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A Comment on "On Humbert Matrix Polynomials of Two Variables"

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ABSTRACT

In this comment we will demonstrate that one of the main formulas given in Ref. [1] is incorrect.

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1. Introduction and Motivation

It is well known that for a family of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ the so-called "generating functions" corresponding to this class of functions are a useful tool for their study, see [2,3]. Usually, a generating function is a function of two variables $F(x,t)$, analytic in some set $D \subset \mathbb{C}^2$, so that

$$F(x,t) = \sum_{n=0}^{\infty} \alpha_n P_n(x) t^n, (x,t) \in D.$$

For example, we have the following generating function of Hermite polynomials $F(x,t) = \exp 2xt - t^2$, because we can write:

$$F(x,t) = \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n, \forall (x,t) \in \mathbb{C}^2.$$

Note that it is important to specify the subset where the function $F(x,t)$ is well defined and analytic. For example, for Legendre polynomials we have

$$F(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, |x| \leq 1, |t| < 1. \quad (1)$$

where it is important to specify the domain of the variables $(|x| \leq 1, |t| < 1)$, because, in other case, for example with the choise $x = t = 1$, formula (1) is meaningless.

The extension to the matrix framework for the classical case of Gegenbauer [4], Laguerre [5], Hermite [6], Jacobi [7] and Chebyshev [8] polynomials has been made in recent years, and properties and applications of different classes for these matrix polynomials are given in several papers, see [9-13] for example. The importance of the generating function for orthogonal matrix polynomials is similar to the scalar case, taking into account the possible additional spectral restrictions (for a matrix $A \in \mathbb{C}^{N\times N}$ we will denote by $\sigma(A)$ the spectrum set $\sigma(A) = \{z; z \text{ is a eigenvalue of } A\}$). For example:

- For a matrix $A \in \mathbb{C}^{N\times N}$ such that $\mathrm{Re}(z) > 0$, $\forall z \in \sigma(A)$, *i.e*, A is say positive stable matrix, the Hermite matrix polynomials sequence $\{H_n(x,A)\}_{n\geq 0}$ is defined by the generating function [6]:

$$F(x,t,A) = \mathrm{e}^{xt\sqrt{A}-t^2 I} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x,A) t^n, (x,t) \in \mathbb{R}^2.$$

- For a matrix $A \in \mathbb{C}^{N\times N}$ such that $-k \notin \sigma(A)$ for every integer $k > 0$, and λ is a complex number with $\mathrm{Re}(\lambda) > 0$, the Laguerre matrix polynomials sequence $\{L_n^{(A,\lambda)}(x)\}_{n\geq 0}$ is defined by the generating function [5]:

$$F(x,t,A) = (1-t)^{-(A+I)} \exp\left(\frac{-\lambda xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n, \forall\, x,t \in \mathbb{C}, |t| < 1.$$

2. The Detected Error

Recently, in Ref. [1], the Humbert matrix polynomials of two variables are defined using the generating matrix function given in Formula (7):

$$\left(1 - (mxt - t^m) - (mys - s^m)\right)^{-A} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_{n,k,m}(x,y,A) t^n s^k, \quad (7)$$

where $A \in \mathbb{C}^{N \times N}$ is a positive stable matrix, *i.e.*, satisfies $\mathrm{Re}(\lambda) > 0$ for all eigenvalue $\lambda \in \sigma(A)$, and m is a positive integer. This Formula (7) turns out to be the key for the development of the properties mentioned in the paper [1]. However, we will see that Formula (7) is incorrect. For this, first we have to observe that for a matrix A, we define

$$t^A = \mathrm{e}^{A\log(t)}$$

where e^{Bx} is the exponential matrix. Of course, t^A has sense only for $t \neq 0$. Thus, Expression (7) is meaningless if the term $1-(mxt-t^m)-(mys-s^m)$ is zero. Then, we only need to consider, for example, $m=3$, $y=s=t=1/2$ and $x=1/3$ and with this choice we have $1-(mxt-t^m)-(mys-s^m)=0$. Thus, (7) is meaningless.

Therefore, I ask the authors of Ref. [1] to clarify the domain of choice for the variables t, s in Formula (7) in order to guarantee the validity of the remaining formulas which are derived from (7) and are used in the remainder of [1].

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