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On the Symmetrical System of Rational Difference Equation $x_{n+1} = A + y_{n-k}/y_n$, $y_{n+1} = A + x_{n-k}/x_n$ *

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ABSTRACT

In this paper, we study the behavior of the symmetrical system of rational difference equation:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n},\ y_{n+1} = A + \frac{x_{n-k}}{x_n},\ n = 0,1,\cdots$$

where $A > 0$ and $x_i, y_i \in (0,\infty)$, for $i = -k, -k+1, \cdots, 0$.

Keywords: Symmetrical System; Difference Equation; Boundedness; Period-Two Solution

1. Introduction

Recently there has been a great interest in studying difference equations and systems, and quite a lot of papers about the behavior of positive solutions of system of difference equation. We can read references [1-10].

In [1] C. Cina studied the system:

$$x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},\ n = 0,1,\cdots. \quad (1)$$

In [2] A. Y. Ozban studied the difference equation system:

$$x_{n+1} = \frac{1}{y_{n-k}},\ y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}},\ n = 0,1,\cdots. \quad (2)$$

In [3] A. Y. Ozban studied the behavior of positive solutions of the difference equation system:

$$x_n = \frac{a}{y_{n-3}},\ y_n = \frac{by_{n-3}}{x_{n-q}y_{n-q}},\ n = 0,1,\cdots. \quad (3)$$

In [4] X. Yang, Y. Liu, S. Bai studied the difference equation system:

$$x_{n+1} = \frac{a}{y_{n-p}},\ y_{n+1} = \frac{by_{n-p}}{x_{n-q}y_{n-q}},\ n = 0,1,\cdots. \quad (4)$$

We can see in [1-4], they have the same similar character, which is the system can be reduced into a difference equation with x_n or y_n.

In [5] G. Papaschinopoulos, C. J. Schinas studied the behavior of positive solutions of the difference equation system:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}},\ y_{n+1} = A + \frac{x_n}{y_{n-p}},\ n = 0,1,\cdots. \quad (5)$$

In [6] G. Papaschinopoulos, Basil K. Papadopoulos studied the behavior of positive solutions of the difference equation system:

$$x_{n+1} = A + \frac{x_n}{y_{n-p}},\ y_{n+1} = B + \frac{y_n}{x_{n-p}},\ n = 0,1,\cdots. \quad (6)$$

In [7] E. Camouzis, G. Papaschinopoulos studied the behavior of positive solutions of the difference equation system:

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}},\ y_{n+1} = 1 + \frac{y_n}{x_{n-m}},\ n = 0,1,\cdots. \quad (7)$$

In [8] Yu Zhang, Xiaofan Yang, David J. Evans, Ce Zhu studied the behavior of positive solutions of the difference equation system:

$$x_{n+1} = A + \frac{y_{n-m}}{x_n},\ y_{n+1} = A + \frac{x_{n-m}}{y_n},\ n = 0,1,\cdots. \quad (8)$$

Motivated by systems above, we introduce the sym-

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metrical system:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n},\ y_{n+1} = A + \frac{x_{n-k}}{x_n},\ n = 0,1,\cdots. \qquad (9)$$

with parameter $A > 0$, the initial conditions $x_i, y_i > 0$, for $i = -k, -k+1, \cdots, 0$, and k is a positive integer. We can easily get the system (9) has the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$.

There are two cases we need to consider:

1) If the initial conditions $x_i = y_i$ in the system (9) for $i = -k, -k+1, \cdots, 0$, then $x_n = y_n$ for all $n \ge -k$, thus, the system (9) reduces to the difference equation

$$x_{n+1} = A + \frac{x_{n-k}}{x_n}$$

which was studied by El-owaidy in [11].

2) If $x_i \ne y_i$ for $i \in \{-k, -k+1, \cdots, 0\}$, then the system (9) is similar to the system in [8]. We study the system (9) basing on this condition in this paper.

In this paper, we try to give some results of the system (9) by using the methods in [8]. We consider the following cases of $0 < A < 1$, $A = 1$ and $A > 1$.

2. The Case 0 < *A* < 1

In this section, we give the asymptotic behavior of positive solution to the system (9).

Theorem 2.1. *Suppose* $0 < A < 1$ *and* $\{x_n, y_n\}$ *is an arbitrary positive solution of the system* (9). *Then the following statements hold.*

1) If k is odd, and $0 < x_{2m-1} < 1$, $0 < y_{2m-1} < 1$, $x_{2m} > \frac{1}{1-A}$, $y_{2m} > \frac{1}{1-A}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \cdots, 0$, then

$$\lim_{n\to\infty} x_{2n} = \infty, \lim_{n\to\infty} y_{2n} = \infty, \lim_{n\to\infty} x_{2n+1} = A, \lim_{n\to\infty} y_{2n+1} = A.$$

2) If k is odd, and $0 < x_{2m} < 1$, $0 < y_{2m} < 1$, $x_{2m-1} > \frac{1}{1-A}$, $y_{2m-1} > \frac{1}{1-A}$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \cdots, 0$, then

$$\lim_{n\to\infty} x_{2n} = A, \lim_{n\to\infty} y_{2n} = A, \lim_{n\to\infty} x_{2n+1} = \infty, \lim_{n\to\infty} y_{2n+1} = \infty.$$

3) If k is even, we can not get some useful results.

Proof: 1) Obviously, we can have

$$0 < x_1 = A + \frac{y_{-k}}{y_0} < A + \frac{1}{y_0} < A + (1-A) = 1,$$

$$0 < y_1 = A + \frac{x_{-k}}{x_0} < A + \frac{1}{x_0} < A + (1-A) = 1,$$

$$x_2 = A + \frac{y_{1-k}}{y_1} > A + y_{1-k} > y_{1-k} > \frac{1}{1-A},$$

$$y_2 = A + \frac{x_{1-k}}{x_1} > A + x_{1-k} > x_{1-k} > \frac{1}{1-A}.$$

By introduction, we can get

$$0 < x_{2n+1} < 1,\ 0 < y_{2n+1} < 1,\ x_{2n} > \frac{1}{1-A},\ y_{2n} > \frac{1}{1-A},$$

for $n = 0,1,2,\cdots$.

So for $n \ge \frac{k+2}{2}$,

$$\begin{aligned} x_{2n} &= A + \frac{y_{2n-(k+1)}}{y_{2n-1}} > A + y_{2n-(k+1)} \\ &= 2A + \frac{x_{2n-(2k+2)}}{x_{2n-k-2}} > 2A + x_{2n-(2k+2)}. \end{aligned}$$

By limiting the inequality above, we can get $\lim_{n\to\infty} x_{2n} = +\infty$. Similarly, we can also get $\lim_{n\to\infty} y_{2n} = +\infty$.

Taking limits on the both sides of the following two equations

$$x_{2n+1} = A + \frac{y_{2n-k}}{y_{2n}},\ y_{2n+1} = A + \frac{x_{2n-k}}{x_{2n}}$$

we can obtain $\lim_{n\to\infty} x_{2n+1} = A$, $\lim_{n\to\infty} y_{2n+1} = A$.

The proof of 2) is similar, so we omit it.

3. The Case *A* = 1

In this section, we try to get the boundedness, persistence, and periodicity of positive solutions of the system (9).

Theorem 3.1. *Suppose* $A = 1$. *Then every positive solution of the system* (9) *is bounded and persists.*

Proof. $\{x_n, y_n\}_{n=-k}^{\infty}$ is a positive solution of the system (9).

Obviously, $x_n > 1, y_n > 1$, for $n \ge 1$. So we can get

$$x_i, y_i \in \left[L, \frac{L}{L-1}\right], i = 1,2,\cdots,k+1,$$

where $L = \min\left\{a, \frac{b}{b-1}\right\} > 1$, $a = \min\{x_i, y_i\}$, $b = \max\{x_i, y_i\}$, for $1 \le i \le k+1$.

Then we can obtain

$$\begin{aligned} L &= 1 + \frac{L}{L/(L-1)} \le x_{k+2} \\ &= 1 + \frac{y_1}{y_{k+1}} \le 1 + \frac{L/(L-1)}{L} = \frac{L}{L-1} \end{aligned}$$

$$\begin{aligned} L &= 1 + \frac{L}{L/(L-1)} \le y_{k+2} \\ &= 1 + \frac{x_1}{x_{k+1}} \le 1 + \frac{L/(L-1)}{L} = \frac{L}{L-1} \end{aligned}$$

By introduction, we have

$$x_i, y_i \in \left[L, \frac{L}{L-1}\right], i = 1, 2, \cdots. \tag{10}$$

Hence, we complete the proof.

Theorem 3.2. *Suppose* $A = 1$, $\{x_n, y_n\}_{n=-k}^{\infty}$ *is a positive solution of the system* (9). *Then*

$$\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n,$$
$$\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n.$$

Proof: By (10), we can get

$$l_1 = \liminf_{n\to\infty} x_n \ge L > 1,$$

$$l_2 = \liminf_{n\to\infty} y_n \ge L > 1.$$

$$U_1 = \limsup_{n\to\infty} x_n > 1,$$

$$U_2 = \limsup_{n\to\infty} y_n > 1.$$

By system (9), we can have

$$U_1 \le 1 + \frac{U_2}{l_2}, U_2 \le 1 + \frac{U_1}{l_1}, l_1 \ge 1 + \frac{l_2}{U_2}, l_2 \ge 1 + \frac{l_1}{U_1}$$

which implies $U_1 l_2 \le l_2 + U_2 \le l_1 U_2 \le l_1 + U_1 \le l_2 U_1$.

Hence, we can obtain

$$l_1 + U_1 = l_2 + U_2, l_1 U_2 = l_2 U_1,$$

which can be changed into

$$l_1 + (-U_2) = l_2 + (-U_1), l_1(-U_2) = l_2(-U_1).$$

Obviously, $l_1 = l_2, U_1 = U_2$, we complete the proof.

Theorem 3.3. *Suppose* $A = 1$.

1) If k is odd, then every positive solution of the system (9) with prime period two takes the form

$$(a,a), \left(\frac{a}{a-1}, \frac{a}{a-1}\right), (a,a), \left(\frac{a}{a-1}, \frac{a}{a-1}\right), \cdots. \tag{11}$$

or

$$\left(a, \frac{a}{a-1}\right), \left(\frac{a}{a-1}, a\right), \left(a, \frac{a}{a-1}\right), \left(\frac{a}{a-1}, a\right), \cdots. \tag{12}$$

with $1 < a \ne 2$.

2) If m is even, there do not exist positive nontrival solution of the system (9) with prime period two.

Proof: 1) As k is odd.

We set $\{x_n, y_n\}$ is the solution of the system (9) with prime period two. Then there are four positive number $A, B, C, D > 1$ such that

$$x_{2n-k} = A, y_{2n-k} = B, x_{2n+1-k} = C, y_{2n+1-k} = D, n = 0, 1, \cdots.$$

If $A = C$, by the system (9) we can get $B = D = 2$, which is contradiction with the condition $a \ne 2$, hence $A \ne C$. Similarly, we can get $B \ne D$. Then we obtain

$$\liminf_{n\to\infty} x_n = \min\{A, C\},$$

$$\liminf_{n\to\infty} y_n = \min\{B, D\}.$$

$$\limsup_{n\to\infty} x_n = \max\{A, C\},$$

$$\limsup_{n\to\infty} y_n = \max\{B, D\}.$$

From Theorem 3.2, we can get

$$\min\{A, C\} = \min\{B, D\}$$

$$\max\{A, C\} = \max\{B, D\}$$

Next, we consider the following possibilities:

Case 1: Either(I) $A < C$ and $B < D$ or (II) $A > C$ and $B > D$. Then $A = B$, $C = D$.

Case 2: Either(I) $A < C$ and $B > D$ or (II) $A > C$ and $B < D$. Then $A = D$, $B = C$.

Therefore by the system (9), we can get 1) holds.

2) Obviously, if k is even, the system (9) just has trival solution with prime period two.

We complete the proof.

4. The Case $A > 1$

Theorem 4.1. *Suppose* $A > 1$. *Then every positive solution of the system* (9) *is bounded and persists.*

Proof. Let $\{x_n, y_n\}$ be a positive solution of the system (9). Obviously, $x_n > A > 1$, $y_n > A > 1$, for $n \ge 1$. So we can get

$$x_i, y_i \in \left[L, \frac{L}{L-A}\right], i = 1, 2, \cdots, k+1,$$

where $L = \min\left\{a, \frac{b}{b-A}\right\} > 1$, $a = \min\{x_i, y_i\}$, $b = \max\{x_i, y_i\}$, for $1 \le i \le k+1$. Then we can obtain

$$L = A + \frac{L}{L/(L-A)} \le x_{k+2}$$
$$= A + \frac{y_1}{y_{k+1}} \le A + \frac{L/(L-A)}{L} = \frac{L}{L-A}$$

$$L = A + \frac{L}{L/(L-A)} \le y_{k+2}$$
$$= A + \frac{x_1}{x_{k+1}} \le A + \frac{L/(L-A)}{L} = \frac{L}{L-A}$$

By introduction, we have

$$x_i, y_i \in \left[L, \frac{L}{L-A}\right], i = 1, 2, \cdots. \tag{13}$$

We complete the proof.

Theorem 4.2. *Suppose* $A > 1$. *Then every positive solution of the system* (9) *converges to the equilibrium as* $n \to \infty$.

Proof: By (13), we can get

$$l_1 = \liminf_{n\to\infty} x_n \ge L > A > 1,$$

$$l_2 = \liminf_{n\to\infty} y_n \ge L > A > 1.$$

$$U_1 = \limsup_{n\to\infty} x_n > A > 1,$$

$$U_2 = \limsup_{n\to\infty} y_n > A > 1.$$

By system (9), we can have

$$U_1 \le A + \frac{U_2}{l_2}, U_2 \le A + \frac{U_1}{l_1}, l_1 \ge A + \frac{l_2}{U_2}, l_2 \ge A + \frac{l_1}{U_1}$$

which imply

$$AU_1 + l_1 \le U_1 l_2 \le Al_2 + U_2,$$

$$AU_2 + l_2 \le U_2 l_1 \le Al_1 + U_1,$$

$$l_1 + AU_1 - (Al_1 + U_1) \le Al_2 + U_2 - (l_2 + AU_2),$$

$$(A-1)(U_1 - l_1 + U_2 - l_2) \le 0.$$

By the condition $A > 1$, we can get

$$U_1 - l_1 + U_2 - l_2 = 0$$

Besides, $U_1 - l_1 \ge 0$ and $U_2 - l_2 \ge 0$, so we can get $U_1 - l_1 = 0$ and $U_2 - l_2 = 0$.
i.e.

$$U_1 = l_1, U_2 = l_2$$

we complete the proof.

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