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On (2, 3, *t*)-Generations for the Rudvalis Group Ru

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ABSTRACT

A group G is said to be $(2,3,t)$-generated if it can be generated by an involution x and an element y so that $o(y)=3$ and $o(xy)=t$. In the present article, we determine all $(2,3,t)$-generations for the Rudvalis sporadic simple group Ru, where t is any divisor of $|\mathrm{Ru}|$.

Keywords: Generating Triple; Sporadic Group; Simple Group; Rudvalis Group; $(2,3,t)$-Generation

1. Introduction

A group G is said to be (p,q,r)-generated if it can be generated by two of its elements x and y so that $o(x)=p$, $o(y)=q$ and $o(xy)=r$. It is well known that every finite simple group can be generated by just two of its elements. Since the classification of all finite simple groups, more recent work in group theory has involved the study of internal structure of these group and generation type problems have played an important role in these studies. Recently, there has been considerable amount of interest in such type of generations. A $(2,3)$-generated group is a homomorphic image of the projective special linear group $PSL(2,Z)$. It has been known since 1901 (see [1]) that the alternating groups A_n $n \neq 6,7,8$ are $(2,3)$-generated. Macbeath [2] proved that projective special linear groups $PSL(2,q)$, $q \neq 9$ are $(2,3)$-generated. With the exception of Matheiu groups M_{11}, M_{22}, M_{23}, and Maclarin's group *McL*, all sporadic simple groups are $(2,3)$-generated (Woldar [3]). Guralnick showed that any non-abelian finite simple group can be generated by an involution and a Sylow 2-subgroup. In addition, a large number of Lie groups and classical linear groups are $(2,3)$-generated as well. Recently, Liebeck and Shalev proved that all finite classical groups (with some exceptions) are $(2,3)$-generated.

We say that a group G is $(2,3,t)$-generated (or $(2,3)$-generated) if it can be generated by just two of its elements x and y such that x is an involution, $|y|=3$ and $|xy|=t$. Moori in [4] computed all $(2,3,p)$-generations for the smallest Fischer group Fi_{22}, where p is a prime divisor of $|Fi_{22}|$. Further, Ganief and Moori determined the $(2,3,t)$-generations for the Janko's third sporadic simple group J_3 (see [5]). Recently, the author with others computed $(2,3,t)$-generations for the Held's sporadic simple group *He*, Tits simple group $2F_4(2)'$ and Conway's two sporadic simple groups Co_3 and Co_2 (see [6-8]). Darafsheh and Ashrafi [9] computed generating pairs for the sporadic group Ru. In the present article, we compute all the $(2,3,t)$-generations for the Rudvalis simple group Ru, where t is any divisor of $|\mathrm{Ru}|$.

2. Preliminaries

In this article, we use same notation as in [6]. In particular, for C_1, C_2 and C_3 conjugacy classes of elements the group Ru and g_3 is a fixed representative of C_3, we define $\Delta_G(C_1,C_2,C_3)$ to be the number of distinct pairs $(g_1,g_2)\in(C_1\times C_2)$ such that $g_1g_2=g_3$. We can compute the structure of G, $\Delta_G(C_1,C_2,C_3)$ for the conjugacy classes C_1, C_2 and C_3 from the character table of G by the following formula

$$\Delta_G(C_1,C_2,C_3)=\frac{|C_1||C_2|}{|G|}\times\sum_{i=1}^{m}\frac{\chi_i(g_1)\chi_i(g_2)\overline{\chi_i(g_3)}}{[\chi_i(1_G)]},$$

where $\chi_1,\chi_2,\cdots,\chi_m$ are the irreducible complex characters of the group G. Further let, $\Delta_G^*(C_1,C_2,C_3)$ denotes the number of distinct tuples $(g_1,g_2)\in(C_1\times C_2)$ such that $g_1g_2=g_3$ and $G=\langle g_1,g_2,\cdots,g_{k-1}\rangle$. If $\Delta_G^*(C_1,C_2,C_3)>0$, then clearly G is (C_1,C_2,C_3)-generated. If H any subgroup of G containing the fixed element $g_3 \in C_3$, then $\Sigma_H(C_1, C_2, C_3)$ denotes the number of distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2)$ such that $g_1 g_2 = g_3$ and $\langle g_1, g_2 \rangle \le H$ where $\Sigma_H(C_1, C_2, C_3)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H over all H-conjugacy classes $c_1, c_2, \cdots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \le i \le k-1$.

A general conjugacy class of elements of order n in G is denoted by nX. For examples, $2A$ represents the first conjugacy class of involutions in a group G. Most of the time, it will clear from the context to which conjugacy classes lX, mY and nZ we are referring. In such case, we suppress the conjugacy classes, using $\Delta^*(G)$ and $\Delta(G)$ as abbreviated notation for $\Delta^*_G(lX, mY, nZ)$ and $\Delta_G(lX, mY, nZ)$, respectively.

Lemma 2.1 ([10]) *Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which* $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|$, $z \in nZ$. *Then* $\Delta^*(G) = 0$ *and therefore G is not* (lX, mY, nZ)*-generated.*

Theorem 2.2 ([5]) *Let G be a finite group and H a subgroup of G containing a fixed element x such that* $\gcd(o(x), [N_G(H):H]) = 1$. *Then the number h of conjugates of H containing x is* $\chi_H(x)$, *where* χ_H *is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

3. Main Results

The Rudvalis group Ru is a sporadic simple group of order

$$145926144000 = 2^{14} \times 3^3 \times 5^3 \times 7 \times 13 \times 29.$$

Wilson [11] completely determined the maximal subgroups of the group Ru. It has exactly 15 conjugacy classes of maximal subgroups (see **Table 1**) as also listed in the $\mathbb{ATLAS}$ of Finite Group (see [12]). It has precisely two classes of involution, namely $2A$ and $2B$ and a unique class $3A$ of elements of order 3 in Ru.

It is a well known that if G is $(2,3,t)$-generated finite simple group, then $1/2 + 1/3 + 1/t < 1$. It follows that we need to consider the cases when

$$t \in \{7, 8, 10, 12, 13, 14, 15, 16, 20, 24, 26, 29\}.$$

The cases when t is prime has already been discussed in [9], so we need to investigate the cases when $t \ne 7, 13, 29$. Next, we investigate each case separately starting with the conjugacy class $15A$ of Ru.

Lemma 3.1 *The sporadic simple Rudvalis group Ru is* $(2X, 3A, 15A)$*-generated for all* $X \in \{A, B\}$.

Proof: The maximal subgroups of the group Ru having elements of order 15, up to isomorphism, are H_3, H_8, H_{11} and H_{15}.

First we consider the case $(2A, 3A, 15A)$. Using $\mathbb{GAP}$ [13], we compute the structure constant $\Delta_{\text{Ru}}(2A, 3A, 15A) = 190$. From the above list of maximal subgroups, Ru class $2A$ does not meet the maximal subgroup H_9. The fusion map of the maximal subgroups H_3 into the group Ru yields

$$2a \to 2A,\ 3a \to 3A,\ 3b \to 3A,\ 15a \to 15A,\ 15b \to 15A$$

where $2a$, $3a$, $3b$, $15a$, $15b$ and $2A$, $3A$, $15A$ are conjugacy classes of elements in the groups H_3 and Ru, respectively. With the help of this fusion map, we calculate the structure $\Sigma_{H_3}(2A, 3A, 15A) = 15$. Further, since a fixed element $z \in 15A$ in Ru is contained in a two conjugates of the maximal subgroup H_3, the total contribution from the maximal subgroup H_3 to the structure constant $\Delta_{Ru}(2A, 3A, 15A)$ is 2 × 15. Similarly by considering the fusion maps from the maximal subgroups H_8, H_{11} and H_{15} we compute that $\Sigma_{H_8}(2A, 3A, 15A) = 5$, $\Sigma_{H_{11}}(2A, 3A, 15A) = 10$ and $\Sigma_{H_{15}}(2A, 3A, 15A) = 0$. Since the fixed element z is contained in two conjugates of H_8 and in a unique conjugate copy of H_{11}, we obtain

Table 1. Maximal subgroups of rudvalis group Ru.

Group	Order	Group	Order
$H_1 \cong 2F_4(2)' \times 2$	$2^{12} \times 3^3 \times 5^2 \times 13$	$H_2 \cong 2^6 : U_3(3) : 2$	$2^{12} \times 3^3 \times 7$
$H_3 \cong (2^2 \times S_z(8)) : 3$	$2^8 \times 3 \times 5 \times 7 \times 13$	$H_4 \cong 2^{3+8} : L_3(2)$	$2^{14} \times 3 \times 7$
$H_5 \cong U_3(5) \times 2$	$2^5 \times 3^2 \times 5^3 \times 7$	$H_6 \cong 2 \times 2^{4+6} : S_5$	$2^{14} \times 3 \times 5$
$H_7 \cong L_2(25) \times 2^2$	$2^5 \times 3 \times 5^2 \times 13$	$H_8 \cong A_8$	$2^6 \times 3^2 \times 5 \times 7$
$H_9 \cong L_2(29)$	$2^2 \times 3 \times 5 \times 7 \times 29$	$H_{10} \cong 5^2 : 4S_5$	$2^2 \times 3 \times 5 \times 7 \times 29$
$H_{11} \cong 3 \times A_6 \times 2^2$	$2^5 \times 3^3 \times 5$	$H_{12} \cong 5_+^{1+2} : 2^5$	$2^5 \times 5^3$
$H_{13} \cong L_2(13) \times 2$	$2^3 \times 3 \times 7 \times 13$	$H_{14} \cong A_6 \times 2^2$	$2^5 \times 3^2 \times 5$
$H_{15} \cong 5 : (4 \times A_5)$	$2^4 \times 3 \times 5^2$		

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2A,3A,15A)\\&\ge \Delta_{\mathrm{Ru}}(2A,3A,15A)-2\Sigma_{H_3}(2A,3A,15A)\\&\quad-2\Sigma_{H_8}(2A,3A,15A)-\Sigma_{H_{11}}(2A,3A,15A)\\&=190-2(15)-2(5)-1(10)>0.\end{aligned}$$

Hence, the group Ru is $(2A,3A,15A)$-generated.

Next, consider the case $(2B,3A,15A)$. We compute the algebra structure constant as $\Delta_{\mathrm{Ru}}(2B,3A,15A)=510$. From the maximal subgroups of Ru, we see that the maximal subgroups that may contain $(2B,3A,15A)$-generated proper subgroups are isomorphic to H_3, H_9, H_{11} and H_{15}. By considering the fusion maps from the these maximal subgroups into the group Ru and the values of h which we compute using Theorem 1, we obtain

$$\Sigma_{H_3}(2B,3A,15A)=45,$$

$$\Sigma_{H_9}(2B,3A,15A)=30,$$

$$\Sigma_{H_{11}}(2B,3A,15A)=0=\Sigma_{H_{15}}(2B,3A,15A).$$

Hence,

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2B,3A,15A)\\&\ge \Delta_{\mathrm{Ru}}(2B,3A,15A)-2\Sigma_{H_1}(2B,3A,15A)\\&\quad-4\Sigma_{H_2}(2B,3A,15A)\\&=510-2(45)-4(30)>0.\end{aligned}$$

Therefore, $(2B,3A,15A)$ is generating triple of the group Ru and the proof is complete. □

Lemma 3.2 *The Rudavalis group Ru is* $(2X,3A,8Z)$*-generated if and and only if* $(X,Z)\in\{(B,B),(B,C)\}$*, where* $X\in\{A,B\}$ *and* $Z\in\{A,B,C\}$.

Proof: Our main proof will consider a number of cases.

Case $(2B,3A,8B)$: From the list of maximal subgroups of Ru (**Table 1**) we observe that, up to isomorphism, H_4 and H_6 are the only maximal subgroups that admit $(2B,3A,8B)$-generated subgroups. From the structure constant we calculate $\Delta(2B,3A,8B)=576$, $\Sigma(H_4)=0$ and $\Sigma(H_6)=64$. Since a fixed element $z\in 8B$ in Ru is contained in three conjugate copies of subgroup H_6, we have $\Delta^*(Ru)\ge\Delta(Ru)-3\Sigma(H_6)>0$, and therefore Ru is $(2B,3A,8B)$-generated.

Case $(2B,3A,8C)$: For this triple we calculate the structure constant $\Delta(\mathrm{Ru})=672$. Up to isomorphism, H_4, H_6 and H_{14} are the only maximal subgroups subgroups of Ru that meet the conjugacy classes $2B$, $3A$ and $8C$. We compute that $\Sigma(H_{14})=8$ and $\Sigma(H_4)=0=\Sigma(H_6)$. A fixed element a fixed element of order 8 in Ru-class $8C$ is contained in eight copies of the subgroup H_{14}. We obtain $\Delta^*(Ru)\ge\Delta(Ru)-8\Sigma(H_{14})>0$, showing that $(2B,3A,8C)$ is a generating triple of the group Ru.

Cases $(2A,3A,8C)$, $(2A,3A,8B)$, $(2B,3A,8A)$: In order to investigate these triples, we construct the group Ru explicitly by using its *standard generators* given by Wilson [14]. The Rudvalis group Ru has a 28 dimensional irreducible representation over $\mathbb{GF}(2)$. Using $\mathbb{GAP}$, we generate the group $\mathrm{Ru}=\langle a,b\rangle$, where a and b are 28 × 28 matrices over $\mathbb{GF}(2)$ with orders 2 and 4 respectively such that ab has order 13. We see that $a\in 2B$, $b\in 4A$ and $ab\in 13A$. We produce $c=ab^2a$,

$$d=\left(ab^2(ab)^3(ba)^2b^2ab^2\right)^8,$$

$$e=(ab)^3(ba)^2b^2ab^2,$$

$$f=\left((ab)^2(ba)^2b^2ab^2\right)^4,$$

$l=e^{acb}$, $m=e^d$ and $n=f^b$ such that $c,e,l\in 2A$, $d,f\in 3A$, $cd\in 12A$, $f\in 12B$. Let $P=\langle m,n\rangle$ then $m\in 2A$, $n\in 3A$ and $mn\in 8C$ such that $P<\mathrm{Ru}$ with $|P|=432$ and $N_{\mathrm{Ru}}(P)=P$. By investigating maximal subgroups of P and looking at the fusion maps of these maximal subgroups into the groups P and Ru we calculate $\Sigma_P(2A,3A,8C)=24$. Consequently we obtain

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2A,3A,8C)\\&=\Delta_{\mathrm{Ru}}(2A,3A,8C)-24\Sigma_P(2A,3A,8C)\\&<|C_{\mathrm{Ru}}(8C)|.\end{aligned}$$

Hence by Lemma 2.1, we obtain $\Delta^*_{\mathrm{Ru}}(2A,3A,8C)=0$, proving that Ru is not generated by the triple $(2A,3A,8C)$. By using the standard generators a and b of Ru together with above produced elements c, d, e, f, l, m, n in a similar way we explicitly generate $(2X,3A,8X)$-subgroups and observed that Ru is also not generated by the triple $(2X,3A,8X)$. This completes the lemma. □

Lemma 3.3 *The sporadic group Ru is* $(2X,3A,10Z)$*-generated for all* $X,Z\in\{A,B\}$.

Proof: We will investigate each triple separately.

Case $(2A,3A,10A)$: We compute the algebra structure constant for this triple as $\Delta_{\mathrm{Ru}}(2A,3A,10A)=340$. Amongst the maximal subgroups of Ru having nonempty intersection with the classes $2A$, $3A$ and $10A$ are isomorphic to H_1, H_5, H_6, H_7, H_{10} and H_{15}. Now, by considering the fusion maps from these maximal subgroups into the group Ru we calculate $\Sigma(H_1)=120$, $\Sigma(H_6)=20$, $\Sigma(H_7)=15$ and $\Sigma(H_5)=\Sigma(H_{10})=\Sigma(H_{15})=0$. Since a fixed element of order 10 is contained in two conjugate copies of H_1, four conjugate copies of H_7 and a unique conjugate copy of H_6. Therefore

$$\begin{aligned}\Delta^*(\mathrm{Ru})&\ge\Delta(\mathrm{Ru})-2\Sigma(H_1)-\Sigma(H_6)-4\Sigma(H_7)\\&=340-2(120)-20-4(15)>0,\end{aligned}$$

proving the generation of Ru by the triple $(2A,3A,10A)$.

Case $(2A,3A,10B)$: The only maximal subgroups of Ru that may contain $(2A,3A,10B)$-generated subgroups, up to isomorphisms, are H_3, H_{11}, H_{14} and H_{15}. However, we calculate that $\Sigma(H_3)=\Sigma(H_{11})=\Sigma(H_{14})=\Sigma(H_{15})=0$. That is, no maximal subgroup of Ru is $(2A,3A,10B)$-generated and we obtain $\Delta^*(\mathrm{Ru})=\Delta(\mathrm{Ru})=240>0$, showing that $(2A,3A,10B)$ is a generating triple for the group Ru.

Case $(2B,3A,10A)$: From the list of maximal subgroup (see **Table 1**) of Ru, observe that, up to isomorphism, H_6, H_7 and H_{15} are only maximal subgroups that may admit $(2B,3A,10A)$-generated subgroups. We compute in $\mathbb{GAP}$ that $\Sigma(H_6)=40$, $\Sigma(H_7)=0$ and $\Sigma(H_{15})=5$. Thus by we have

$$\Delta^*(\mathrm{Ru})\geq\Delta(\mathrm{Ru})-\Sigma(H_6)-4\Sigma(H_{15})$$
$$=520-40-4(5)>0.$$

This shows that Ru is $(2B,3A,10A)$-generated.

Case $(2B,3A,10B)$: For this triple we compute $\Delta(2B,3A,10B)=540$ and H_3, H_{11}, H_{14} and H_{15} are the only maximal subgroups of Ru that meet the classes in this triple. We calculate $\Sigma(H_{11})=30$, $\Sigma(H_{14})=10$, $\Sigma(H_3)=0=\Sigma(H_{15})$ and a fixed element $z\in 10B$ of order 10 in the group Ru is contained in exactly two copies of each of H_{11} and H_{14}. Therefore,

$$\Delta^*(\mathrm{Ru})\geq\Delta(\mathrm{Ru})-2\Sigma(H_{11})-2\Sigma(H_{14})>0.$$

Hence the group Ru is $(2B,3A,10B)$-generated and the lemma is complete. □

Lemma 3.4 *Let* $X,Z\in\{A,B\}$. *The group Ru is* $(2X,3A,12Z)$*-generated if and only if* $X=B$.

Proof: We will consider each case separately.

Case $(2B,3A,12A)$: The maximal subgroups of Ru with order divisible by 12 and non-empty intersection with the classes 2*B* and 3*A* are isomorphic to H_4, H_6, H_7 and H_{11}. We calculate that $\Sigma(H_4)=24$, $\Sigma(H_6)=48$ while $\Sigma(H_7)=0=\Sigma(H_{11})$. It follows

$$\Delta^*(\mathrm{Ru})\geq\Delta(\mathrm{Ru})-2\Sigma(H_4)-\Sigma(H_6)>0,$$

proving the generation by this triple.

Case $(2B,3A,12B)$: Up to isomorphism H_3, H_4, H_6, H_7, H_{11}, H_{13} and H_{15} are maximal subgroups of Ru that may contain $(2B,3A,12B)$-generated proper subgroups. We compute that $\Delta(2B,3A,12B)=672$, $\Sigma(H_3)=72$, $\Sigma(H_4)=24$, $\Sigma(H_6)=48$, $\Sigma(H_{13})=12$ and $\Sigma(H_7)=\Sigma(H_{11})=0=\Sigma(H_{15})$. It follows that

$$\Delta^*(\mathrm{Ru})\geq\Delta(\mathrm{Ru})-4\Sigma(H_3)-2\Sigma(H_4)-\Sigma(H_6)-\Sigma(H_{13})$$
$$=672-4(72)-2(24)-48-2(12)>0$$

Hence the group Ru is $(2B,3A,12B)$-generated.

Case $(2A,3A,12Z)$: We show that the group Ru is not $(2A,3A,12Z)$-generated by using the 28-dimensional irreducible representation of Ru over $\mathbb{GF}(2)$ as we used in Lemma 3.2 above. We generate $K=\langle l,f\rangle$ with $l\in 2A$, $f\in 3A$ such that $f\in 12B$. We have $|K|=768$ and $|N_{Ru}(K)|=1536$. By investigating the group *K*, we see that $\Sigma_K(2A,3A,12B)=96$ and consequently

$$\Delta^*_{\mathrm{Ru}}(2A,3A,12B)=288-4\Sigma_K(2A,3A,12B)$$
$$<|C_{Ru}(12B)|.$$

Hence by Lemma 2.1 Ru is not generated by the triple $(2A,3A,12B)$. Similar technique and arguments show that $(2A,3A,12B)$ is also not a generating triple for Ru. The lemma is complete. □

Lemma 3.5 *The group Ru is* $(2X,3A,14Z)$*-generated for all* $X\in\{A,B\}$ *and* $Z\in\{A,B,C\}$.

Proof: We calculate the structure constants $\Delta_{\mathrm{Ru}}(2A,3A,14Z)=280$, $\Delta_{\mathrm{Ru}}(2B,3A,14Z)=532$. From **Table 1**, the only maximal subgroups of Ru that meet the classes 2*A*, 3*A* and 14*Z* are isomorphic to H_3, H_4 and H_{13}. Further, H_4 is the only maximal subgroup that contribute to the structure constant $\Delta_{\mathrm{Ru}}(2A,3A,14Z)=280$ as $\Sigma(H_4)=56$ and $\Sigma(H_3)=0=\Sigma(H_{13})$. Thus, we have $\Delta^*(\mathrm{Ru})\geq\Delta(\mathrm{Ru})-2\Sigma(H_4)>0$ and the generation of Ru by this triple follows.

Next, we consider the triple $(2B,3A,14Z)$. For this triple, the maximal subgroups that meet the Ru classes 2*B*, 3*A* and 14*Z*, up to isomorphism, are H_3, H_4, H_9 and H_{13}. Our computation shows that $\Sigma(H_9)=28$, $\Sigma(H_{13})=14$ and $\Sigma(H_3)=0=\Sigma(H_4)$. Thus, for this triple $\Delta^*(\mathrm{Ru})\geq\Delta(\mathrm{Ru})-2\Sigma(H_9)-2\Sigma(H_{13})>0$ proving that $(2B,3A,14Z)$ is a generating triple for the group Ru. This completes the proof. □

Lemma 3.6 *The Rudvalis group Ru is* $(2X,3A,16Z)$*-generated if and only if* $X=B$, *where* $X,Z\in\{A,B\}$.

Proof: We calculate structure constant $\Delta_{\mathrm{Ru}}(2A,3A,16Z)=288$ and $\Delta_{\mathrm{Ru}}(2B,3A,16Z)=576$. First we consider the case when $X=B$. From the fusion maps of maximal subgroups in **Table 1** into the Rudvalis group Ru, we observed that the $(2B,3A,16Z)$-generated proper subgroups are contained in the maximal subgroups isomorphic to H_4 and H_6. Further, since $\Sigma_{\mathrm{Ru}}(2B,3A,16Z)=0=\Sigma_{\mathrm{Ru}}(2B,3A,16Z)$ we obtain that H_4 and H_6 are not $(2B,3A,16Z)$-generated. Hence we have $\Delta^*_{\mathrm{Ru}}(2B,3A,16Z)=\Delta_{\mathrm{Ru}}(2B,3A,16Z)=576>0$ and generation of Ru by the triple $(2B,3A,16Z)$ follows.

For the triple $(2A,3A,16Z)$, we use random element generation method as described in Conder [15] to show that Ru is not $(2A,3A,16Z)$-generated. Since $\Delta_{\mathrm{Ru}}(2A,3A,16Z)=288$. This is, there are 288 pairs (x,y) with $x\in 2A$, $y\in 3A$ and $xy=z\in 16Z$. We apply a procedure (an analogous procedure given in Conder [15] for CAYLEY), in the computer algebra system $\mathbb{MAGMA}$ (see [16]). It turns out that all 288 pairs generate proper subgroup of Ru and so $(2A,3A,16Z)$ is not a generating triple of Ru. □

Lemma 3.7 *The group Ru is* $(2X,3A,20Z)$*-generated where* $X\in\{A,B\}$ *and* $Z\in\{A,B,C\}$.

Proof: The maximal subgroups of the group Ru which contains element of order 20, up to isomorphism, are H_1, H_5, H_{10} and H_{15} (see **Table 1**). We now consider each case separately.

Case $(2A,3A,20A)$: We observed that 20*A* class of Ru does not meet the maximal subgroup H_{15}. So, the maximal subgroups of Ru having non-empty intersection with the classes 2*A*, 3*A* and 20*A* are, up to isomorphism, H_1, H_5, H_6 and H_{10}. We compute that $\Sigma_{H_1}(2A,3A,20A)=0$, $\Sigma_{H_5}(2A,3A,20A)=25$, $\Sigma_{H_6}(2A,3A,20A)=20$ and $\Sigma_{H_{10}}(2A,3A,20A)=25$. Thus,

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2A,3A,20A)\\&\geq\Delta_{\mathrm{Ru}}(2A,3A,20A)-2\Sigma_{H_5}(2A,3A,20A)\\&\quad-\Sigma_{H_6}(2A,3A,20A)-\Sigma_{H_{10}}(2A,3A,20A)\\&=220-2(25)-20-4(25)>0,\end{aligned}$$

and therefore Ru is $(2A,3A,20A)$-generated.

Case $(2A,3A,20Y)$ *where* $Y\in\{B,C\}$: The only maximal subgroups of Ru that may contain $(2A,3A,20Y)$-generated proper subgroups are isomorphic to H_6 and H_{15}. Further since $\Sigma_{H_6}(2A,3A,20Y)=20$ and $\Sigma_{H_{15}}(2A,3A,20Y)=0$ we have

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2A,3A,20Y)\\&\geq\Delta_{\mathrm{Ru}}(2A,3A,20A)-\Sigma_{H_6}(2A,3A,20Y)\\&=220-20>0,\end{aligned}$$

and so Ru is $(2A,3A,20B)$-, and $(2A,3A,20C)$-generated.

Case $(2B,3A,20A)$: We compute the algebra structure constant as $\Delta_{\mathrm{Ru}}(2A,3A,20A)=520$. The only maximal subgroup of Ru having non-empty intersection with the classes 2*B*, 3*A* and 20*A* is isomorphic to H_6. Since $\Sigma_{H_6}(2B,3A,20A)=40$ and a fixed element $z\in Ru$ is contained in a unique conjugate of M_6, we have

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2B,3A,20A)\\&=\Delta_{\mathrm{Ru}}(2B,3A,20B)-\Sigma_{H_6}(2B,3A,20A)\\&=220-1(40)>0,\end{aligned}$$

proving that Ru is $(2B,3A,20A)$-generated.

Case $(2B,3A,20Y)$ *where* $Y\in\{B,C\}$: First we calculate that the values of structure constant $\Delta_{\mathrm{Ru}}(2B,3A,20Y)=600$. Again, the only maximal subgroups of Ru which may contain $(2B,3A,20Y)$-generated proper subgroups are isomorphic to H_6 and H_{15}. We compute $\Sigma_{H_6}(2B,3A,20Y)=40$ and $\Sigma_{H_{15}}(2B,3A,20Y)=0$. Hence we obtain

$$\begin{aligned}&\Delta^*_{\mathrm{Ru}}(2B,3A,20Y)\\&=\Delta_{\mathrm{Ru}}(2B,3A,20A)-\Sigma_{H_6}(2B,3A,20Y)\\&=600-40>0,\end{aligned}$$

and the generation of Ru with the triples $(2B,3A,20B)$ and $(2B,3A,20C)$ follows. This completes the lemma. □

Lemma 3.8 *The Rudvalis group Ru is* $(2X,3A,24Z)$*-generated, where* $X,Z\in\{A,B\}$.

Proof: The maximal subgroups of Ru having elements of order 24 are isomorphic to H_6, H_7, H_{10} and H_{11} (see **Table 1**).

First we consider the triple $(2A,3A,24Z)$. By looking at the fusion maps from the above maximal subgroups into the group Ru, in each case, we obtain

$$\Sigma(H_6)=0=\Sigma(H_7)=\Sigma(H_{10})=\Sigma(H_{11})=0.$$

Now, since $\Delta(\mathrm{Ru})=240$ we have $\Delta^*(\mathrm{Ru})=\Delta(\mathrm{Ru})>0$. Thus, Ru is $(2A,3A,24A)$-, and $(2A,3A,24B)$-generated.

Next, for the triple $(2B,3A,24Z)$, we compute $\Delta_{\mathrm{Ru}}(2B,3A,24Z)=624$. From the above maximal subgroups H_{10} does the Ru-class 2*B*. For the maximal subgroups H_6 and H_{11} we obtain $\Sigma(H_6)=0=\Sigma(H_{11})$. For the maximal subgroup H_7 we calculate $\Sigma(H_7)=24$. Hence $\Delta^*(\mathrm{Ru})=\Delta(\mathrm{Ru})=624-24>0$, proving the generation of Ru by the triples $(2B,3A,24A)$ and $(2B,3A,24B)$. This completes the proof. □

Lemma 3.9 *The Rudvalis simple group Ru is* $(2X,3A,26Z)$*-generated, where* $X\in\{A,B\}$ *and* $Z\in\{A,B,C\}$.

Proof: Up to isomorphism, H_3 and H_7 are the only maximal subgroups of Ru which contain an element of order 26 (see **Table 1**).

Case $(2A,3A,26Z)$: In this case we compute $\Delta_{\mathrm{Ru}}(2A,3A,26Z)=312$. However, in both cases we obtain $\Sigma(H_3)=0=\Sigma(H_7)$. This implies that there is no contribution from these maximal subgroups to the structure constant $\Delta(\mathrm{Ru})$. Hence generation of Ru by the triple follows as we have $\Delta^*(\mathrm{Ru})=\Delta(\mathrm{Ru})>0$.

Case $(2A,3A,26Z)$: For this triple we have $\Delta_{\mathrm{Ru}}(2B,3A,26Z)=572$. By considering the fusion maps from the maximal subgroups H_3 and H_7 into Ru we calculate $\Sigma(H_3)=0$ and $\Sigma(H_7)=26$. Thus

$$\Delta^*(\mathrm{Ru})=\Delta(\mathrm{Ru})-2\Sigma(H_7)=572-2(26)>0,$$

and we conclude that Ru is $(2B,3A,26Z)$-generated, which completes the proof.

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