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Inferences under a Class of Finite Mixture Distributions Based on Generalized Order Statistics

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ABSTRACT

The main purpose of this paper is to obtain estimates of parameters, reliability and hazard rate functions of a heterogeneous population represented by finite mixture of two general components. The doubly Type II censoring of generalized order statistics scheme is used. Maximum likelihood and Bayes methods of estimation are used for this purpose. The two methods of estimation are compared via a Monte Carlo Simulation study.

Keywords: Generalized Order Statistics; Bayes Estimation; Heterogeneous Population; Monte Carlo Integration; Monte Carlo Simulation

1. Introduction

Let the random variable (rv) T follow a class including some known lifetime models, its cumulative distribution function (CDF) is given by

$$F(t) = 1 - \exp\left[-\theta\lambda(t)\right], t > 0, (\theta > 0), \tag{1}$$

and its probability density function (PDF) is given by

$$f(t) = \theta\lambda'(t)\exp\left[-\theta\lambda(t)\right], t > 0, (\theta > 0), \tag{2}$$

where $\lambda'(t)$ is the derivative of $\lambda(t)$ with respect to t and $\lambda(t) \equiv \lambda(t;\alpha)$ is a nonnegative continuous function of t and α may be a vector of parameters, such that $\lambda(t) \to 0$ as $t \to 0^+$ and $\lambda(t) \to \infty$ as $t \to \infty$.

The reliability function (RF) and hazard rate function (HRF) are given, respectively, by

$$R(t) = \exp\left[-\theta\lambda(t)\right], \tag{3}$$

$$H(t) = \theta\lambda'(t), \tag{4}$$

where

$$H(.) = f(.)/R(.).$$

Bayesian inferences based on finite mixture distribution have been discussed by several authors. Bayesian estimation of the mixing parameter, mean and reliability function of a mixture of two exponential lifetime distributions based on right censored samples considered by [1,2] estimated the survival and hazard functions of a finite mixture of two Gompertz components by using type I and type II censored samples, using the maximum likelihood (ML) and Bayes methods. Based on type I censored samples from a finite mixture of two truncated type I generalized logistic components, [3] computed the Bayes estimates of parameters, reliability and hazard rate functions. [4] considered estimation for the mixed exponential distribution based on record statistics. [5] considered Bayes inference under a finite mixture of two compound Gompertz components model. [6] studied some properties of the mixture of two inverse Weibull distributions and obtained the estimates of the unknown parameters via the EM Algorithm.

[7] introduced the generalized order statistics (gos's). Ordinary order statistics, ordinary record values and sequential order statistics are, among others, special cases of gos's. The gos's have been considered extensively by many authors, among others, they are [8-20].

Mixtures of distributions arise frequently in life testing, reliability, biological and physical sciences. Some of the most important references that discussed different types of mixtures of distributions are a monograph by [21-23].

The PDF, CDF, RF and HRF of a finite mixture of two components of the class under study are given, respectively,

$$f(t) = p_1 f_1(t) + p_2 f_2(t), \tag{5}$$

$$F(t)=p_1F_1(t)+p_2F_2(t), \tag{6}$$

$$R(t)=p_1R_1(t)+p_2R_2(t), \tag{7}$$

$$H(t)=f(t)/R(t), \tag{8}$$

where, for $j=1,2$, the mixing proportions p_j are such that $0\le p_j\le 1, p_1+p_2=1$ and $f_j(t),F_j(t),R_j(t)$ are given from (1), (2), (3) after using θ_j and $\lambda_j(t)$ instead of θ and $\lambda(t)$.

The property of identifiability is an important consideration on estimating the parameters in a mixture of distributions. Also, testing hypothesis, classification of random variables, can be meaning fully discussed only if the class of all finite mixtures is identifiable. Idenifiability of mixtures has been discussed by several authors, including [24-26].

Our aim of this paper is the estimation of the parameters and functions of these parameters of a class of finite mixture distributions based on doubly Type II censoring gos's using ML and Bayes methods. Illustrative example of Gompertz distribution is given and compared with the results obtained by previous researchers.

2. Maximum Likelihood Estimation

Let

$$T_{s;n,m,k},T_{s+1;n,m,k},\cdots,T_{r;n,m,k},1\le s<r\le n,k>0,$$

be the (r - s) gos's drawn from a mixture of two components of the class (2). Based on this doubly censored sample, the likelihood function can be written [27] as

$$L(\theta|t)=\begin{cases} c_1\left\{\prod_{i=s}^{r}\left[R(t_i)\right]^m f(t_i)\right\}\left[R(t_r)\right]^{\gamma_{r+1}} \\ \quad\times\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[R(t_s)\right]^{(s-\ell-1)(m+1)}, & m\ne -1,\\ c_2\left[R(t_r)\right]^k\left[\ln\ R(t_s)\right]^{s-1}\prod_{i=s}^{r}H(t_i), & m=-1,\end{cases} \tag{9}$$

where $t=(t_s,\cdots,t_r)$, $\theta\in\Theta$, Θ is the parameter space, and

$$\left.\begin{aligned} &c_1=\frac{(-1)^{s-1}C_{r-1}}{(m+1)^{s-1}(s-1)!},\ c_2=\frac{(-1)^{s-1}k^r}{(s-1)!},\\ &C_{r-1}=\prod_{j=1}^{r}\gamma_j,\ \gamma_r=k+(n-r)(m+1),\\ &\omega_\ell^{(s)}=(-1)^\ell\binom{s-1}{\ell}. \end{aligned}\right\}$$

For definition and various distributional properties of gos's, see [7, 28].

The likelihood function (9) and maximum likelihood estimates (MLE's) can be obtained by using (1) and (5) in two cases, regarding to m value, as follows.

2.1 MLE's When $m\ne -1$

In this case, substituting (1), (5) in (9), the likelihood function takes the form

$$\begin{aligned} L(\theta|\boldsymbol{t})&=c_1\left\{\prod_{i=s}^{r}\left[p_1R_1(t_i)+p_2R_2(t_i)\right]^m\right.\\ &\quad\left.\times\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\right\}\\ &\quad\times\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{\gamma_{r+1}}\\ &\quad\times\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{(s-\ell-1)(m+1)}. \end{aligned} \tag{10}$$

Take the logarithm of (10), we have

$$\begin{aligned} \ell(\theta)&\equiv\ln L(\theta|\boldsymbol{t})=\ln c_1+m\sum_{i=s}^{r}\ln\left[p_1R_1(t_i)+p_2R_2(t_i)\right]\\ &+\sum_{i=s}^{r}\ln\left[p_1f_1(t_i)+p_2f_2(t_i)\right]+\gamma_{r+1}\ln\left[p_1R_1(t_r)+p_2R_2(t_r)\right]\\ &+\ln\left\{\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{(s-\ell-1)(m+1)}\right\}, \end{aligned} \tag{11}$$

where $p_1=p$, $p_2=1-p$.

Differentiating (11) with respect to the parameters p,θ_j and α_j (involved in λ) and equating to zero gives the following likelihood equations

$$\left.\begin{aligned} &\frac{\partial\ell}{\partial p}=m\sum_{i=s}^{r}\vartheta^*(t_i)+\sum_{i=s}^{r}\vartheta(t_i)+\gamma_{r+1}\vartheta^*(t_r)\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)\vartheta^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)}\\ &\quad=0,\\ &\frac{\partial\ell}{\partial\theta_j}=-m\sum_{i=s}^{r}p_j\lambda_j(t_i)\psi_j^*(t_i)+\sum_{i=s}^{r}p_j\xi_j(t_i)\psi_j(t_i)\\ &\quad-\gamma_{r+1}p_j\lambda_j(t_r)\psi_j^*(t_r)\\ &\quad-\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)p_j\lambda_j(t_s)\psi_j^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)}=0,\quad j=1,2,\\ &\frac{\partial\ell}{\partial\alpha_j}=-m\sum_{i=s}^{r}p_j\theta_j\frac{\partial\lambda_j(t_i)}{\partial\alpha_j}\psi_j^*(t_i)+\sum_{i=s}^{r}p_j\xi_j^*(t_i)\psi_j(t_i)\\ &\quad-\gamma_{r+1}p_j\theta_j\frac{\partial\lambda_j(t_r)}{\partial\alpha_j}\psi_j^*(t_r)\\ &\quad-\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)p_j\theta_j\frac{\partial\lambda_j(t_s)}{\partial\alpha_j}\psi_j^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)}=0,\quad j=1,2 \end{aligned}\right\} \tag{12}$$

where, for j = 1,2

$$\left.\begin{aligned}&\xi_j(t_i)=\left[\frac{1}{\theta_j}-\lambda_j(t_i)\right],\ \xi_j^*(t_i)=\left[\frac{1}{\lambda_j'(t_i)}\frac{\partial\lambda_j'(t_i)}{\partial\alpha_j}-\theta_j\frac{\partial\lambda_j(t_i)}{\partial\alpha_j}\right],\ \vartheta(t_i)=\frac{f_1(t_i)-f_2(t_i)}{f(t_i)},\\&\vartheta^*(t_i)=\frac{R_1(t_i)-R_2(t_i)}{R(t_i)},\ \psi_j(t_i)=\frac{f_j(t_i)}{f(t_i)},\quad \psi_j^*(t_i)=\frac{R_j(t_i)}{R(t_i)},\ \tau_\ell(t_s)=\omega_\ell^{(s)}R(t_s)^\delta,\delta=(s-\ell-1)(m+1)\end{aligned}\right\}\tag{13}$$

The solution of the five nonlinear likelihood Equations (12) using numerical method, yields the MLE's $\hat{p},\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha}_1$ and $\hat{\alpha}_2$.

2.2. MLE's When $m=-1$

The likelihood function takes the form

$$L(\theta|\boldsymbol{t})=c_2\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^k\times\left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)^{s-1}\times\prod_{i=s}^{r}\frac{\left[p_1f_1(t_i)+p_2f_2(t_i)\right]}{\left[p_1R_1(t_i)+p_2R_2(t_i)\right]}.\tag{14}$$

So, from (14)

$$\begin{aligned}\ell(\theta)\propto{}&k\ln\left[p_1R_1(t_r)+p_2R_2(t_r)\right]+(s-1)\ln\left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)\\&+\sum_{i=s}^{r}\left(\ln\left[p_1f_1(t_i)+p_2f_2(t_i)\right]-\ln\left[p_1R_1(t_i)+p_2R_2(t_i)\right]\right).\end{aligned}\tag{15}$$

Differentiating (15) with respect to the parameters p,θ_j and α_j and equating to zero gives the following likelihood equations

$$\left.\begin{aligned}&\frac{\partial\ell}{\partial p}=k\vartheta^*(t_r)+(s-1)\vartheta^*(t_s)\varphi(t_s)+\sum_{i=s}^{r}\left(\vartheta(t_i)-\vartheta^*(t_i)\right)=0,\\&\frac{\partial\ell}{\partial\theta_j}=-kp_j\lambda_j(t_r)\psi_j^*(t_r)-(s-1)p_j\lambda_j(t_s)\psi_j^*(t_s)\varphi(t_s)+\sum_{i=s}^{r}\left(p_j\xi_j(t_i)\psi_j(t_i)+p_j\lambda_j(t_i)\psi_j^*(t_i)\right)\\&\quad=0,\qquad j=1,2,\\&\frac{\partial\ell}{\partial\alpha_j}=-kp_j\theta_j\frac{\partial\lambda_j(t_r)}{\partial\alpha_j}\psi_j^*(t_r)-(s-1)p_j\theta_j\frac{\partial\lambda_j(t_s)}{\partial\alpha_j}\psi_j^*(t_s)\varphi(t_s)+\sum_{i=s}^{r}\left(p_j\xi_j^*(t_i)\psi_j(t_i)+p_j\theta_j\frac{\partial\lambda_j(t_i)}{\partial\alpha_j}\psi_j^*(t_i)\right)\\&\quad=0,\qquad j=1,2\end{aligned}\right\}\tag{16}$$

where

$$\varphi(t_i)=\frac{1}{\ln R(t_i)}.\tag{17}$$

The solution of the five nonlinear likelihood Equations (16) using numerical method, yields the MLE's $\hat{p},\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha}_1$ and $\hat{\alpha}_2$.

3. Bayes Estimation

In this section, Bayesian estimation for the parameters of a class of finite mixture distributions is considered under squared error and Linex (Linear-Exponential) loss functions.

We shall use the conjugate prior density, that was suggested by [29], in the following form

$$\begin{aligned}&\pi(\theta;\nu)\propto C(\theta;\nu)\exp\left[-D(\theta;\nu)\right],\\&\theta=(p,\theta_1,\theta_2,\alpha_1,\alpha_2),\nu\in\Omega,\end{aligned}\tag{18}$$

where Ω is the hyperparameter space.

3.1. Bayes Estimates When $m\neq-1$

It follows, from (10) and (18), that the posterior density function is given by

$$\begin{aligned}&\pi^*(\theta|\boldsymbol{t})=A_1C(\theta;\nu)\exp\left[-D(\theta;\nu)\right]\\&\times\left\{\prod_{i=s}^{r}\left[p_1R_1(t_i)+p_2R_2(t_i)\right]^m\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\right\}\\&\times\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{\gamma_{r+1}}\\&\times\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^\delta\end{aligned}\tag{19}$$

where

$$A_1^{-1}=\int_\theta\pi(\theta;\nu)L(\theta|\boldsymbol{t})\mathrm{d}\theta.\tag{20}$$

The Bayes estimator of a function, say $u(\theta)$, under squared error and Linex loss functions is given, respectively, by

$$\hat{u}_{BS} = E\left(u(\theta)\middle|\boldsymbol{t}\right) = \int_{\theta} u(\theta)\pi^*\left(\theta\middle|\boldsymbol{t}\right)\mathrm{d}\theta, \tag{21}$$

$$\begin{aligned}\hat{u}_{BL} &= \frac{1}{a}\ln\left[E\left(\mathrm{e}^{-au(\theta)}\middle|\boldsymbol{t}\right)\right] \\ &= -\frac{1}{a}\ln\left[\int_{\theta}\mathrm{e}^{-au(\theta)}\pi^*\left(\theta\middle|\boldsymbol{t}\right)\mathrm{d}\theta\right],\end{aligned} \tag{22}$$

where the integral is taken over the five dimensional space and $a \neq 0$.

To compute the integral, we can use the Monte Carlo Integration (MCI) method in the form

$$\hat{u}_{BS} = \frac{\sum_{j=1}^{M} u\left(\theta^j\right)L\left(\theta^j\middle|\boldsymbol{t}\right)}{\sum_{j=1}^{M} L\left(\theta^j\middle|\boldsymbol{t}\right)}, \tag{23}$$

$$\hat{u}_{BL} = -\frac{1}{a}\ln\left[\frac{\sum_{j=1}^{M}\mathrm{e}^{-au\left(\theta^j\right)}L\left(\theta^j\middle|\boldsymbol{t}\right)}{\sum_{j=1}^{M} L\left(\theta^j\middle|\boldsymbol{t}\right)}\right], \tag{24}$$

where $\theta^j, j = 1, 2, \cdots, M$ is generated from the PDF $\pi(\theta;\nu)$, for more details see [30].

Under squared error and Linex loss functions, we can obtain the Bayes estimator of the parameter *p*, by generating

$$\left(p^j, \theta_1^j, \theta_2^j, \alpha_1^j, \alpha_2^j\right) j = 1, 2, \cdots, M$$

from the prior (18) and setting $u(\theta) = p$ in (23) and (24). The Bayes estimates of $\theta_1, \theta_2, \alpha_1$ and α_2 can be similarly computed.

3.2. Bayes Estimates When $m = -1$

The posterior density function can be obtained from (14) and (18), as

$$\begin{aligned}\pi^*\left(\theta\middle|\boldsymbol{t}\right) &= A_2 C(\theta;\nu)\exp\left[-D(\theta;\nu)\right] \\ &\times\left[p_1 R_1(t_r) + p_2 R_2(t_r)\right]^k \\ &\times\left(\ln\left[p_1 R_1(t_s) + p_2 R_2(t_s)\right]\right)^{s-1} \\ &\times\prod_{i=s}^{r}\frac{\left[p_1 f_1(t_i) + p_2 f_2(t_i)\right]}{\left[p_1 R_1(t_i) + p_2 R_2(t_i)\right]}\end{aligned} \tag{25}$$

where

$$A_2^{-1} = \int_{\theta}\pi(\theta;\nu)L\left(\theta\middle|\boldsymbol{t}\right)\mathrm{d}\theta. \tag{26}$$

Under squared error and Linex loss functions, we can obtain the Bayes estimator of the parameter *p*, by generating

$$\left(p^j, \theta_1^j, \theta_2^j, \alpha_1^j, \alpha_2^j\right) j = 1, 2, \cdots, M$$

from the prior (18) and setting $u(\theta) = p$ in (23) and (24). The Bayes estimates of $\theta_1, \theta_2, \alpha_1$ and α_2 can be similarly computed.

4. Example

4.1. Gompertz Components

4.1.1. Maximum Likelihood Estimation

Suppose that, for $j = 1, 2$ and $t > 0, \theta_j = 1,$

$$\lambda_j(t) = \frac{1}{\alpha_j}\left[\mathrm{e}^{\alpha_j t} - 1\right]$$

so

$$\dot{\lambda}_j(t) = \mathrm{e}^{\alpha_j t}.$$

In this case, the j^{th} subpopulation is Gompertz distribution with parameter $\alpha_j > 0.$

For $m \neq -1$ by substituting $\lambda_j(t)$ and $\dot{\lambda}_j(t)$ in (12), we have the following nonlinear equations

$$\left.\begin{aligned}\frac{\partial\ell}{\partial p} &= m\sum_{i=s}^{r}\vartheta^*(t_i) \\ &+\sum_{i=s}^{r}\vartheta(t_i) + \gamma_{r+1}\vartheta^*(t_r) \\ &+\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)\vartheta^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)} = 0, \\ \frac{\partial\ell}{\partial\alpha_j} &= -m\sum_{i=s}^{r}p_j\varpi_j(t_i)\psi_j^*(t_i) \\ &+\sum_{i=s}^{r}p_j\xi_j^*(t_i)\psi_j(t_i) \\ &-\gamma_{r+1}p_j\varpi_j(t_r)\psi_j^*(t_r) \\ &-\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)p_j\varpi_j(t_s)\psi_j^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)} = 0, \quad j = 1, 2\end{aligned}\right\}, \tag{27}$$

where, for $j = 1, 2$

$$\left.\begin{aligned}\xi_j(t_i) &= 1 - \frac{1}{\alpha_j}\left(\mathrm{e}^{\alpha_j t} - 1\right), \\ \xi_j^*(t_i) &= t_i - \varpi_j(t_i), \\ \varpi_j(t_i) &= \frac{1}{\alpha_j^2}\left[1 + \left(\alpha_j t_i - 1\right)\mathrm{e}^{\alpha_j t_i}\right]\end{aligned}\right\}, \tag{28}$$

$\hat{p}, \hat{\alpha}_1$ and $\hat{\alpha}_2$ are the solution of the above nonlinear equations.

Also, for $m = -1$ substituting $\lambda_j(t)$ and $\dot{\lambda}_j(t)$ in (13), (16) and (17), we have the following nonlinear equations:

$$\left.\begin{aligned}&\frac{\partial \ell}{\partial p}=k\vartheta^*(t_r)+(s-1)\vartheta^*(t_s)\varphi(t_s)+\sum_{i=s}^{r}\left(\vartheta(t_i)-\vartheta^*(t_i)\right)=0,\\&\frac{\partial \ell}{\partial \alpha_j}=-kp_j\varpi_j(t_r)\psi_j^*(t_r)-(s-1)p_j\varpi_j(t_s)\psi_j^*(t_s)\varphi(t_s)+\sum_{i=s}^{r}\left(p_j\xi_j^*(t_i)\psi_j(t_i)+p_j\varpi_j(t_i)\psi_j^*(t_i)\right)=0,\quad j=1,2\end{aligned}\right\},\tag{29}$$

$\hat{p},\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the solution of the above nonlinear equations.

Special cases

1) Upper order statistics

If we put $m=0$ and $k=1$ in (10),

$$\left(\gamma_r=n-r+1\right),$$

the likelihood function takes the form

$$L(\theta|\boldsymbol{t})=c_1\prod_{i=s}^{r}\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{n-r}\times\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{s-\ell-1}.\tag{30}$$

Substituting $m=0,k=1$ and $\gamma_r=n-r+1$ in (27), we have the following nonlinear equations

$$\left.\begin{aligned}&\frac{\partial \ell}{\partial p}=\sum_{i=s}^{r}\vartheta(t_i)+(n-r)\vartheta^*(t_r)+\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)\vartheta^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)}=0,\\&\frac{\partial \ell}{\partial \alpha_j}=\sum_{i=s}^{r}p_j\xi_j^*(t_i)\psi_j(t_i)-(n-r)p_j\varpi_j(t_r)\psi_j^*(t_r)-\frac{\sum_{\ell=0}^{s-1}\delta\tau_\ell(t_s)p_j\varpi_j(t_s)\psi_j^*(t_s)}{\sum_{\ell=0}^{s-1}\tau_\ell(t_s)}=0,\quad j=1,2\end{aligned}\right\},\tag{31}$$

where $\delta=s-\ell-1$.

The solution of the nonlinear likelihood equations (31) gives the MLE's $\hat{p},\hat{\alpha}_1$ and $\hat{\alpha}_2$.

2) Upper record values

If we put $k=1$ in (14), $\left(\gamma_r=1\right)$ the likelihood function takes the form

$$L(\theta|\boldsymbol{t})=c_2\left[p_1R_1(t_r)+p_2R_2(t_r)\right]\left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)^{s-1}\times\prod_{i=s}^{r}\frac{\left[p_1f_1(t_i)+p_2f_2(t_i)\right]}{\left[p_1R_1(t_i)+p_2R_2(t_i)\right]}.\tag{32}$$

Substituting $k=1$ in (29), we have the following nonlinear equations

$$\left.\begin{aligned}&\frac{\partial \ell}{\partial p}=(s-1)\vartheta^*(t_s)\varphi(t_s)+\sum_{i=s}^{r}\vartheta(t_i)-\sum_{i=s}^{r-1}\vartheta^*(t_i)=0,\\&\frac{\partial \ell}{\partial \alpha_j}=\sum_{i=s}^{r}p_j\xi_j^*(t_i)\psi_j(t_i)-(s-1)p_j\varpi_j(t_s)\psi_j^*(t_s)\varphi(t_s)+\sum_{i=s}^{r-1}p_j\varpi_j(t_i)\psi_j^*(t_i)=0,\quad j=1,2\end{aligned}\right\},\tag{33}$$

The solution of the nonlinear likelihood Equations (33) gives the MLE's $\hat{p},\hat{\alpha}_1$ and $\hat{\alpha}_2$.

4.1.2. Bayes Estimation

Let p,α_1 and α_2 are independent random variables such that $p\sim Beta(b_1,b_2)$ and for $j=1,2$, α_j to follow a left truncated exponential density with parameter d_j $\left(LTE(d_j)\right)$, as used by [2]. A joint prior density function is then given by

$$\pi(\theta;\nu)\propto p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right],\tag{34}$$

where

$$0<p_1<1,\alpha_j>1,(b_1,b_2,d_1,d_2)>0$$

and $p_2=1-p_1$.

For $m\neq-1$, the posterior density function $\pi^*(\theta|\boldsymbol{t})$ then takes the form

$$\begin{aligned}\pi^*(\theta|\boldsymbol{t})&\propto p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right]\\&\times\left\{\prod_{i=s}^{r}\left[p_1R_1(t_i)+p_2R_2(t_i)\right]^m\times\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\right\}\\&\times\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{\gamma_{r+1}}\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{\delta}\end{aligned}\tag{35}$$

For m = −1 the posterior density function $\pi^*(\theta|\boldsymbol{t})$ then takes the form

$$\pi^*(\theta|\boldsymbol{t}) \propto p_1^{b_1-1} p_2^{b_2-1} \exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right]\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^k \times \left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)^{s-1}\prod_{i=s}^{r}\frac{\left[p_1f_1(t_i)+p_2f_2(t_i)\right]}{\left[p_1R_1(t_i)+p_2R_2(t_i)\right]}. \tag{36}$$

Under squared error and Linex loss functions, we can obtain the Bayes estimator of the parameter p by generating $\left(p^j,\alpha_1^j,\alpha_2^j\right)$ $j=1,2,\cdots,M$ from the prior (34) and setting $u(\theta)=p$ in (23) and (24). The Bayes estimates of α_1 and α_2 can be similarly computed.

Special cases

1) Upper order statistics

If we put $m=0$ and $k=1$ in (35), $(\gamma_r=n-r+1)$, the posterior density function takes the form

$$\pi^*(\theta|\boldsymbol{t}) \propto p_1^{b_1-1} p_2^{b_2-1} \exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right]\prod_{i=s}^{r}\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\times\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{n-r}\sum_{\ell=0}^{s-1}\omega_\ell^{(s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{\delta}. \tag{37}$$

2) Upper record values

If we put $k=1$ in (36), $(\gamma_r=1)$, the posterior density function takes the form

$$\pi^*(\theta|\boldsymbol{t}) \propto p_1^{b_1-1} p_2^{b_2-1} \exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right]\left[p_1R_1(t_r)+p_2R_2(t_r)\right]\times\left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)^{s-1}\prod_{i=s}^{r}\frac{\left[p_1f_1(t_i)+p_2f_2(t_i)\right]}{\left[p_1R_1(t_i)+p_2R_2(t_i)\right]}. \tag{38}$$

Under squared error and Linex loss functions, we can obtain the Bayes estimator of the parameter p, by generating $\left(p^j,\alpha_1^j,\alpha_2^j\right)$ $j=1,2,\cdots,M$ from the prior (34) and setting $u(\theta)=p$ in (23) and (24). The Bayes estimates of α_1 and α_2 can be similarly computed.

5. Simulation Study

A comparison between ML and Bayes estimators, under either a squared error or a Linex loss functions, is made using a Monte Carlo simulation study in the two cases upper order statistics and upper record values according to the following steps:

1) For a given values of the prior parameters (b_1,b_2) generate a random value p from the $Beta(b_1,b_2)$ distribution.

2) For a given values of the prior parameters d_j, for $j=1,2,$ generate a random value α_j from the $LTE(d_j)$ distribution.

3) Using the generated values of p,α_1 and α_2, we generate a random sample of size $n=(20,30,50)$ from a mixture of two $Gomp(\alpha_j)$ components, $j=1,2,$ as follows:

- generate two observations u_1,u_2 from Uniform$(0,1)$.
- if $u_1\le p,$ then

$$t=\frac{1}{\alpha_1}\log\left[1-\alpha_1\log(1-u_2)\right],$$

otherwise

$$t=\frac{1}{\alpha_2}\log\left[1-\alpha_2\log(1-u_2)\right].$$

- repeat above steps n times to get a sample of size n.

4) The sample obtained in Step 3 is ordered.

5) The MLE's of the parameters p,α_1 and α_2 are obtained by solving the nonlinear Equations (31), by using Mathematica 6.

6) Using the generated values of p,α_1 and α_2, we generate upper record values of size $n=(5,8,10)$ from a mixture of two $Gomp(\alpha_j)$, $j=1,2$ components.

7) The MLE's of the parameters p,α_1 and α_2 are obtained by solving the nonlinear Equations (33), by using Mathematica 6.

8) The Bayes estimates under squared error and Linex loss functions (BES, BEL), of p,α_1 and α_2 are computed, by using MCI forms (23) and (24), respectively.

9) The squared deviations $(\hat{\varepsilon}-\varepsilon)^2$ are computed for different samples and censoring sizes, where ε stands for the parameter and $\hat{\varepsilon}$ its estimate (ML or Bayes).

10) The above Steps (3)-(9) are repeated 1000 times. The averages and the estimated risks (ER) are computed over the 1000 repetitions by averaging the estimates and the squared deviations, respectively.

The computational (our) results were computed by using Mathematica 6.0. In all above cases the prior parameters chosen as $b_1 = 4.5, b_2 = 3.2, d_1 = 1.2,\ d_2 = 2.3$, which yield the generated values of $p = 0.524855$, $\alpha_1 = 1.47589$ and $\alpha_2 = 2.7255$ (as the true values). The true values of $R(t)$ and $H(t)$ when $t = 0.5$, are computed to be $R(0.5) = 0.414044$ and $H(0.5) = 2.80864$. The value of the shape parameter a of the Linex loss function is $a = 2$. The averages and the estimated risks (ER) are displayed in **Tables 1-4**. **Figures 1** and **2** represent the estimated risks of the estimates in the case of upper order statistics. **Figures 3** and **4** represent the estimated risks of the estimates

Table 1. (Upper order statistics) Averages and Estimated Risks (ER) of the estimates of $\theta = (p, \alpha_1, \alpha_2)$ for different samples and censoring sizes.

$n\ (s,r)$	Method	$\bar{\bar{p}}\ \ ER(\bar{\bar{p}})$	$\bar{\bar{\alpha}}_1\ \ ER(\bar{\bar{\alpha}}_1)$	$\bar{\bar{\alpha}}_2\ \ ER(\bar{\bar{\alpha}}_2)$
20 (2, 14)	ML	0.707005 (0.19356)	2.31845 (1.46886)	2.39962 (0.850168)
	BL	0.616284 (0.00864557)	1.09815 (0.14386)	1.13638 (2.52742)
	BS	0.643168 (0.0141694)	1.10888 (0.136649)	1.16379 (2.44375)
30 (2, 21)	ML	0.714324 (0.188562)	2.25684 (1.18964)	2.32837 (0.716555)
	BL	0.632902 (0.011701)	1.022 (0.20618)	1.12869 (2.55009)
	BS	0.683938 (0.0254091)	1.02377 (0.204632)	1.16001 (2.45121)
50 (2, 35)	ML	0.666667 (0.181185)	2.17853 (0.806393)	2.26631 (0.516371)
	BL	0.547597 (0.000574537)	1.0264 (0.20208)	1.06996 (2.74081)
	BS	0.552577 (0.000797349)	1.02767 (0.200941)	1.07139 (2.73608)
20 (1, 14)	ML	0.710467 (0.196735)	2.36646 (1.64159)	2.44219 (0.907496)
	BL	0.601193 (0.005991)	1.10452 (0.138821)	1.1209 (2.57694)
	BS	0.623959 (0.00994848)	1.1156 (0.131404)	1.15205 (2.48091)
$n\ (s,r)$	Method	$\bar{\bar{p}}\ \ ER(\bar{\bar{p}})$	$\bar{\bar{\alpha}}_1\ \ ER(\bar{\bar{\alpha}}_1)$	$\bar{\bar{\alpha}}_2\ \ ER(\bar{\bar{\alpha}}_2)$
30 (1, 21)	ML	0.725016 (0.195162)	2.20851 (1.03578)	2.28107 (0.692991)
	BL	0.607509 (0.00766407)	1.065 (0.169001)	1.10095 (2.63928)
	BS	0.630063 (0.0116613)	1.06868 (0.166023)	1.10867 (2.61438)
50 (1, 35)	ML	0.682439 (0.183833)	2.15281 (0.758847)	2.24968 (0.522389)
	BL	0.458447 (0.00464071)	1.04342 (0.187036)	1.05501 (2.79062)
	BS	0.464408 (0.00402117)	1.04364 (0.186845)	1.05637 (2.7861)
20 (1, 20)	ML	0.695272 (0.178062)	1.89954 (0.505669)	2.38476 (0.90054)
	BL	0.562142 (0.00151982)	1.62688 (0.0668917)	1.82389 (0.85426)
	BS	0.589223 (0.00429)	1.97985 (0.360228)	2.83692 (0.411247)
30 (1, 30)	ML	0.671677 (0.1622)	1.79666 (0.353318)	2.45167 (1.18287)
	BL	0.556202 (0.00130063)	1.64213 (0.0834779)	1.83633 (0.838901)
	BS	0.585439 (0.004034)	1.96737 (0.352081)	2.67792 (0.375141)
50 (1, 50)	ML	0.626802 (0.144386)	1.70131 (0.209794)	2.49075 (0.791618)
	BL	0.563246 (0.00174849)	1.651 (0.0792463)	1.8386 (0.821578)
	BS	0.588489 (0.00434187)	1.93554 (0.292391)	2.65172 (0.359827)

Table 2. (Upper order statistics) averages and estimated risks (ER) of the estimates of $R(t)$ and $H(t)$ for different sample and censoring sizes.

$n\ (s,r)$	Method	$\bar{\bar{R}}(t)\ ER\left(\bar{\bar{R}}(t)\right)$	$\bar{\bar{H}}(t)\ ER\left(\bar{\bar{H}}(t)\right)$
20 (2, 14)	ML	0.381661 (0.00906794)	3.68514 (7.66423)
	BL	0.511856 (0.00958999)	1.74308 (1.13692)
	BS	0.511923 (0.00960227)	1.7486 (1.12574)
30 (2, 21)	ML	0.388581 (0.00706446)	3.43017 (2.94528)
	BL	0.51866 (0.0109467)	1.68457 (1.26369)
	BS	0.51867 (0.0109488)	1.6854 (1.26185)
50 (2, 35)	ML	0.396371 (0.00370546)	3.18718 (1.07172)
	BL	0.518651 (0.0109425)	1.68541 (1.26163)
	BS	0.518651 (0.0109426)	1.68546 (1.26153)
20 (1, 14)	ML	0.376971 (0.0100604)	3.85478 (12.0168)
	BL	0.511715 (0.00955799)	1.74519 (1.13219)
	BS	0.51177 (0.00956825)	1.7498 (1.12281)
30 (1, 21)	ML	0.393384 (0.00610191)	3.31504 (2.11435)
	BL	0.515905 (0.0103771)	1.70996 (1.2072)
	BS	0.515913 (0.0103787)	1.71063 (1.20575)
50 (1, 35)	ML	0.398975 (0.00348857)	3.14314 (0.93894)
	BL	0.518317 (0.0108729)	1.68844 (1.25485)
	BS	0.518318 (0.0108731)	1.68851 (1.2547)
20 (1, 20)	ML	0.409763 (0.00335753)	2.962 (0.742865)
	BL	0.394839 (0.00226178)	2.58386 (0.154358)
	BS	0.398332 (0.0020448)	2.95451 (0.319495)
30 (1, 30)	ML	0.413582 (0.00272424)	2.881 (0.51324)
	BL	0.398846 (0.00209693)	2.62951 (0.174472)
	BS	0.401559 (0.00195636)	2.92977 (0.308405)
50 (1, 50)	ML	0.415782 (0.00169979)	2.82688 (0.290495)
	BL	0.402934 (0.00139306)	2.67367 (0.135707)
	BS	0.404751 (0.001321)	2.87568 (0.186433)

Table 3. (Upper record values) Averages and Estimated Risks (ER) of the estimates of $\theta = (p, \alpha_1, \alpha_2)$ for different sample and censoring sizes.

$n\ (s,r)$	Method	$\bar{\bar{p}}\ ER(\bar{\bar{p}})$	$\bar{\bar{\alpha}}_1\ ER(\bar{\bar{\alpha}}_1)$	$\bar{\bar{\alpha}}_2\ ER(\bar{\bar{\alpha}}_2)$
5 (2, 4)	ML	0.631933 (0.200897)	2.67986 (2.83816)	2.89873 (1.4118)
	BL	0.558404 (0.00128567)	1.74504 (0.108009)	1.98448 (0.579141)
	BS	0.587338 (0.00404486)	2.28454 (0.793391)	3.46485 (0.776266)
8 (2, 7)	ML	0.564415 (0.193903)	2.61393 (2.11598)	3.0953 (5.09421)
	BL	0.545122 (0.000661206)	1.99517 (0.36226)	2.21236 (0.334018)
	BS	0.574543 (0.00269061)	2.59249 (1.47151)	3.70339 (1.32978)
10 (2, 9)	ML	0.751296 (0.19713)	2.14647 (0.632337)	2.93067 (23.1693)
	BL	0.553808 (0.000924086)	1.86508 (0.227293)	2.11734 (0.431788)
	BS	0.580802 (0.00320725)	2.30151 (0.800718)	3.30646 (0.509574)
5 (1, 4)	ML	0.626796 (0.2019)	2.62149 (2.54316)	2.85242 (1.32977)
	BL	0.561015 (0.00133488)	1.74589 (0.104393)	1.92257 (0.667007)
	BS	0.5882	2.27422	3.24856
$n\ (s,r)$	Method	$\bar{\bar{p}}\ ER(\bar{\bar{p}})$	$\bar{\bar{\alpha}}_1\ ER(\bar{\bar{\alpha}}_1)$	$\bar{\bar{\alpha}}_2\ ER(\bar{\bar{\alpha}}_2)$
8 (1, 7)	ML	0.553391 (0.194865)	2.58457 (1.69481)	2.9875 (2.2188)
	BL	0.553064 (0.00100325)	2.00503 (0.360188)	2.14377 (0.404423)
	BS	0.582772 (0.0035541)	2.56838 (1.36471)	3.76664 (1.4979)
10 (1, 9)	ML	0.737692 (0.195127)	2.12728 (0.627068)	2.69955 (5.2213)
	BL	0.551621 (0.000954124)	1.86627 (0.218747)	2.08316 (0.468689)
	BS	0.583116 (0.00359548)	2.24422 (0.690718)	3.34005 (0.632348)
5 (1, 5)	ML	0.69058 (0.195468)	2.25104 (1.41082)	2.6748 (1.11396)
	BL	0.555479 (0.00102412)	1.72972 (0.116361)	1.91339 (0.699269)
	BS	0.584415 (0.00363311)	2.21588 (0.7209)	3.26564 (0.576103)
8 (1, 8)	ML	0.671123 (0.196826)	2.28674 (1.0187)	2.85737 (4.12699)
	BL	0.538713 (0.000392836)	1.89649 (0.283578)	2.10966 (0.470518)
	BS	0.56659 (0.00190609)	2.4252 (1.13393)	3.41334 (0.749679)
10 (1, 10)	ML	0.789522 (0.194147)	2.00076 (0.478516)	2.67385 (1.92451)
	BL	0.547329 (0.000585634)	1.82498 (0.181567)	2.03864 (0.53141)
	BS	0.575719 (0.00265766)	2.23599 (0.660433)	3.23598 (0.476606)

Table 4. (Upper record values) Averages and Estimated Risks (ER) of the estimates of $R(t)$ and $H(t)$ for different sample and censoring sizes.

$n\ (s,r)$	Method	$\bar{\hat{R}}(t)\ ER\left(\bar{\hat{R}}(t)\right)$	$\bar{\hat{H}}(t)\ ER\left(\bar{\hat{H}}(t)\right)$
5 (2, 4)	ML	0.342028 (0.0172756)	5.66515 (237.48)
	BL	0.344783 (0.00674596)	2.68139 (0.0733731)
	BS	0.353593 (0.00543755)	3.59821 (1.31366)
8 (2, 7)	ML	0.335959 (0.0132904)	4.4366 (10.0859)
	BL	0.312673 (0.0125119)	3.05881 (0.196192)
	BS	0.320432 (0.0109407)	4.07878 (2.75144)
10 (2, 9)	ML	0.392196 (0.00425724)	3.26384 (1.69679)
	BL	0.349632 (0.00541351)	2.86293 (0.118538)
	BS	0.355924 (0.00461905)	3.46669 (0.874738)
5 (1, 4)	ML	0.348583 (0.0161086)	4.99658 (64.6011)
	BL	0.352821 (0.00530487)	2.69339 (0.0650445)
	BS	0.360458 (0.00428379)	3.5342 (1.30239)

$n\ (s,r)$	Method	$\bar{\hat{R}}(t)\ ER\left(\bar{\hat{R}}(t)\right)$	$\bar{\hat{H}}(t)\ ER\left(\bar{\hat{H}}(t)\right)$
8 (1, 7)	ML	0.338792 (0.012543)	4.31724 (6.94347)
	BL	0.31614 (0.0115378)	2.99744 (0.157374)
	BS	0.323935 (0.0100044)	3.95132 (2.20381)
10 (1, 9)	ML	0.394473 (0.00356277)	3.19236 (1.10688)
	BL	0.350405 (0.00522127)	2.87756 (0.100219)
	BS	0.356443 (0.00444454)	3.48105 (0.768389)
5 (1, 5)	ML	0.379553 (0.0101978)	3.97702 (60.4934)
	BL	0.356758 (0.00535798)	2.66213 (0.0984434)
	BS	0.364529 (0.00435279)	3.41487 (1.22003)
8 (1, 8)	ML	0.372725 (0.0076614)	3.64809 (3.4103)
	BL	0.331679 (0.00880409)	2.93711 (0.172037)
	BS	0.338926 (0.0076069)	3.74408 (1.62913)
10 (1, 10)	ML	0.406745 (0.00339404)	3.01535 (1.01302)
	BL	0.353242 (0.004766)	2.84017 (0.097151)
	BS	0.359484 (0.0040121)	3.38218 (0.568505)

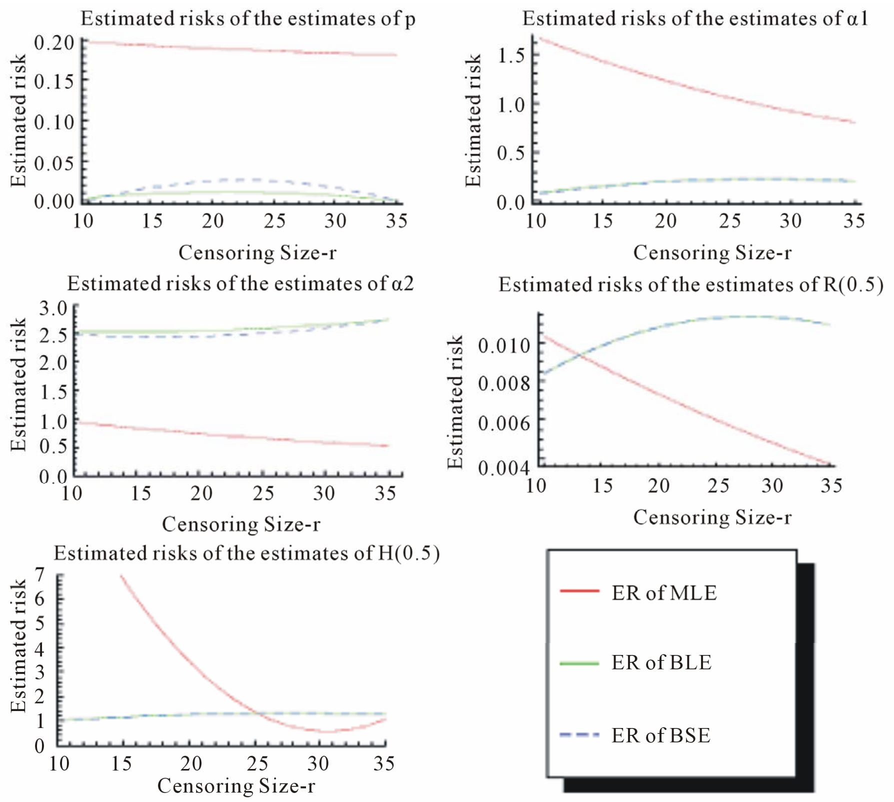

Figure 1. Estimated Risks (ER) of the estimates of $\theta = (p, \alpha_1, \alpha_2)$, $R(t)$ and $H(t)$ based on doubly Type II censored samples (*s* = *2*).

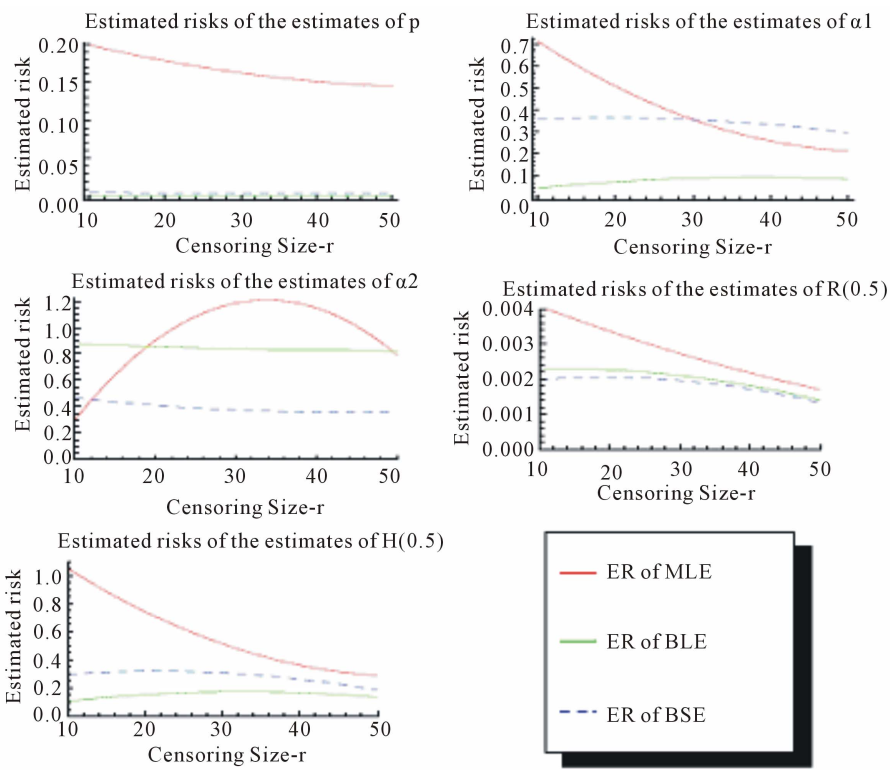

Figure 2. Estimated Risks (ER) of the estimates of $\theta = (p, \alpha_1, \alpha_2)$, $R(t)$ and $H(t)$ complete samples.

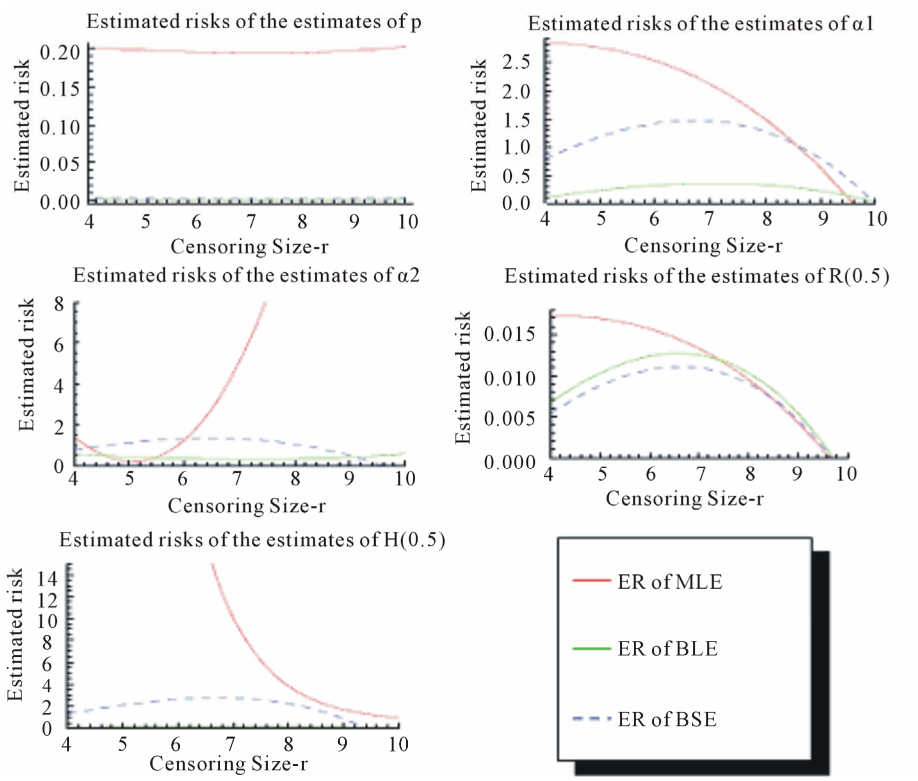

Figure 3. Estimated Risks (ER) of the estimates of $\theta=(p,\alpha_1,\alpha_2)$, $R(t)$ and $H(t)$ based on doubly Type II censored samples (*s* = *2*).

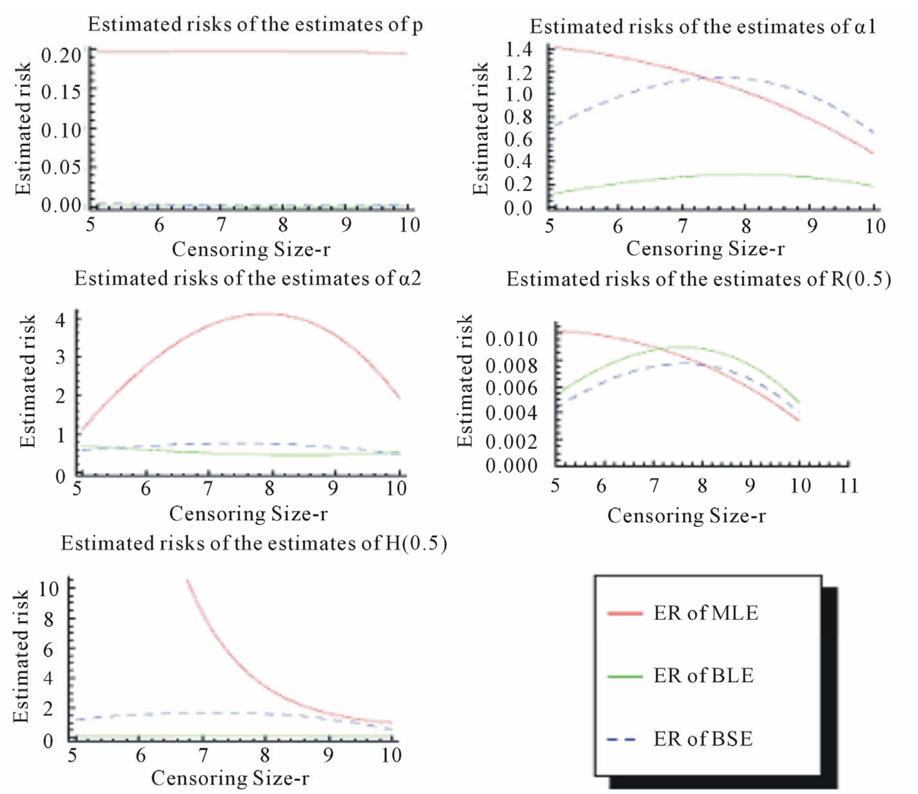

Figure 4. Estimated Risks (ER) of the estimates of $\theta=(p,\alpha_1,\alpha_2)$, $R(t)$ and $H(t)$ complete samples.

in the case of upper record values.

6 Concluding Remarks

1) Estimation of the parameters of the finite mixture model of two Gompertz distributions are considered from a Bayesian approach based on gos's. A compareson between ML and Bayes estimators, under either a squared error loss or a Linex loss, is made by using a Monte Carlo simulation study in both two cases considering order statistics and upper record values cases.

2) From **Tables 1** and **2**, we see that in most of the considered cases, the ER's of the estimates decrease as *n* increases. In complete sample case, the Bayes estimates of *p*, α_1 and *HRF* under Linex loss function have the smallest ER's as compared with their corresponding estimates under squared error loss function or MLE', while the ER's of the Bayes estimates of α_2 and *RF* under squared error loss functions are the smallest estimated risks. For censored samples, the Bayes estimates of *p* under Linex loss function have the smallest ER's as compared with their corresponding estimates under squared error loss function or MLE's. While, the Bayes estimates (against the proposed prior) of α_1 and *HRF* under squared error loss function have the smallest ER's as compared with their corresponding estimates. It is observed that MLE's for *HRF* perform best when sample size *n* is increased. Also, we note that the MLE's of α_2 and *RF* have the smallest ER's as compared with Bayes estimates.

3) From **Tables 3** and **4**, we see that the Bayes estimates (against the proposed prior) of the parameters and *HRF* under Linex loss function have the smallest ER's as compared with their corresponding estimates under squared error loss function or MLE's. While, the Bayes estimates of α_2 (for complete sample) and *RF* under squared error loss function have the smallest ER's as compared with both Bayes estimates under Linex loss function or the MLE's. Also, it is observed that MLE's for *RF* perform best when sample size *n* is increased.

4) If the mixing proportion *p* is known, [2] estimated the parameters α_1, α_2, reliability and hazard rate functions based on Types I and II censored samples.

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