (d) \end{aligned} \quad (24)$$

It can be noted that Equations (21a) and (21d) can be solved as

$$\begin{aligned} Q_{11}(t) &= \int G_{11}(t) dt \\ Q_{22}(t) &= \int G_{22}(t) dt \end{aligned} \quad (25)$$

To find the solution of Equation (24b) and (24c), elements of $Q(t)$ and $G(t)$ will have to be expanded in multiple Fourier series as

$$\begin{aligned}
Q_{mn} &= \sum_{p_1}^{+\infty} \sum_{p_2}^{+\infty} h_{mnp_1 p_2} e^{\bar{i} \cdot (\bar{p} \cdot \omega) t} \\
G_{mn} &= \sum_{p_1=-\infty}^{+\infty} \sum_{p_2=-\infty}^{+\infty} v_{mnp_1 p_2} e^{\bar{i} \cdot (\bar{p} \cdot \omega) t}
\end{aligned} \tag{26}$$

where $h_{mnp_1 p_2}$ are the unknown and $v_{mnp_1 p_2}$ known coefficients. Term by term comparison yields

$$h_{mnp_1 p_2} = \frac{v_{mnp_1 p_2}}{\bar{i} (\bar{p} \cdot \omega) + \lambda_m - \lambda_n} \tag{27}$$

Thus, Equation (27) can be solved if

$$\bar{i} (\bar{p} \cdot \omega) + \lambda_m - \lambda_n \neq 0 \tag{28}$$

Equation (28) is called as the reducibility condition. Thus, a quasi-periodic system can be reduced to a constant system provided Equation (28) is satisfied. It is also noted that since the order of quasi-periodicity is ε , we obtain solution of $Q(t)$ to the order ε . It is possible to extend this approach to higher orders of ε , if needed.

3. Applications

In this section, we present two examples—a commutative system and a quasi-periodic Mathieu equation. We apply the procedure discussed in Section two to reduce the quasi-periodic system to a constant one.

3.1. Commutative System

Consider the following commutative system

$$\dot{x} = [\bar{A}(t) + \tilde{A}(t)]x \tag{29}$$

where

$$\bar{A}(t) = \begin{pmatrix} -1 + \alpha \cos^2(t) & 1 - \alpha \sin(t) \cos(t) \\ -1 - \alpha \sin(t) \cos(t) & -1 + \alpha \sin^2(t) \end{pmatrix}$$

$$\tilde{A}(t) = \begin{pmatrix} \sin(\pi t) & \cos(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \text{ and } \alpha \text{ is a system parameter}$$

The L-F transformation for matrix $Q_1(t)$ is

$$Q_1(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}. \tag{30}$$

and

$$R = \begin{pmatrix} (\alpha - 1) & 0 \\ 0 & -1 \end{pmatrix}.$$

Applying the L-F Transformation $x = Q_1 z$ to Equation (31) yields

$$\dot{z} = [R + \varepsilon Q_1^{-1} \tilde{A}(t) Q_1] z \tag{31}$$

It is noted that for this special case matrix R is in the Jordan form $R = J$. Thus

$$\dot{z} = \underbrace{\begin{pmatrix} (\alpha - 1) & 0 \\ 0 & -1 \end{pmatrix}}_J + \varepsilon \underbrace{\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \cdot \begin{pmatrix} \sin(\pi t) & \cos(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \cdot \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}}_{G(t)} z \tag{32}$$

Equation (32) can be reduced to a constant coefficient system with the procedure discussed in section two. First a quasi-periodic near identity transformation $z = s + \varepsilon Q(t)s$ can be substituted in Equation (32) where

$Q(t)$ is the quasi-periodic matrix with unknown coefficients. The elements of this matrix have the form given by Equation (26) with $\omega_1 = 1, \omega_2 = \pi$. At this point, collecting the terms of order ε yields the homological equation

$$JQ(t) + G(t) - Q(t)J - \dot{Q}(t) = 0 \quad (33)$$

Expanding $Q(t)$ and $G(t)$ in the form given by Equation (26) and collecting the terms via harmonic balance yields the elements of $Q(t)$ if the resonance condition given by Equation (28) is satisfied. Thus, we reduce the quasi-periodic system given by Equation (26) to a constant coefficient system given by Equation (31)

$$\dot{s} = Js \quad (34)$$

To compare the results we integrate Equation (29) numerically and generate the time traces and phase plane. For the reduced system given by Equation (31), it is possible to find a closed form solution as

$$s(t) = e^{Jt} s_0 \quad (35)$$

where s_0 is the initial condition. Using the transformation $z = s + \varepsilon Q(t)s$ it is possible to obtain the closed form solution in $x(t)$.

In **Figure 1**, the solutions of the original system and the reduced order system are compared. Time traces shown in red were obtained via numerical integration of Equation (29) and time traces in blue were obtained by solving the reduced constant coefficient system in the closed form (given by Equation (35)) and mapping it back to original coordinates $x(t)$ via quasi-periodic near identity and L-F transformation. It can be observed that the time traces match quite well. In **Figure 2**, phase portraits of the original and reduced system are compared. It can be seen that the phase portraits also match quite well.

3.2. Quasiperiodic Mathieu Equation

Consider a Quasiperiodic Mathieu equation given by

$$\ddot{\bar{x}} + (\delta + \cos \omega_1 t + \varepsilon \cos \omega_2 t) \bar{x} = 0 \quad (36)$$

where $\delta = 0.5, \omega_1 = \pi$ and $\omega_2 = 1$ are incommensurate frequencies and ε is the small parameter. Equation (36) can be represented in the state space form as given by Equation (13) where

$$x = \begin{bmatrix} \bar{x} \\ \dot{\bar{x}} \end{bmatrix}, A_0 = \begin{bmatrix} 0 & 1 \\ -\delta & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ -\cos \omega_1 t & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ -\cos \omega_2 t & 0 \end{bmatrix} \quad (37)$$

It is noted that the parametric excitation is strong and the L-F and modal transformation is applied to get Equation (38).

$$\dot{\bar{x}} = \left[\bar{J} + \varepsilon M^{-1} Q_1^{-1}(t) A_2(t) Q_1(t) M \right] \bar{z} \quad (38)$$

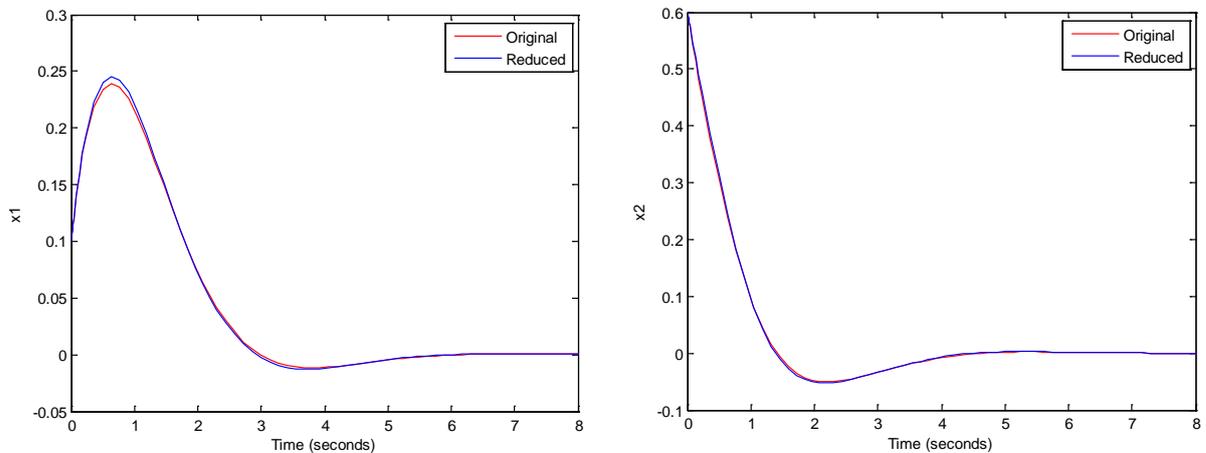


Figure 1. Time Trace Comparison.

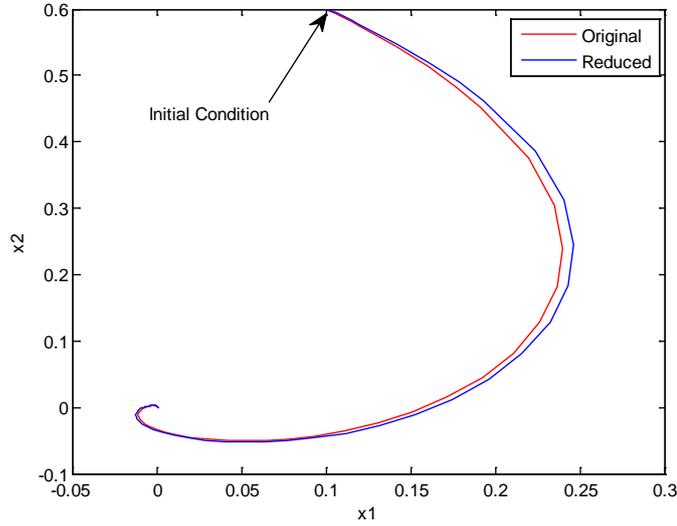


Figure 2. Phase portrait comparison.

where $\bar{J} = \begin{bmatrix} +0.26i & 0 \\ 0 & -0.26i \end{bmatrix}$ and the detailed expression for $\bar{G}(t) = M^{-1}Q_1^{-1}(t)A_2(t)Q_1(t)M$ was obtained by Mathematica™. At this stage, as before, a quasi-periodic near identity transformation $\bar{z} = (I + \varepsilon\bar{Q}(t))\bar{s}$ can be substituted. After collecting the terms of order ε Equation (35) can be obtained

$$\bar{J}\bar{Q}(t) + \bar{G}(t) - \bar{Q}(t)J - \dot{\bar{Q}}(t) = 0 \tag{39}$$

Expanding $\bar{Q}(t)$ and $\bar{G}(t)$ in the form given by Equation (23) and collecting the terms via harmonic balance, elements of $\bar{Q}(t)$ can be found out. It is noted that in this case the reducibility condition (given by Equation (25)) was satisfied. Thus, we could reduce the quasi-periodic system given by Equation (36) to a constant coefficient system given by Equation (40)

$$\begin{Bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{Bmatrix} = \begin{bmatrix} +0.26i & 0 \\ 0 & -0.26i \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} \tag{40}$$

The solution of Equation (40) can be found in the closed form as

$$\bar{s}(t) = e^{Jt} \bar{S}_0 \tag{41}$$

where $\bar{s} = [\bar{s}_1, \bar{s}_2]^T$ and \bar{S}_0 is the vector of typical initial conditions. To compare the results we integrated Equation (36) numerically and generated the time traces and phase portrait. For the reduced system, we used Equation (41) and applied the quasi-periodic transformation $\bar{z} = (I + \varepsilon\bar{Q}(t))\bar{s}$, L-F and modal transformation to obtain the time traces and phase portrait of vector $x = [\bar{x}, \bar{x}]$. These solutions are compared in Figures 3 and 4. Figure 3 shows the time traces of the states and Figure 4 shows the phase portrait of the “Original” and “Reduced” states. It can be seen that these solutions match quite well.

4. Conclusions

In this work, a new approach for reducibility of quasi-periodic system analysis is presented using the L-F transformation and quasi-periodic near identity transformation. In this process, one obtains the reducibility conditions and the quasi-periodic system can be converted to a constant coefficient system provided the reducibility conditions are satisfied.

The resulting homological is expanded using multiple Fourier series which can be solved for the unknown Fourier coefficients of the near-identity transformation coefficients via harmonic balance. Two examples presented a commutative system and quasi-periodic Mathieu equation. In both cases, the parametric excitation is

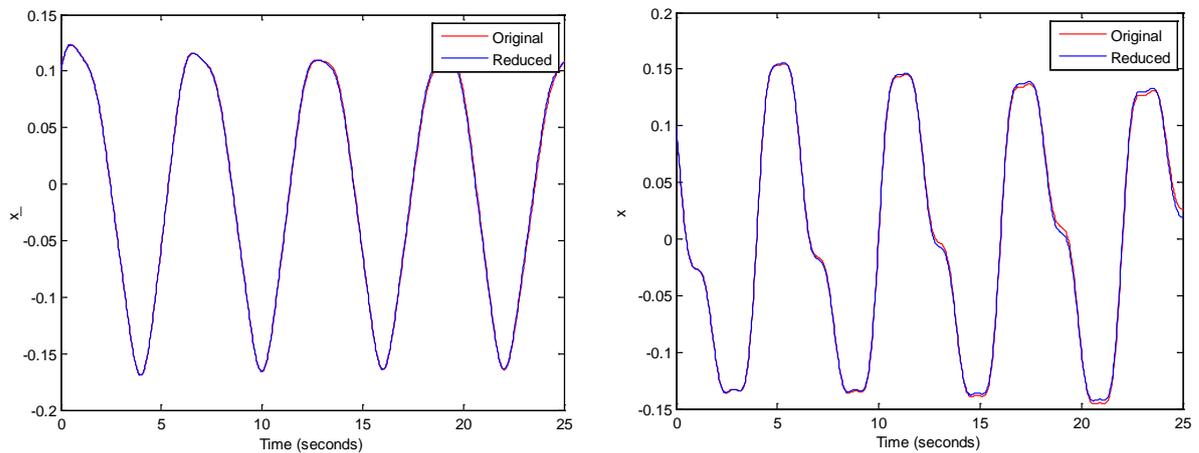


Figure 3. Time Trace Comparison.

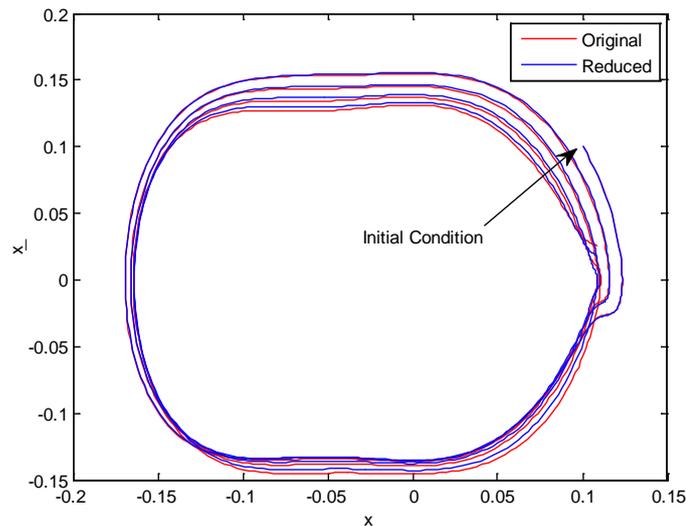


Figure 4. Phase Plane Comparison.

strong. Simulations and phase plane were plotted to compare results from numerical integration and closed form solution. It can be seen that the results matched quite well. Thus, if the reducibility conditions are satisfied then the quasi-periodic systems with strong parametric excitation can be reduced to a constant form using the L-F transformation. It is possible to analyze or control this reduced order time invariant system and map the results back using appropriate transformations to study and control original quasi-periodic system.

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