

Laplace Transform Method for Unsteady Thin Film Flow of a Second Grade Fluid through a Porous Medium

M. Ali¹, M. Awais^{2*}

¹Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

²Department of Mathematics and Statistics, University of Victoria, Victoria, Canada

Email: *mawais@uvic.ca

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ABSTRACT

In this article, we have effectively used the Numerical Inversion of Laplace transform to study the time-dependent thin film flow of a second grade fluid flowing down an inclined plane through a porous medium. The solution to the governing equation is obtained by using the standard Laplace transform. However, to transform the obtained solutions from Laplace space back to the original space, we have used the Numerical Inversion of Laplace transform. Graphical results have been presented to show the effects of different parameters involved and to show how the fluid flow evolves with time.

KEYWORDS

Numerical Inversion of Laplace Transform; Unsteady Thin Film; Second Grade Fluid

1. Introduction

Laplace transform is a very useful tool for solving the differential equations. However, to analytically compute the inverse Laplace transform of the solutions obtained by the use of Laplace transform is a very important but complicated step. To overcome this issue, several algorithms for numerical inversion of Laplace transform have been proposed in literature [1-4]. Here we implement the idea of numerical inversion of Laplace transform presented by Weeks [1] and the one by Juraj and Lubomir [2].

The inverse Laplace transform $G(s)$ of a function $g(t)$ is given by the following contour integral

$$g(t) = \frac{1}{2\pi i} \lim_{\omega \rightarrow \infty} \int_{\sigma-i\omega}^{\sigma+i\omega} G(s) e^{st} ds. \quad (1)$$

Here $s = \sigma + i\omega$ and σ is greater than the real part of any singularity in the transformed function $G(s)$. Weeks method [1] for numerical inversion of Laplace transform is based on the use of the Laguerre functions and is given as

$$\begin{cases} g(t) = \lim_{N \rightarrow \infty} g_N(t), \\ g_N(t) = e^{\sigma t} \sum_{n=0}^{N-1} a_n e^{-bt} L_n(2bt), \\ L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n). \end{cases}$$

Whereas, the method of Juraj and Lumboir [2] is based on the use of infinite series for $\cosh(z)$ and $\sinh(z)$ along with the application of residue theorems to compute the contour integral given in (1). It gives us the following expression, which is also implemented here, for the numerical inversion of Laplace transform

$$\begin{cases} e^{st} = \frac{e^a}{2 \sinh(a-st)}, \\ g(t, a) = \frac{e^a}{t} \left[\frac{1}{2} G\left(\frac{a}{t}\right) \sum_{n=1}^{\infty} (-1)^n \operatorname{Re} \left\{ G\left(\frac{a}{t} + in \frac{\pi}{t}\right) \right\} \right]. \end{cases}$$

Non-Newtonian fluids have been a famous topic of research because of their diverse use in many industrial processes. Various complex fluids such as oils, polymer

*Corresponding author.

melts, different types of drilling muds and clay coatings and many emulsions are included in the category of non-Newtonian fluids. One of the very important models suggested for non-Newtonian fluids is called the second-grade fluid. Flows of second grade fluid have been studied by many [5-9]. Also, the flow of thin films has vast applications in industry. Such flows have applications in microchip productions, in biology, in chemistry and many other fields. Very often, thin film flows are examined by using steady flows. Some investigators have recently obtained some results for thin film flows of non-Newtonian fluids [10-14]. An investigation of thin film flow of a second order fluid is made by Huang and Li [15]. Miladinova *et al.* [11] studied the thin film flow of a power law fluid falling down an inclined plane and found a numerical solution. Analysis of lubrication flow of an upper-convected Maxwell fluid was carried out by Zhang and Li [12]. The field equation of micropolar fluids has been studied by Gan and Ji [16]. Siddiqui *et al.* [13,14] investigated thin film flow of non-Newtonian fluids by using the homotopy perturbation method. Linear and nonlinear stability analysis of the thin micropolar film flowing down a rotating cylinder was made by Chen [17]. M. Awais and S. Nadeem [18-20] investigated solutions for thin film flow of non-Newtonian fluids using a homotopy analysis method. However, studying unsteady thin film flow problems has always been a difficult and challenging task. Here, we solve an unsteady thin film flow problem of a second grade fluid by using the Laplace transform and then implement the methods of numerical inversion of Laplace transform described earlier to obtain the solutions in the original space and to find the effects of different fluid parameters. It also helps us to better understand the evolution of the flow with time.

2. Basic Equations

The momentum balance equation in the presence of a body force B and the equation of conservation of mass are respectively given as

$$\rho \frac{Dv}{Dt} = \text{div}(\tau) + \rho B, \tag{2}$$

$$\nabla \cdot v = 0, \tag{3}$$

with ρ being the density, v the velocity, τ the stress tensor and $\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$ represents the material derivative. The stress tensor τ in Equation (2) for a second grade fluid has form

$$\tau = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{4}$$

where I is the identity tensor, p is the pressure, μ is the dynamic viscosity, α_1 is the elastic coefficient

and α_2 is the transverse viscosity coefficient and A_1 and A_2 represent the Rivlin-Ericksen tensors. The tensors A_1 and A_2 are defined by the following expressions

$$A_1 = (\nabla v) + (\nabla v)^T, \tag{5}$$

$$A_2 = \frac{DA_1}{Dt} + A_1(\nabla v) + (\nabla v)^T A_1. \tag{6}$$

The material constants μ , α_1 and α_2 also satisfy

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0, \tag{7}$$

due to Clausius-Duhem inequality and the condition that the Helmholtz free energy is minimum when the fluid is at rest [21].

3. Governing Equations

We consider an unsteady gravity driven thin film flow of a second grade fluid of uniform film thickness δ flowing over a semi-infinite plate, inclined at an angle θ with the horizontal, through a porous medium, with stationary ambient air and negligible surface tension. The velocity field thus becomes $v(y,t) = (u(y,t), 0, 0)$. Using such a velocity field in Equations (2)-(7) yields the following governing equation

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \rho g \sin \theta - \frac{\phi u}{k}. \tag{8}$$

Symbols ϕ and k , respectively, represent the porosity and permeability of the porous space. The associated boundary conditions are

$$\begin{cases} u(y,t) = 0 & \text{at } y = 0, \\ \frac{\partial u}{\partial y} = 0 & \text{at } y = \delta. \end{cases} \tag{9}$$

Here we would like to point out the presence of α_1 in Equation (8). For time independent velocity field, the presence of the fluid parameter α_1 is completely wiped out from the governing equation. The first boundary condition in Equation (9) is due to the no-slip assumption at the boundary $y = 0$. Now considering the following nondimensional variables

$$\eta = \frac{y}{\delta}, \quad f = \frac{v}{\delta^2 g} u, \quad t^* = \frac{t}{T}, \quad \text{with } T = \frac{\delta^2}{\nu}. \tag{10}$$

This gives us the following

$$\frac{\partial f}{\partial t^*} = \frac{\partial^2 f}{\partial \eta^2} + \alpha \frac{\partial^3 f}{\partial \eta^2 \partial t^*} + \sin \theta - \phi' f, \tag{11}$$

where $\alpha = \frac{\alpha_1}{\rho \delta^2}$ and $\phi' = \frac{\phi \delta^2}{\nu k \rho}$. The boundary conditions take the form

$$\begin{cases} f(\eta, t^*) = 0 & \text{at } \eta = 0, \\ \frac{\partial f}{\partial \eta} = 0 & \text{at } \eta = 1. \end{cases} \quad (12)$$

Taking the Laplace transform of Equations (11)-(12), we get, after rearranging

$$\frac{\partial^2 \hat{f}}{\partial \eta^2} - \beta \hat{f} = \gamma, \quad (13)$$

with $\beta = \frac{\phi + s}{\alpha s + 1}$ and $\gamma = -\frac{\sin \theta}{s(\alpha s + 1)}$. Also, the transformed boundary conditions are

$$\begin{cases} \hat{f}(\eta, s) = 0 & \text{at } \eta = 0, \\ \frac{\partial \hat{f}}{\partial \eta} = 0 & \text{at } \eta = 1. \end{cases} \quad (14)$$

Equations (13)-(14) have the following solution in the Laplace space

$$2\hat{f}(\eta, s) = \frac{\gamma}{\beta(e^{\sqrt{\beta}} + e^{-\sqrt{\beta}})} \left[e^{\sqrt{\beta}(\eta-1)} + e^{-\sqrt{\beta}(\eta-1)} \right] - \frac{\gamma}{\beta}. \quad (15)$$

Now we use the method of numerical inversion of the Laplace transform [1,2] for the solution given in Equation (15) to give us the following plots.

4. Results and Discussion

The dimensionless velocity field for an unsteady thin film flow of a second grade fluid is plotted in **Figure 1**. The values of different parameters used are $\alpha = 0.1$ and $\phi = 0.1$. It is evident from these graphical results that the fluid velocity increases as the time progresses. **Figure 2** provides us with the behaviour of the velocity profile due to changing α . The values of different parameters involved are $t^* = 0.1$ and $\phi = 0.1$. A drop in the value of velocity is observed due to an increase of α .

Figure 3, which depicts the change in velocity profile, $t^* = 0.1$ and $\alpha = 0.1$, due to changing porosity also shows a decrease in velocity for an increase in ϕ .

5. Conclusion

We have shown an efficient application of the numerical inverse Laplace algorithms [1,2] to study an unsteady thin film flow problem of a second grade fluid which is flowing through a porous medium along an inclined plane. This shows that the numerical inversion of the Laplace transform is a very effective and useful technique. Many unsteady problems of fluid flows, which are very hard to solve otherwise, can be dealt easily by the use of the Laplace transform. To accurately convert the solutions back to the original space, we can make use of

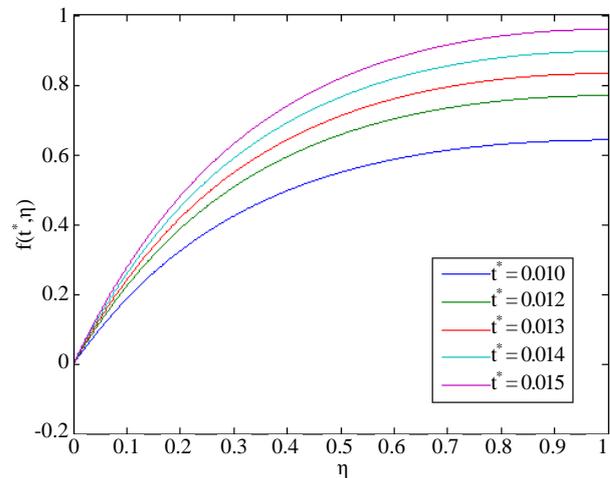


Figure 1. Plots for dimensionless velocity for different values of t^* .

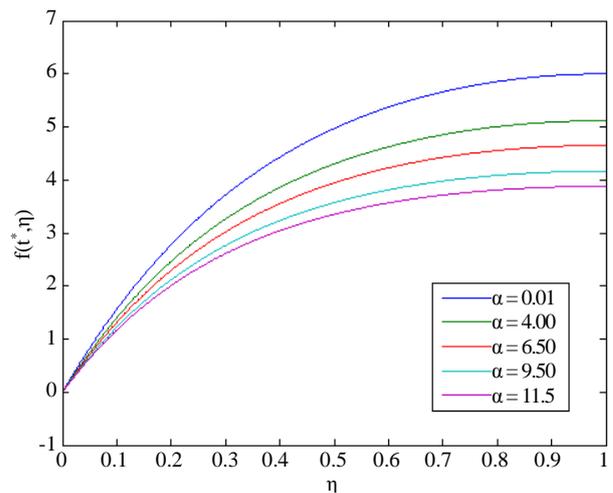


Figure 2. Plots for dimensionless velocity for different values of α .

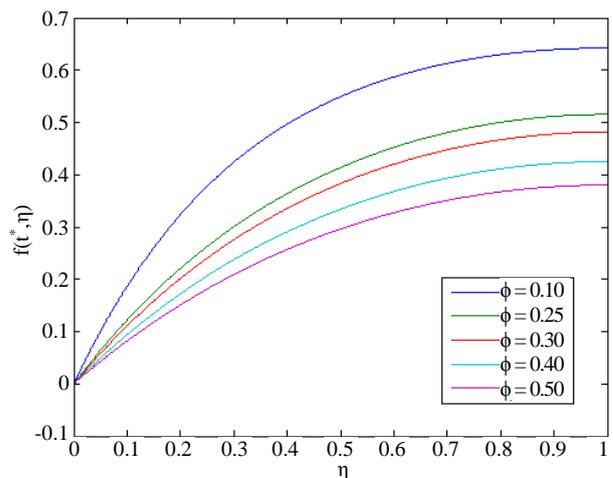


Figure 3. Plots for dimensionless velocity for different values of ϕ .

these efficient algorithms [1,2] available for the numerical inversion of the Laplace transform.

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