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A New Characterization of Totally Umbilical Hypersurfaces in de Sitter Space

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ABSTRACT

It is shown that a compact spacelike hypersurface which is contained in the chronological future (or past) of an equator of de Sitter space is a totally umbilical round sphere if the *k*th mean curvature function H_k is a linear combination of $H_{k+1},\cdots, H_n$. This is a new angle to characterize round spheres.

KEYWORDS

de Sitter Space; Spacelike Hypersurface; Higher Order Mean Curvatures

1. Introduction

Let R_1^{n+2} be the (*n* + 2)-dimensional Lorentz-Minkowski space, namely, the real vector space R^{n+2} endowed with the Lorentzian inner product $\langle , \rangle$ given by

$$\langle v,w\rangle=\sum_{i=1}^{n+1}v_iw_i-v_{n+2}w_{n+2},\ v=(v_1,\cdots,v_{n+2}),\ w=(w_1,\cdots,w_{n+2})\in R^{n+2}.$$

Then the *n*-dimensional de Sitter space is defined by $S_1^{n+1}=\{x\in R_1^{n+2}\,|\,\langle x,x\rangle=1\}$. It is well known that, for $n\ge 2$ the de Sitter space S_1^{n+1} is the standard simply connected Lorentzian space form of positive constant sectional curvature. A smooth immersion $\psi: M^n\to S_1^{n+1}\subset R_1^{n+2}$ of an n-dimensional connected manifold M^n is said to be a spacelike hypersurface if the induced metric via ψ is a Riemannian metric on M^n, which, as usual, is also denoted by $\langle , \rangle$.

The interest for the study of spacelike hypersurfaces in de Sitter space is motivated by the fact that such hypersurfaces exhibit nice Bernstein-type properties. In 1977, Goddard [1] conjectured that the only complete spacelike hypersurfaces with constant mean curvature in S_1^{n+1} should be the totally umbilical ones. This conjecture motivated the work of an important number of authors who considered the problem of characterizing the totally umbilical spacelike hypersurfaces of de Sitter space. In [2], Montiel showed that the only compact spacelike hypersurfaces in S_1^{n+1} with constant mean curvature H_1 were the totally umbilical round spheres. More recently, Cheng and Ishikawa [3] have shown that the totally umbilical round spheres are the only compact spacelike hypersurfaces in de Sitter space with constant scalar curvature $S<n(n-1)$.

The natural generalization of mean and scalar curvature for a spacelike hypersurface in de Sitter space are the *k*th mean curvature H_k for $k=1,\cdots,n$. Actually, H_1 is the mean curvature and H_2 is, up to a constant, the scalar curvature of the hypersurface. In [4], Aledo, jointly with Alias and Romero, developed some integral formulas for compact spacelike hypersurfaces in S_1^{n+1} and applied them in order to characterize the totally umbilical round spheres of S_1^{n+1}.

Theorem 1([4], Theorem 7) *Let* $\psi: M^n \to S_1^{n+1} \subset R_1^{n+2}$ *be a compact spacelike hypersurface in de Sitter space which is contained in the chronological future* (*or past*) *of an equator of* S_1^{n+1}. *If* H_k *is constant for some* $k, 1 \le k \le n$, *then* M^n *is a totally umbilical round sphere*.

Since $H_0 = 1$ by definition, the result above can be read as follows: if H_k / H_0 is constant for some $k, 1 \le k \le n$, then M^n is a totally umbilical round sphere. In [5], Alias extended Theorem 1 in the following way.

Theorem 2 ([5]) *Let* $\psi: M^n \to S_1^{n+1} \subset R_1^{n+2}$ *be a compact spacelike hypersurface in de Sitter space which is contained in the chronological future* (*or past*) *of an equator of* S_1^{n+1}. *If* H_l *does not vanish on* M^n *and the ratio* H_k / H_l *is constant for some* $k, l, 1 \le l < k \le n$, *then* M^n *is a totally umbilical round sphere*.

In [6] the authors considered that H_k is the linear combination of $H_1, \cdots, H_{k-1}$, and proved:

Theorem 3 ([6]) *Let* $\psi: M^n \to S_1^{n+1} \subset R_1^{n+2}$ *be a compact spacelike hypersurface in de Sitter space which is contained in the chronological future* (*or past*) *of an equator of* S_1^{n+1}. *If there are nonnegative constants* $C_1, C_2, \cdots, C_{l-1}$, *at least one* C_i *is positive, such that* $H_l = \sum_{i=1}^{l-1} C_i H_i$ *holds on* M^n, *then* M^n *is a totally umbilical round sphere*.

In this paper, we will show another characterization of totally umbilical round sphere, which extends Theorems 1 and 2 above.

Theorem 4 *Let* $\psi: M^n \to S_1^{n+1} \subset R_1^{n+2}$ *be a compact spacelike hypersurface in de Sitter space which is contained in the chronological future* (*or past*) *of an equator of* S_1^{n+1}. *If* H_n *does not vanish on* M^n,

- for some fixed k, $1 \le k \le n-1$, there exist constants $C_j \ge 0$ such that $H_k = \sum_{j=k+1}^{n} C_j H_j$ on M^n, then M^n is a totally umbilical round sphere.
- there are $n-1$ constants $C_j \ge 0$ such that $H_0 = \sum_{j=1}^{n-1} C_j H_j$ on M^n, then M^n is a totally umbilical round sphere.

Remark.

- Note in some special cases the condition H_n does not vanish should can be dropped, for examples, only one coefficient $C_j > 0$ case. However in general cases we can not drop it now.
- The corresponding theorem characterizes ellipsoids also holds in affine differential geometry.

2. Preliminaries

Throughout this paper we will deal with compact spacelike hypersurfaces in de Sitter space. Recall that every compact spacelike hypersurfaces M^n in S_1^{n+1} is diffeomorphic to an *n*-sphere [4] and, in particular, it is orientable. Then, there exists a timelike unit normal field N globally defined on M^n. We will refer to N as the Gauss map of the immersion and we will say that M^n is oriented by N.

We will denote by $A: \chi(M) \to \chi(M)$ the shape operator of M^n in S_1^{n+1} with respect to N, which is given by

$$A(X) = -dN(X).$$

Associated to the shape operator of M^n there are n algebraic invariants, which are the elementary symmetric functions σ_k of its principal curvatures $k_1, \cdots, k_n$, given by

$$\sigma_k(k_1, \cdots, k_n) = \sum_{i_1 < \cdots < i_k} k_{i_1} \cdots k_{i_k}, \quad 1 \le k \le n.$$

The *k*th mean curvature H_k of the spacelike hypersurfaces is then defined by

$$\binom{n}{k} H_k = (-1)^k \sigma_k(k_1, \cdots, k_n) = \sigma_k(-k_1, \cdots, -k_n).$$

When $k = 1$, $H_1 = -(1/n)\operatorname{tr}(A)$ is the mean curvature of M^n. On the other hand, when $k = n$, $H_n = (-1)^n \det(A)$ defines the Gauss-Kronecker curvature of the spacelike hypersurface, and for $k = 2$, H_2 is, up to a constant, the scalar curvature S of M^n, since $S = n(n-1)(1 - H_2)$ (for details see [4]).

The proof of our theorem makes an essential use of the following integral formulas for compact spacelike hypersurfaces in S_1^{n+1}, which is developed in [4].

Lemma 5 (Minkowski formulas) *Let* $\psi: M^n \to S_1^{n+1} \subset R_1^{n+2}$ *be a compact spacelike hypersurface immersed into de Sitter space and let* $a \in R_1^{n+2}$ *a fixed arbitrary vector*. *For each* $r = 0, \cdots, n-1$ *the following formula holds*:

$$\int_M \left(-H_r \langle a,\psi\rangle + H_{r+1}\langle a,N\rangle\right)\mathrm{d}V = 0,$$

where $\mathrm{d}V$ is the n-dimensional volume element of M^n with respect to the induced metric and the chosen orientation.

3. Proof of the Theorem 4

Let us assume, for instance, that the hypersurface $\psi : M^n \to S_1^{n+1} \subset R_1^{n+2}$ is contained in the future of the equator determined by a unit timelike vector $a \in R_1^{n+2}$ (the case of the past is similar). That means that

$$\psi(M) \subset \left\{x \in S_1^{n+1} : \langle a,x\rangle < 0\right\}.$$

Let us orient M^n by the Gauss map N which is in the same time-orientation as a, so that $\langle a,N\rangle \le -1 < 0$. Since the height function $\langle a,\psi\rangle$ is negative on M^n, by compactness there exists a point $p_0 \in M$ where it attains its maximum

$$\langle a,\psi(p_0)\rangle = \max_{p\in M} \langle a,\psi(p)\rangle < 0.$$

Therefore, its gradient vanishes at that point, $\nabla\langle a,\psi\rangle(p_0) = 0$, and its Hessian satisfies

$$\nabla^2 \langle a,\psi\rangle(p_0)(v,w) = -\langle a,\psi(p_0)\rangle\langle v,w\rangle - \langle a,N(p_0)\rangle\langle A_{p_0}(v),w\rangle \le 0$$

for all $v,w \in T_{p_0}M$ (for the details see the proof of Theorem 7 in [4]). On the other hand, since

$$\langle a,N\rangle^2 = 1 + \langle a,\psi\rangle^2 + \left|\nabla\langle a,\psi\rangle\right|^2$$

and

$$\nabla\langle a,\psi\rangle(p_0) = 0,$$

then

$$-\langle a,N(p_0)\rangle = \sqrt{1+\langle a,\psi(p_0)\rangle^2}.$$

Therefore, choosing $\{e_1,\cdots,e_n\}$ a basis of principal directions at the point p_0 we conclude that

$$k_i(p_0) \le \frac{\langle a,\psi(p_0)\rangle}{\sqrt{1+\langle a,\psi(p_0)\rangle^2}} < 0 \tag{1}$$

for each $i = 1,\cdots,n$. In particular, $H_j(p_0)(1 \le j \le n)$ are positive. The mean curvature functions H_n is positive on M^n (recall that H_n does not vanish on M^n by assumption). Therefore, from the proof of Lemma 1 in [7] and taking into account the sign convention in our definition of the higher order mean curvature, it follows that every H_j is positive for $j = 1,\cdots,n$ and

$$\frac{H_1}{H_0} \ge \frac{H_2}{H_1} \ge \cdots \ge \frac{H_n}{H_{n-1}}, \tag{2}$$

with equality at any stage only at umbilical points.

Let us start proving the first statement of Theorem 4. Using

$$H_k = \sum_{j=k+1}^{n} C_j H_j$$

and the Minkowski formulae, we have

$$\int_M H_{k-1}\langle a,\psi\rangle\mathrm{d}V = \int_M H_k\langle a,N\rangle\mathrm{d}V = \sum_{j=k+1}^{n} C_j \int_M H_j\langle a,N\rangle\mathrm{d}V = \sum_{j=k+1}^{n} C_j \int_M H_{j-1}\langle a,\psi\rangle\mathrm{d}V$$

That is,

$$\int_M \langle a,\psi\rangle\left(H_{k-1} - \sum_{j=k+1}^{n} C_j H_{j-1}\right)\mathrm{d}V = 0.$$

Now we claim that

$$H_{k-1} - \sum_{j=k+1}^{n} C_j H_{j-1} \le 0 \tag{3}$$

on M, with equality if and only if $k_1 = \cdots = k_n$. Assume that (3) is true. Then, since $\langle a, \psi \rangle < 0$ on M, we conclude that

$$H_{k-1} - \sum_{j=k+1}^{n} C_j H_{j-1} \equiv 0,$$

which implies that M is an totally umbilical round sphere.

It remains to prove (3). Using the assumption of theorem 4, that is $H_k = \sum_{j=k+1}^{r} C_j H_j$ and (2), we have

$$H_{k-1} - \sum_{j=k+1}^{n} C_j H_{j-1} = \sum_{j=k+1}^{n} C_j \left(\frac{H_{k-1}}{H_k} H_j - H_{j-1} \right) \le 0. \tag{4}$$

Now we prove the second statement. Using

$$H_0 = \sum_{j=1}^{n-1} C_j H_j$$

and the Minkowski formulae, we have

$$\int_M H_1 \langle a, N \rangle \mathrm{d}V = \int_M H_0 \langle a, \psi \rangle \mathrm{d}V = \sum_{j=1}^{n-1} C_j \int_M H_j \langle a, \psi \rangle \mathrm{d}V = \sum_{j=1}^{n-1} C_j \int_M H_{j+1} \langle a, N \rangle \mathrm{d}V$$

That is,

$$\int_M \langle a, N \rangle \left(H_1 - \sum_{j=1}^{n-1} C_j H_{j+1} \right) \mathrm{d}V = 0.$$

Now we claim that

$$H_1 - \sum_{j=1}^{n-1} C_j H_{j+1} \ge 0 \tag{5}$$

on M, with equality if and only if $k_1 = \cdots = k_n$. Assume that (5) is true. Then, since $\langle a, N \rangle < 0$ on M, we conclude that

$$H_1 - \sum_{j=1}^{n-1} C_j H_{j+1} \equiv 0,$$

which implies that M is an totally umbilical round sphere. It remains to prove (5). As in the first proof, using the assumption of theorem 4, that is $H_0 = \sum_{j=1}^{n-1} C_j H_j$, and (2) we known

$$H_1 - \sum_{j=1}^{n-1} C_j H_{j+1} = \sum_{j=1}^{n-1} C_j \left(\frac{H_1}{H_0} H_j - H_{j+1} \right) \ge 0.$$

This completes the proof of the Theorem 4.

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