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On Complete Bicubic Fractal Splines

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Abstract

Fractal geometry provides a new insight to the approximation and modelling of experimental data. We give the construction of complete cubic fractal splines from a suitable basis and their error bounds with the original function. These univariate properties are then used to investigate complete bicubic fractal splines over a rectangle Ω. Bicubic fractal splines are invariant in all scales and they generalize classical bicubic splines. Finally, for an original function $f \in C^4[\Omega]$, upper bounds of the error for the complete bicubic fractal splines and derivatives are deduced. The effect of equal and non-equal scaling vectors on complete bicubic fractal splines were illustrated with suitably chosen examples.

Keywords: Fractals, Iterated Function Systems, Fractal Interpolation Functions, Fractal Splines, Surface Approximation.

1. Introduction

Schoenberg [1] introduced "spline functions" to the mathematical literature. In the last 60 years, splines have proved to be enormously important in different branches of mathematics such as numerical analysis, numerical treatment of differential, integral and partial differential equations, approximation theory and statistics. Also, splines play major roles in field of applications, such as CAGD, tomography, surgery, animation and manufacturing. In this paper, we discuss on complete fractal splines that generalize the classical complete splines.

Fractal interpolation functions (FIFs) were introduced by Barnsley [2,3] based on the theory of iterated function system (IFS). The attractor of the IFS is the graph of FIF that interpolates a given set of data points. Fractal interpolation constitutes an advance in the sense that the functions used are not necessarily differentiable and show the rough aspect of real-world signals [3,4]. A specific feature is the fact that the graph of these interpolants possesses a fractal dimension and this parameter provides a geometric characterization of the measured variable which may be used as an index of the complexity of a phenomenon. Barnsley and Harrington [5] first constructed a differentiable FIF or C^r-FIF f that interpolates the prescribed data if values of $f^{(k)}$, $k = 1, 2, \ldots, r$, at the initial end point of the interval are given. In this construction, specifying boundary conditions similar to those of classical splines was found to be quite difficult to handle. The fractal splines with general boundary conditions are studied recently [6,7]. The power of fractal methodology allows us to generalize almost any other interpolation techniques, see for instance [8-10].

Fractal surfaces have proved to be useful functions in scientific applications such as metallurgy, physics, geology, image processing and computer graphics. Massopust [11] was first to put forward the construction of fractal interpolation surfaces (FISs) on triangular domains, where the interpolation points on the boundary of the domain are coplanar. Geronimo and Hardin [12], and Zhao [13] generalized this construction in different ways. The general bivariate FIS on rectangular grids are treated for instance in references [14,15]. Recently, Bouboulis and Dalla constructed fractal interpolation surfaces from FIFs through recurrent iterated function systems [16].

In this paper we approach the problem of complete cubic spline surface from a fractal perspective. In Section 2, we construct cardinal cubic fractal splines through moments and estimate the error bound of the complete cubic spline with the original function. The construction of bicubic fractal splines is carried out in Section 3 through tensor products. Finally, for an original function $f \in C^4[\Omega]$, upper bounds of the error for the complete bicubic fractal splines and derivatives are deduced. The effect of scaling factors on bicubic fractal splines are demonstrated in the last section through various examples.

2. Complete Cubic Fractal Splines

We discuss on fractal interpolation based on IFS theory in Subsection 2.1 and construct cardinal cubic fractal spline through moments in Subsection 2.2. Upper bounds of L_∞-norm of the error of a complete cubic spline FIF with respect to the original function are deduced in Subsection 2.3.

2.1. Fractal Interpolation Functions

Let $\Delta_t : t_0 < t_1 < ... < t_N$ be a partition of the real compact interval $I = [t_0, t_N]$. Let a set of data points $\mathcal{D} = \{(t_n, x_n) \in I \times \mathbb{R} : n = 0,1,2,...,N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n$, $n = 1,2,\ldots,N$ be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n, \quad |L_n(c_1) - L_n(c_2)| \le l\,|c_1 - c_2| \ \ \forall\, c_1, c_2 \in I, \tag{1}$$

for some $0 \le l < 1$. Let $C = I \times \mathbb{R}$ and N continuous mappings, $F_n : C \to \mathbb{R}$, satisfying

$$F_n(t_0, x_0) = x_{n-1},\ F_n(t_N, x_N) = x_n,\ n = 1,2,\ldots,N, \quad |F_n(t,x) - F_n(t,y)| \le |\alpha_n||x-y|,\ t \in I,\ x,y \in \mathbb{R},\ -1 < \alpha_n < 1. \tag{2}$$

Now, define functions
$\forall\, n = 1,2,\ldots,N,\ w_n(t,x) = (L_n(t), F_n(t,x)), w_n : C \to I_n \times \mathbb{R}.$

Proposition 2.1 (Barnsley [2]) *The Iterated Function System* (*IFS*) $\{C; w_n : n = 1,2,...,N\}$ *defined above admits a unique attractor* G. G *is the graph of a continuous function* $f : I \to \mathbb{R}$ *which obeys* $f(t_n) = x_n$ *for* $n = 0,1,2,...,N$.

The above function f is called a Fractal Interpolation Function (FIF) corresponding to the IFS $\{(L_n(t), F_n(t,x))\}_{n=1}^N$. Let $\mathcal{G} = \{g : I \to \mathbb{R} \mid g \text{ iscontinuous}, g(t_0) = x_0 \text{ and } g(t_N) = x_N\}$ $\mathcal{G}$ is a complete metric space respect to the uniform norm. Define, the Read-Bajraktarević operator T on $(\mathcal{G}, \|.\|_\infty)$ by

$$Tg(t) = F_n\left(L_n^{-1}(t), g\left(L_n^{-1}(t)\right)\right), t \in I_n, n = 1,2,\ldots,N. \tag{3}$$

According to (1)-(2), Tg is continuous on the interval $I_n; n = 1,2,\ldots,N$ and at each of the points t_1, $t_2, \ldots, t_{N-1}$. T is a contraction mapping on the metric space $(\mathcal{G}, \|.\|_\infty)$ *i.e.*

$$\|Tf - Tg\|_\infty \le |\alpha|_\infty \|f - g\|_\infty, \tag{4}$$

where $|\alpha|_\infty = \max\{|\alpha_n| : n = 1,2,\ldots,N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point f (say) on $\mathcal{G}$, that is to say, there is $f \in \mathcal{G}$ such that $(Tf)(t) = f(t)\ \forall\, t \in I$. This function is the FIF corresponding to w_n and according to (3), the FIF satisfies the functional equation:

$$f(t) = F_n\left(L_n^{-1}(t), f \circ L_n^{-1}(t)\right),\ t \in I_n,\ n = 1,2,\ldots,N. \tag{5}$$

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t,x) = \alpha_n x + q_n(t) \end{cases} \tag{6}$$

where $-1 < \alpha_n < 1$ and $q_n : I \to \mathbb{R}$ are suitable continuous functions such that (2)are satisfied. α_n is called a vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of IFS. The scale factors give a degree of freedom to the FIF and allow us to modify its properties. If $q_n(t)$ are affine in (6) for $t \in I$, then the FIF is called affine [3]. Based on the principle of construction [6] of a C^r-FIF, $r \in \mathbb{N}$, complete cardinal cubic fractal splines are constructed through their moments in the following.

2.2. Complete Cardinal Cubic Fractal Splines

A cubic spline is called complete if the values of its first derivative are prescribed at the end points. A function $h(t)$ defined on the grid $\Delta_t : t_0 < t_1 < \ldots t_N$ is called an interpolating cubic spline function if the function. 1) is a cubic polynomial on each partial segment $[t_{n-1}, t_n]$, $n = 1,2,\ldots,N$. 2) the function is of class $C^2[t_0, t_2]$. 3) satisfies the conditions $h(t_n) = x_n$, $n = 0,1,\ldots,N$. Two conditions are given in the form of restriction on the spline values and/or the values of its derivatives at the end points of the segment $[t_0, t_N]$.

Definition 2.1 *A function* $f_m(t)$ *is called a cardinal cubic fractal spline if* 1) f_m *is a FIF associated with the set of data points* $(t_n, \delta_{m,n})$ *with mesh* Δ_t, *that is to say*

$$f_m(t_n) = \delta_{m,n} \left.\begin{aligned} &= 1, \quad m = n, \\ &= 0, \quad m \ne n, \end{aligned}\right\} \forall\, m,n = 0,1,2,\ldots,N. \tag{7}$$

Besides, 2) $f_m \in C^2[t_0, t_N]$, 3) *the corresponding IFS* $\omega_{m,n}(t,x) = (L_n(t), F_{m,n}(t,x))$ *is such that* $L_n(t)$ *is defined by* (6) *and* $F_{m,n}(t,x) = a_n^2 \alpha_{m,n} x + a_n^2 q_{m,n}(t), |\alpha_{m,n}| < 1$, *where* $q_{m,n}(t)$ *is a suitable cubic polynomial so that the polynomial associated with the fractal function* f_m'' *on the mesh* Δ_t *is affine.*

In the construction of cardinal cubic fractal splines, we have taken $\alpha_{m,n} = \alpha_n$, $n = 1,2,\ldots,N$; $m = 0,1,2,\ldots,N$.

A derivation of the defining equations for a cubic fractal spline through moments $M_{m,n} = f''_m(t_n)$ $n = 0,1,\dots,N$ can be found in [6], but for completeness and to set the terminology, it is outlined in the appendix.

For a basis of complete cubic fractal spline space on *I*, we need $f'_m(t_0) = f'_m(t_N) = 0$ for $m = 0,1,\dots,N$ in the construction of cubic spline FIF, and two more complete cubic fractal splines f_{-1} and f_{N+1} such that

$$f_{-1}(t_n) = 0, n = 0,1,\dots,N;\ f'_{-1}(t_0) = 1,\ f'_{-1}(t_N) = 0,$$
$$f_{N+1}(t_n) = 0, n = 0,1,\dots,N;\ f'_{N+1}(t_0) = 0,\ f'_{N+1}(t_N) = 1. \quad (8)$$

Denote, $x_{-1} = f(t_{-1}) = f'(t_0)$, $x_{N+1} = f(t_{N+1}) = f'(t_N)$. Let f be the original function providing the data $\{(t_n, x_n)\}_{n=-1}^{N+1}$ and f_c be the complete cubic fractal spline corresponding to this data. Let $\mathcal{U}(I,\Delta_t) = \{h \mid h$ is a complete cubic fractal spline on $\Delta_t\}$. If $h \in \mathcal{U}(I,\Delta_t)$ interpolating the same data $\{(t_n,x_n)\}_{n=0}^{N}$, then due to the uniqueness of fixed point of Read-Bajraktarević operator, $h(t) = \sum_{m=-1}^{N+1} x_m f_m(t)$. Also, none of the f_m is a linear combination of other cardinal splines and hence $\{f_m\}_{m=-1}^{N+1}$ is a basis for $\mathcal{U}(I,\Delta_t)$. Define a complete cubic fractal spline operator $\mathcal{F}: C^2(I) \to \mathcal{U}(I,\Delta_t)$ as $\mathcal{F}(f) = f_c$ such that

$$\begin{aligned} f_c\left(L_n(t)\right) &= \sum_{m=-1}^{N+1} f(t_m) f_m\left(L_n(t)\right) \\ &= \sum_{m=-1}^{N+1} x_m f_m\left(L_n(t)\right), t \in I, n = 1,2,\dots,N. \end{aligned} \quad (9)$$

It is easy to check that $\mathcal{F}$ is linear and bounded operator on $C^2(I)$. According to ((7)) and ((8)), we have

$$f_c(t_0) = f_c(L_1(t_0)) = \sum_{m=-1}^{N+1} x_m f_m(L_1(t_0)) = \sum_{m=-1}^{N+1} x_m \delta_{m,0} = x_0$$

and for $i = 1,2,\dots,N$,

$$\begin{aligned} f_c(t_n) = f_c\left(L_n(t_N)\right) &= \sum_{m=-1}^{N+1} x_m f_m\left(L_N(t_N)\right) \\ &= \sum_{m=-1}^{N+1} x_m f_m(x_n) = \sum_{n=-1}^{N+1} x_m \delta_{m,n} = x_n \end{aligned}$$

Also, $f'_c(t_0) = \sum_{m=-1}^{N+1} x_m f'_m(t_0) = x_{-1} = f'(t_0)$ and $f'_c(t_N) = \sum_{m=-1}^{N+1} x_m f'_m(t_N) = x_{N+1} = f'(t_N)$. If we choose $\alpha_n = 0; n = 1,2,\dots,N$, then from (26), it is clear that right side of cardinal spline f_m reduces to a cubic polynomial in t and hence, in this case f_m reduces to a classical complete cardinal spline S_m such that $S_m(t_n) = \delta_{m,n}$. The classical complete cubic spline $S(t)$ for the data $\{(t_n,x_n)\}_{n=0}^{N}$ is given by

$$S\left(L_n(t)\right) = \sum_{m=-1}^{N+1} x_m S_m\left(L_n(t)\right), t \in I, n = 1,2,\dots,N. \quad (10)$$

2.3. Error Estimation with Univariate Fractal Splines

To estimate error bounds for the complete bicubic fractal spline, we need error bounds between a cubic fractal spline and the original function $f \in C^p(I)$, $p = 2,3,4$. For given moments $\{M_{m,n}\}_{n=0}^{N}$, we can observe that $q_{m,n}$ is a function of the scaling factors $\alpha_n; n = 1,2,\dots,N$ for the cubic fractal spline equation (cf. (26)). We need the following proposition with the assumption $|\alpha_n| \le \kappa < 1$, for fixed κ.

Proposition 2.2 *Let* f_m *and* S_m $(m = -1,0,\dots,N+1)$ *be the cardinal cubic fractal spline and the classical cardinal cubic spline respectively to the same set of data* $\{(t_m,\delta_{m,n})\}_{m=0}^{N}$. *Let* $h_t = \max\{t_n - t_{n-1} : n = 1,2,\dots,N\}$, $|\alpha|_\infty = \max\{|\alpha_n| : n = 1,2,\dots,N\}$, *and* $|I|$ *is the length of the interval* I. *Suppose the cubic polynomial* $q_{m,n}(\alpha_n,t)$ *associated with the IFS corresponding to the cardinal fractal spline* f_m *satisfies*

$$\left|\frac{\partial^{1+u} q_{m,n}(\tau_n,t)}{\partial\alpha_n \partial t^u}\right| \le \Theta_{u,m}$$

for $|\tau_n| \in (0,\kappa a_m^u), t \in I_n, u = 0,1,2$ and $n = 1,2,\dots,N$. Then,

$$\left\|f_m^{(u)} - S_m^{(u)}\right\|_\infty \le \frac{h_t^{2-u}|\alpha|_\infty}{|I|^{2-u} - h_t^{2-u}|\alpha|_\infty}\left(\left\|S_m^{(u)}\right\|_\infty + \Theta_{u,m}\right), \quad (11)$$
$$u = 0,1,2.$$

The proof of the above proposition can be seen in [6]. Now, we derive an upper bound for the error between the classical complete cubic spline and a complete cubic fractal spline for the same set of interpolation data. According to (9) and (10), we get the bottom equation

$$\begin{aligned} \left|f_c^{(u)}\left(L_n(t)\right) - S^{(u)}\left(L_n(t)\right)\right| &= \left|\sum_{m=-1}^{N+1} f(t_m)\left(f_m^{(u)} - S_m^{(u)}\right)\left(L_n(t)\right)\right| \le \sum_{m=-1}^{N+1}\|f\|_1 \left\|f_m^{(u)} - S_m^{(u)}\right\|_\infty \\ &\le \sum_{m=-1}^{N+1}\|f\|_1 \frac{h_t^{2-u}|\alpha|_\infty}{|I|^{2-u} - h_t^{2-u}|\alpha|_\infty}\left(\Lambda_u + \Theta_u\right),\ u = 0,1,2, \end{aligned}$$

where 1-norm of f is $\|f\|_1 = Max\{\|f\|_\infty, \|f'\|_\infty\}$, $\Lambda_u = \max\{\|S_m^{(u)}\|_\infty : m = -1, 0, \ldots, N+1\}$ and $\Theta_u = \max\{\Theta_{u,m} : m = -1, 0, \ldots, N+1\}$. Set, $\Gamma_{u,\alpha,N} = \dfrac{|\alpha|_\infty (N+3)(\Lambda_u + \Theta_u)}{|I|^{2-u} - h_t^{2-u} |\alpha|_\infty}$. Since the above inequality is true for $n = 1, 2, \ldots, N$, we have the following estimate.

$$\left\|f_c^{(u)} - S^{(u)}\right\|_\infty \le \|f\|_1 \Gamma_{u,\alpha,N} h_t^{2-u},\ u = 0,1,2. \qquad (12)$$

We need the error bound between the complete cubic fractal spline f_c and the original function $f \in C^p(I)$, $p = 2,3,4$.

Proposition 2.3 [17] *Let S be the complete cubic spline interpolant of $f \in C^p[t_0, t_N]$ for $p = 2,3$, or 4 with the assumption $h_t \le 1$. Then*

$$\left\|(S - f)^{(u)}\right\|_\infty \le \varepsilon_{p,u} \left\|f^{(p)}\right\|_\infty h_t^{p-u},\ 0 \le u \le \min\{p, 3\}, \qquad (13)$$

where $\varepsilon_{p,u}$ are given in **Table 1** with $\eta_t = h_t / \min(t_n - t_{n-1})$.

If the original function f is such that $f \in C^p[t_0, t_N]$ with p-norm $\|f\|_p = Max\{\|f\|_\infty, \|f'\|_\infty, \ldots, \|f^{(p)}\|_\infty\}$, according to (12) and (13), we have the following upper bound estimation for the error.

$$\left\|(f_c - f)^{(u)}\right\|_\infty \le \|f\|_1 \Gamma_{u,\alpha,N} h_t^{2-u} + \left\|f^{(p)}\right\|_\infty \varepsilon_{p,u} h_t^{p-u},$$

$$0 \le u \le 2,$$

and

$$\left\|(f_c - f)^{(u)}\right\|_\infty \le \|f\|_p \left(\Gamma_{u,\alpha,N} h_t^{2-u} + \varepsilon_{p,u} h_t^{p-u}\right),\ 0 \le u \le 2. \qquad (14)$$

Table 1. Coefficients associated with the error of classical complete cubic spline.

$\varepsilon_{p,u}$	$u=0$	$u=1$	$u=2$	$u=3$
$p=2$	9/8	4	10	-
$p=3$	71/216	31/27	5	$(63+8\eta_t^2)/9$
$p=4$	5/384	$1/24^a$	$3/8^a$	$(\eta_t+\eta_t^{-1})/2^a$

[a]See [18]

3. Fractal Splines in Two Variables

Using univariate complete cubic fractal spline results, we construct complete bicubic fractal splines in Subsection 3.1 through tensor product and the upper bounds of the L_∞-norm of its error with the original functions in Subsection 3.2.

3.1. Construction of Complete Bicubic Fractal Splines

Suppose that $\Delta_t : a = t_0 < t_1 < \ldots < t_N = b$ and $\Delta_s : c = s_0 < s_1 < \ldots < s_J = d$ orm a rectangular mesh $\pi : \Delta_t \times \Delta_s$ for a rectangular region $\Omega = [a,b] \times [c,d]$. Let $f(t,s)$ be a sufficiently smooth function in the domain Ω. Let $\rho_f(t,s)$ be the complete bicubic spline fractal interpolation surface associated with the function $f(t,s)$ and the mesh π. Then, $\rho_f(t,s)$ is a tensor product of univariate cubic fractal splines such that

$$\begin{aligned}
&\rho_f(t_n, s_j) = f(t_n, s_j);\ n = 0,1,\ldots,N;\ j = 0,1,\ldots,J,\\
&\rho_f^{(1,0)}(t_n, s_j) = f^{(1,0)}(t_n, s_j);\ n = 0,N;\ j = 0,1,\ldots,J,\\
&\rho_f^{(0,1)}(t_n, s_j) = f^{(0,1)}(t_n, s_j);\ n = 0,1,\ldots,N;\ j = 0,J,\\
&\rho_f^{(1,1)}(t_n, s_j) = f^{(1,1)}(t_n, s_j);\ n = 0,N,\ j = 0,J,
\end{aligned} \qquad (15)$$

where $\rho_f^{(\mu,\nu)} = \partial^{\mu+\nu}\rho_f / \partial t^\mu \partial s^\nu$. This definition is analogous to that of the classical complete bicubic spline in [19]. In the construction, we need two sets of nodal bases for univariate cubic fractal splines. Let $\{f_m(t)\}_{m=-1}^{N+1}$ be a nodal basis for the complete cubic fractal spline space $\mathcal{U}([a,b],\Delta_t)$ (cf. Section 2) and $\{\tilde{f}_i(s)\}_{i=-1}^{J+1}$ be a nodal basis for the complete cubic fractal spline space $\mathcal{V}([c,d],\Delta_s)$ with a choice of scaling parameters $\beta_j, j = 1,\ldots,J$ and $\tilde{L}_j : [c,d] \to [s_{j-1}, s_j]$, $\tilde{L}_j(s) = c_j s + d_j$, where $|c_j| < 1$ for $j = 1,2,\ldots,J$. Define generic transformations P and Q on $C^2[\Omega]$ respectively as

$$P\left(g\left(L_n(t), \tilde{L}_j(s)\right)\right) = \sum_{m=-1}^{N+1} g\left(t_m, \tilde{L}_j(s)\right) f_m\left(L_n(t)\right) \qquad (16)$$

$$Q\left(h\left(L_n(t), \tilde{L}_j(s)\right)\right) = \sum_{i=-1}^{J+1} h\left(L_n(t), s_i\right) \tilde{f}_i\left(\tilde{L}_j(s)\right) \qquad (17)$$

The $(N+3)(J+3)$-dimensional subspace $\mathcal{U}([a,b],\Delta_t) \otimes \mathcal{V}([c,d],\Delta_s)$ of $C^2[\Omega]$ defined by the bottom equation is called the fractal tensor product of the spaces $\mathcal{U}([a,b],\Delta_t)$ and $\mathcal{V}([c,d],\Delta_s)$ with the basis $\{f_m(t)\tilde{f}_i(s) \mid m = -1,0,1,\ldots,N+1; i = -1,0,1,\ldots,J+1\}$. Now, we define complete cubic spline fractal surface $\rho_f = (P \circ Q) f$ for $f \in C^2[\Omega]$ as

$$\mathcal{U}([a,b],\Delta_t) \otimes \mathcal{V}([c,d],\Delta_s) = \left\{\sum_{m=-1}^{N+1}\sum_{i=-1}^{J+1} y_{m,i} f_m\left(L_n(t)\right) \tilde{f}_i\left(\tilde{L}_j(s)\right) : y_{m,i} \in \mathbb{R}\right\}$$

$$\rho_f\left(L_n(t),\tilde{L}_j(s)\right)=\sum_{m=-1}^{N+1}\sum_{i=-1}^{J+1} f\left(t_m,s_i\right) f_m\left(L_n(t)\right)\tilde{f}_i\left(\tilde{L}_j(s)\right),(t,s)\in\Omega, \tag{18}$$

where $f(t_{-1},s)=\partial f(t_0,s)/\partial t, f(t,s_{-1})=\partial f(t,s_0)/\partial s$ with the analogues for $f(t_{N+1},s)$, $f(t,s_{J+1})$, $f(t_{-1},s_{-1})$ etc. We now show that the function ρ_f satisfies the interpolation conditions. According to (18), for $n\in\{1,2,\ldots,N\}$ and $j\in\{1,2,\ldots,J\}$,

$$\begin{aligned}\rho_f\left(t_n,s_j\right)&=\left(\left(P\circ Q\right)f\right)\left(L_n\left(t_N\right),\tilde{L}_j\left(s_J\right)\right)\\&=\sum_{m=-1}^{N+1}\sum_{i=-1}^{J+1} f\left(t_m,s_i\right) f_m\left(t_n\right)\tilde{f}_i\left(s_j\right)\\&=\sum_{m=-1}^{N+1}\sum_{i=-1}^{J+1} f\left(t_m,s_i\right)\delta_{m,n}\delta_{i,j}=f\left(t_n,s_j\right)\end{aligned}$$

Similarly, $\rho_f(t_0,s_j)=f(t_0,s_j),\ j=1,2,\ldots,J$; $\rho_f(t_n,s_0)=f(t_n,s_0),\ n=1,2,\ldots,N.$ and $\rho_f(t_0,s_0)=f(t_0,s_0)$. Since $f_m'(t_0)=f_m'(t_N)=0$ for $m=0,1,\ldots,N$, using (8), we have

$$\rho_f^{(1,0)}(t_n,s_j)=\sum_{m=-1}^{N+1}\sum_{i=-1}^{J+1} f(t_m,s_i)f_m'(t_n)\tilde{f}_i(s_j)=f^{(1,0)}(t_n,s_j);$$

$n=0,N; j=0,1,\ldots,J.$ Analogously, the rest of conditions of definition (15) are true. Since, $f_m=S_m$ if $\alpha_n=0\ \forall\ n=1,2,\ldots,N$ and $\tilde{f}_i=\tilde{S}_i$ if $\beta_j=0\ \forall\ j=1,2,\ldots,J$, we can retrieve classical complete bicubic spline S_f to the original function f from (18).

3.2. Upper Bounds of L_∞-Norm of Error

We will prove the L_∞-norm error of complete bicubic splines with the original function by using the following notations analogous to those of Proposition 2.2 for the *t*-variable.

Suppose $\tilde{q}_{i,j}$; $j=1,2,\ldots,J$ are cubic polynomials associated with the IFSs for $\tilde{f}_i$ such that

$$\left|\frac{\partial^{1+v}\tilde{q}_{i,j}(\tilde{\tau}_j,s)}{\partial\beta_j\partial s^v}\right|\le\tilde{\Theta}_{v,i}\quad\text{for}\quad i=-1,0,1,\ldots,J+1,$$

$|\tilde{\tau}_j|\in(0,\kappa^* c_j^v)$ with $|\beta_j|\le\kappa^*<1$ for fixed real κ^*. Let, $\tilde{\Lambda}_v=\max\{\|\tilde{S}_i^{(v)}\|_\infty : i=-1,0,1,\ldots,J+1\}$, $\tilde{\Theta}_v=\max\{\tilde{\Theta}_{v,i} : i=-1,0,1,\ldots,J+1\}$ and

$$\tilde{\Gamma}_{v,\beta,J}=\frac{|\beta|_\infty (J+3)(\tilde{\Lambda}_v+\tilde{\Theta}_v)}{|J|^{2-v}-h_s^{2-v}|\beta|_\infty}.$$

Suppose $C^4[\Omega]=\{g:\Omega\to\mathbb{R}: g^{(u,v)}\in C[\Omega], 0\le u+v\le 4\}$ and the norm corresponding to this space is $\|g\|_4=\max\{\|g^{(u,v)}\|_\infty; 0\le u+v\le 4\}$.

Theorem 3.1 *Let ρ_f be the complete bicubic fractal spline to the original function $f\in C^4[\Omega]$. Then for an arbitrary sequence of partitions,*

$$\begin{aligned}\left\|\left(f-\rho_f\right)^{(u,v)}\right\|_\infty &\le\|f\|_4\left\{\Gamma_{u,\alpha,N}h_t^{2-u}+\varepsilon_{4-v,u}h_t^{4-u-v}\right.\\&+\tilde{\Gamma}_{v,\beta,J}h_s^{2-v}+\varepsilon_{4-u,v}h_s^{4-u-v}\\&\left.+\left(\Gamma_{u,\alpha,N}h_t^{2-u}+\varepsilon_{4-v,u}h_t^{4-u-v}\right)\left(\tilde{\Gamma}_{v,\beta,J}h_s^{2-v}+\varepsilon_{4-u,v}h_s^{4-u-v}\right)\right\},\\&0\le u,v\le 2.\end{aligned}$$

Proof. In order to calculate the error, we will use the generic transformations P and Q. Suppose that

$$f-\rho_f=\left(f-Q(f)\right)-\left\{f-Q(f)-\left(P(f)-\rho_f\right)\right\}+\left(f-P(f)\right) \tag{19}$$

Consider

$$\left(f-P(f)\right)\left(L_n(t),\tilde{L}_j(s)\right)=f\left(L_n(t),\tilde{L}_j(s)\right)-\sum_{m=-1}^{N+1} f\left(t_m,\tilde{L}_j(s)\right)f_m\left(L_n(t)\right)$$

For $s=s^*$ fixed, $(P(f))^{(0,v)}(L_n(t),\tilde{L}_j(s^*))$ is the spline of $f^{(0,v)}(L_n(t),\tilde{L}_j(s^*))$ (with respect to the first variable) and we can apply (14) for $p=4-v$ since $f^{(0,v)}(.,\tilde{L}_j(s^*))\in C^{(4-v)}[a,b]$. For $0\le u\le 2$,

$$\begin{aligned}&\left\|f^{(u,v)}\left(.,\tilde{L}_j\left(s^*\right)\right)-P\left(f\right)^{(u,v)}\left(.,\tilde{L}_j\left(s^*\right)\right)\right\|_\infty\\&\le\left\|f^{(0,v)}\left(.,\tilde{L}_j\left(s^*\right)\right)\right\|_{4-v}\left(\Gamma_{u,\alpha,N}h_t^{2-u}+\varepsilon_{4-v,u}h_t^{4-v-u}\right)\\&\le\|f\|_4\left(\Gamma_{u,\alpha,N}h_t^{2-u}+\varepsilon_{4-v,u}h_t^{4-v-u}\right)\end{aligned} \tag{20}$$

Since the last term of (20) does not depend on s^*,

$$\left\|f^{(u,v)}-P\left(f\right)^{(u,v)}\right\|_\infty\le\|f\|_4\left(\Gamma_{u,\alpha,N}h_t^{2-u}+\varepsilon_{4-v,u}h_t^{4-v-u}\right) \tag{21}$$

In the same way,

$$\left\|f^{(u,v)}-Q(f)^{(u,v)}\right\|_\infty\le\|f\|_4\left(\tilde{\Gamma}_{v,\beta,J}h_s^{2-v}+\varepsilon_{4-u,v}h_s^{4-u-v}\right) \tag{22}$$

Consider $P(f)-\rho_f$ and using their definitions,

$$\begin{aligned}&\left(P(f)-\rho_f\right)\left(L_n(t),\tilde{L}_j(s)\right)\\&=\sum_{m=-1}^{N+1}\left\{f\left(t_m,\tilde{L}_j(s)\right)-\sum_{i=-1}^{J+1} f\left(t_m,s_i\right)\tilde{f}_i\left(\tilde{L}_j(s)\right)\right\}f_m\left(L_n(t)\right)\\&=\sum_{m=-1}^{N+1} R_1\left(t_m,\tilde{L}_j(s)\right)f_m\left(L_n(t)\right)=P\left(R_1\right)\left(L_n(t),\tilde{L}_j(s)\right)\end{aligned}$$

where $R_1=f-Q(f)$. Hence, we have $f-Q(f)-(P(f)-\rho_f)=R_1-P(R_1)$. For $s=s^*$ fixed, $P(R_1)^{(0,v)}$ is the cubic spline FIF of $R_1^{(0,v)}$ with respect to the first variable and we can apply (14) taking $p=4-v$.

$$\left\| R_1^{(u,v)}\left(.,\tilde{L}_j\left(s^*\right)\right) - P(R_1)^{(u,v)}\left(.,\tilde{L}_j(s^*)\right) \right\|_\infty \le \left\| R_1^{(0,v)}\left(.,\tilde{L}_j\left(s^*\right)\right) \right\|_{4-v} \left(\Gamma_{u,\alpha,N} h_t^{2-u} + \varepsilon_{4-v,u} h_t^{4-v-u}\right) \quad (23)$$

For $0 \le v \le 2$,

$$R_1^{(0,v)}\left(.,\tilde{L}_j\left(s^*\right)\right) = f^{(0,v)}\left(.,\tilde{L}_j\left(s^*\right)\right) - Q\left(f\right)^{(0,v)}\left(.,\tilde{L}_j\left(s^*\right)\right)$$

and similarly to (22) for $0 \le u \le 2$,

$$\left\| R_1^{(u,v)}\left(.,\tilde{L}_j\left(s^*\right)\right) \right\|_\infty \le \|f\|_4 \left(\tilde{\Gamma}_{v,\beta,J} h_s^{2-v} + \varepsilon_{4-u,v} h_s^{4-u-v}\right)$$

and by (23), we get the first equation of the bottom ones.

The inequality of Theorem 3.1 follows from (21)-(24).

Remark. From Theorem 3.1, it can be observed that the convergence of the bicubic fractal spline ρ_f is slower than that of the case of the classical bicubic spline S_f (see [20]). Since the classical bicubic spline is a particular case of bicubic fractal spline, Theorem 3.1 generalizes the classical result. If there exist positive reals k and l such that $\Gamma_{u,\alpha,N} < \frac{1}{(N+3)^k}$ and $\Gamma_{v,\beta,J} < \frac{1}{(J+3)^l}$, then the complete bicubic fractal spline ρ_f converges to the original function f in C^2-norm for uniform partitions.

4. Examples

First, we construct three different bases for complete cubic fractal spline space with $I = [0,3]$, $N = 3$ and three different sets of scaling vectors. These scaling vectors play important role over classical splines in overall shape of fractal approximants. The scaling vectors are chosen for a basis of complete fractal splines as 1) Set I: $\alpha_n = 0.9, i = 1,2,3$; 2) Set II: $\alpha_n = -0.9, i = 1,2,3$; 3) Set III: $\alpha_1 = -0.5, \alpha_2 = 0.9, \alpha_3 = 0.7$. In our examples, $L_1(t) = \frac{1}{3}t, L_2(t) = \frac{1}{3}t + \frac{1}{3}$, and $L_3(t) = \frac{1}{3}t + \frac{2}{3}$. We compute the moments $M_{m,n}$, $n = 1,2,3$; $m = -1,0,\ldots,4$ from Equations (26)-(28). These values of moments for three sets of bases are given in **Table 2**. These moments are then used in IFS $\{\mathbb{R}^2; w_n(t,x) = (L_n(t), F_n(x,t)), n = 1,2,3\}$ to compute $F_n(x,t)$. From (5) and (25), we have the bottom equation. where x_n depends on the cardinal condition of the basis elements f_m, *i.e.*, $x_n = \delta_{m,n}$, $n = 0,1,2,3; m = -1,0,\ldots,4$.

Using the above IFS, we compute basis elements for complete cubic fractal spline space with non-zero and zero scaling vectors. When scaling factors are same in Set I and Set II, the values of moments of f_{-1} and f_4; f_0 and f_3; f_1 and f_2 follow a particular pattern (see **Table 2**). This pattern is very close to the moments pattern of classical complete cardinal splines (with zero scale vector). That's why pair-wise similarity between complete cardinal fractal splines f_{-1} and f_4; f_0 and f_3; f_1 and f_2 in such cases (see **Figure 1(a)** and **Figure 1(b)**) observed as in classical complete cardinal splines (see **Figure 1(d)**). But for unequal scaling factors in Set III, there is no pattern between moments of complete cardinal fractal splines and hence, their shapes are completely different (see **Figure 1(c)**). The unequal scaling factors provides an additional advantage of complete cardinal fractal splines over their classical counterparts in smooth object modelling in engineering applications like computer graphics, CAD/CAM.

The non-zero scale vectors gives irregular shape to fractal splines because f_n'' are typical fractal functions, *i.e.*, fractal dimension of graph of f_n'' is non-integer. These cardinal fractal splines differ from their classical interpolants in the sense that they obey a functional relation related to self-similarity on smaller scales. Hence, cardinal fractal splines are defined globally on the entire domain. Moreover, classical complete splines are defined piecewisely between consecutive nodes and hence, their shapes can be defined locally. Importantly, if $\alpha_n = 0$, $n = 1,2,3$, then we can retrieve the basis elements (see **Figure 1(d)**) for the classical complete cubic spline space. Using these three sets of vertical scaling vectors, we have constructed complete bicubic splines in the following.

Some complete bicubic fractal splines are constructed using the tensor product of univariate fractal splines for the interpolation data given in **Table 3**. In all our examples, we assume the same boundary conditions for complete

$$\left\| R_1^{(u,v)} - P\left(R_1\right)^{(u,v)} \right\|_\infty \le \|f\|_4 \left(\tilde{\Gamma}_{v,\beta,J} h_s^{2-v} + \varepsilon_{4-u,v} h_s^{4-u-v}\right).\left(\Gamma_{u,\alpha,N} h_t^{2-u} + \varepsilon_{4-v,u} h_t^{4-v-u}\right). \quad (24)$$

$$F_n\left(t,x\right) = \frac{1}{9}\left\{ \alpha_n x + \frac{\left(M_{m,n} - \alpha_n M_{m,3}\right)t^3}{18} + \frac{\left(M_{m,n-1} - \alpha_n M_{m,0}\right)\left(3-t\right)^3}{18} - \frac{\left(M_{m,n} - \alpha_n M_{m,3}\right)t}{2} - \frac{\left(M_{m,n-1} - \alpha_n M_{m,0}\right)\left(3-t\right)}{2} + \left(9x_{n-1} - \alpha_n x_0\right)\frac{3-t}{3} + \left(9x_n - \alpha_n x_3\right)\frac{t}{3} \right\}, n = 1,2,3,$$

Table 2. Moments for cardinal complete cubic fractal splines.

Basis Elements	Moments	Set I	Set II	Set III	Classical Case
f_{-1}	$M_{-1,0}$	−16.4058	−2.5546	−2.4586	−3.4667
	$M_{-1,1}$	−10.7536	1.4714	0.6318	0.9333
	$M_{-1,2}$	−12.5797	0.4584	−1.4137	−0.2667
	$M_{-1,3}$	−10.9275	0.4844	−0.5122	0.1333
f_0	$M_{0,0}$	−24.4348	−2.8448	−2.7653	−4.4000
	$M_{0,1}$	−14.5217	3.5448	2.0938	2.8000
	$M_{0,2}$	−19.4783	0.3500	−2.2451	−0.8000
	$M_{0,3}$	−15.5652	0.7396	−0.2621	0.4000
f_1	$M_{1,0}$	27.8261	3.3903	3.5365	5.6000
	$M_{1,1}$	11.3913	−5.7266	−3.6048	−5.2000
	$M_{1,2}$	22.6087	1.8319	3.6812	3.2000
	$M_{1,3}$	12.1739	−1.2850	−2.0094	−1.6000
f_2	$M_{2,0}$	12.1739	−1.2850	−2.0094	−1.6000
	$M_{2,1}$	22.6087	1.8319	1.3664	3.2000
	$M_{2,2}$	11.3913	−5.7266	−1.0007	−5.2000
	$M_{2,3}$	27.8261	3.3903	9.3235	5.6000
f_3	$M_{3,0}$	−15.5652	0.7396	1.2382	0.4000
	$M_{3,1}$	−19.4783	0.3500	0.1447	−0.8000
	$M_{3,2}$	−14.5217	3.5448	−0.4354	2.8000
	$M_{3,3}$	−24.4348	−2.8448	−7.0520	−4.4000
f_4	$M_{4,0}$	10.9275	−0.4844	−0.7738	−0.1333
	$M_{4,1}$	12.5797	−0.4584	−0.1938	0.2667
	$M_{4,2}$	10.7536	−1.4714	1.0400	−0.9333
	$M_{4,3}$	16.4058	2.5546	5.0306	3.4667

Table 3. Interpolation data for complete bicubic splines.

$f(t_n, s_j)$	$s_0 = 1$	$s_1 = 2$	$s_2 = 3$	$s_3 = 3$
$t_0 = 1$	−1	11	−5	10
$t_1 = 2$	2	14	3	15
$t_2 = 3$	0	9	1	17
$t_3 = 4$	1	10	3	13

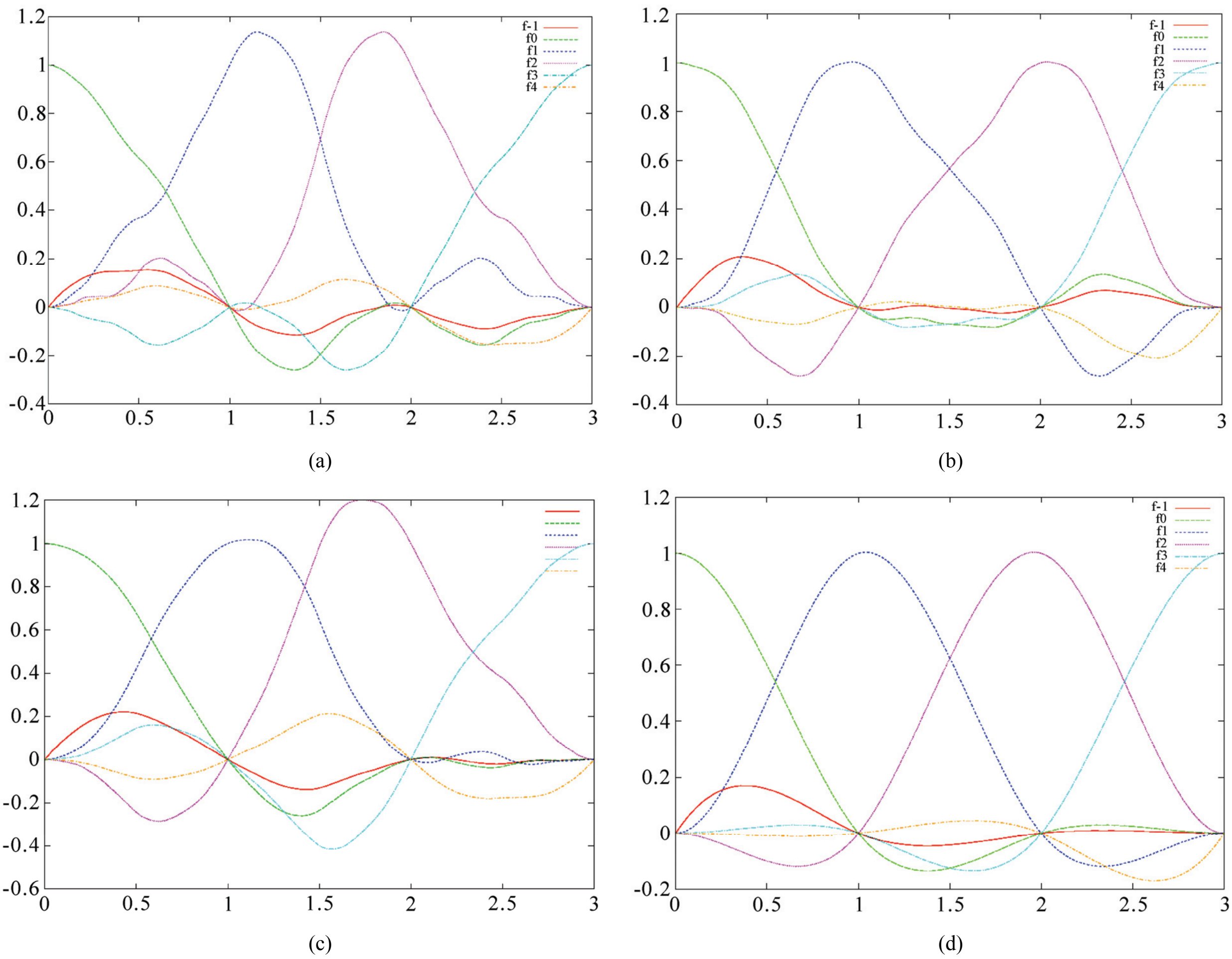

Figure 1. Bases for complete cubic fractal spline space. (a) Cardinal cubic fractal splines with Set I; (b) Cardinal cubic fractal splines with Set II; (c) Cardinal cubic fractal splines with Set III; (d) Classical cardinal cubic splines with $\alpha_n = 0, n = 1,2,3$.

bicubic splines: $f^{(1,0)}(t_n, s_j) = 5; n = 0,3, j = 0,1,2,3$, $f^{(0,1)}(t_n, s_j) = 3; n = 0,1,2,3, j = 0,3,$ and $f^{(1,1)}(t_n, s_j) = 2;$ $j = 0,3$. The scaling vectors are same in both directions in our first three examples, *i.e.*, $\alpha_n = \beta_j$ for $n = j$ and we take these scale vectors as Set I, Set II and Set III defined in above for univariate case. The cardinal splines $f_m = \tilde{f}_i$ if $i = m$ in these cases for $i, m = -1, 0, \ldots, 4$. Based on (18), the points of complete bicubic fractal splines are generated and plotted in **Figsures 2-4**. The effect of change in scaling factors from 0.9 to −0.9 on complete bicubic spline can be seen from **Figsures 2-3**. The difference in the shape of complete bicubic spline for an unequal scaling factors can be observed by comparing **Figure 4** with **Figsures 2-3**.

Next, we take scaling vectors in *t*-direction as Set I and in *s*-direction as Set III and the corresponding complete bicubic spline generated in **Figure 5**. It has similarity with both **Figure 2** and **Figure 4** due to self-similarity relation in *t* and *s* directions respectively. For **Figure 6**, we chose scaling vectors in *t*-direction as Set III and in *s*-direction as Set II. The distinct deviation in *s*-direction of complete bicubic spline is present in this case as in **Figure 3**. Finally, we chose $\alpha_n = \beta_j = 0 \quad \forall n, j$. Since $f_m = S_m$ and $\tilde{f}_i = \tilde{S}_i$ in this case, we retrieve the classical complete bicubic spline S_f in **Figure 7**. An infinite number of complete bicubic splines can be constructed interpolation the same data by choosing different sets of scaling vectors. Hence, the presence of scaling vectors in bicubic fractal splines provides an additional advantage over classical bicubic splines in smooth surface modelling. Since bicubic fractal splines are invariant in all scales, it can also be applied to bivariate image compression and zooming problems in image processing.

5. Conclusions

We introduced bases for complete cubic fractal splines through cardinal fractal splines in the present work. These cardinal fractal splines constructed through moments

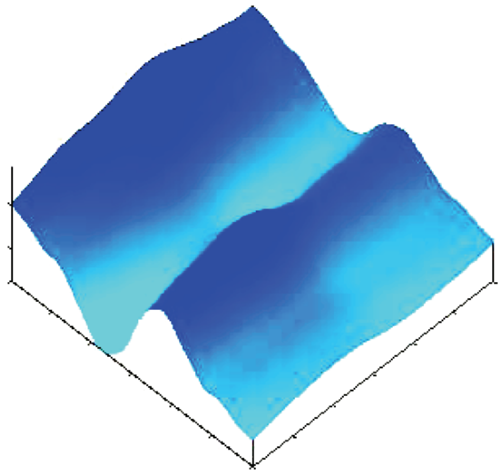

Figure 2. Complete bicubic fractal spline with scale vectors Set I in both directions.

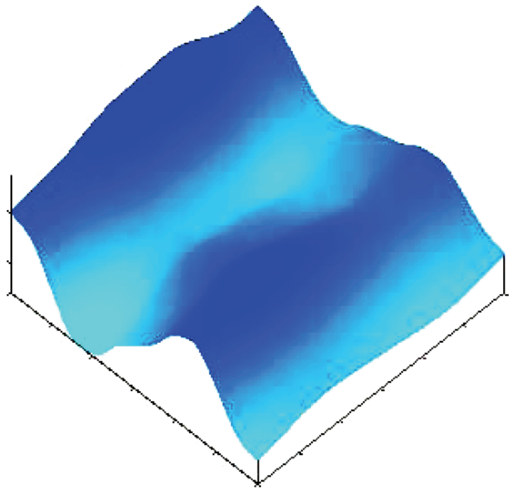

Figure 3. Complete bicubic fractal spline with scale vectors Set II in both directions.

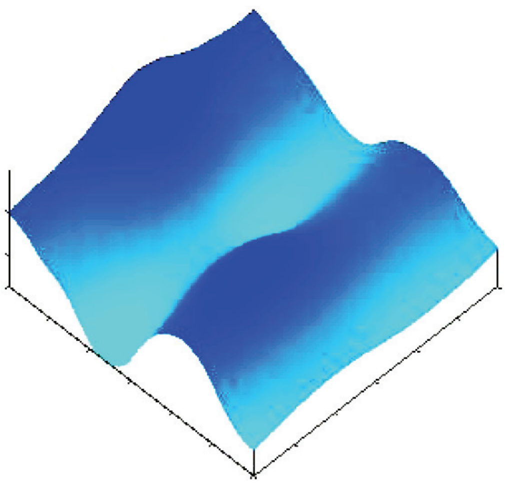

Figure 4. Complete bicubic fractal spline with scale vectors Set III in both directions.

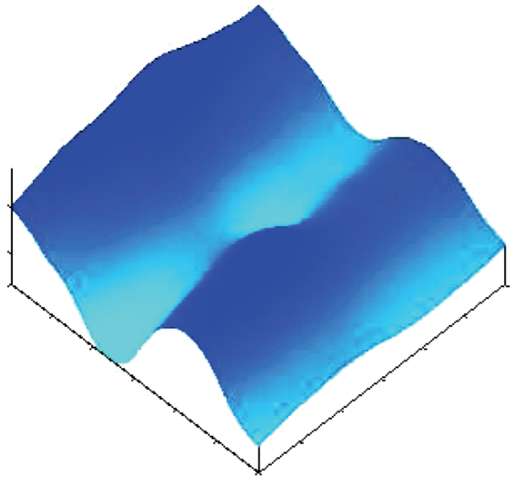

Figure 5. Complete bicubic fractal spline with scale vectors Set I, Set III.

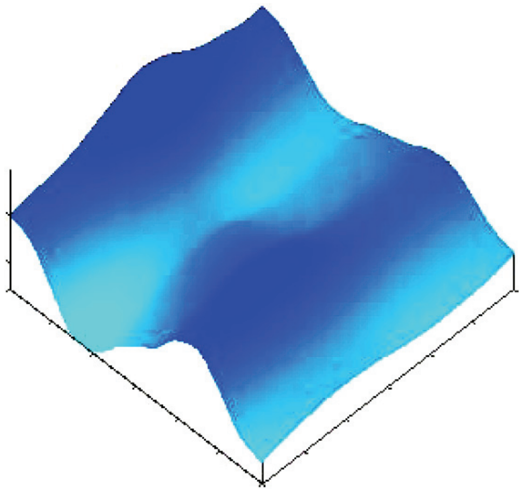

Figure 6. Complete bicubic fractal spline with scale vectors Set III, Set II.

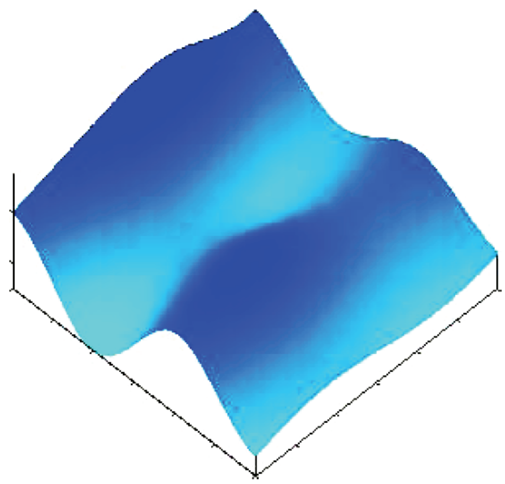

Figure 7. Classical complete bicubic spline with zero scale vector.

as in the classical case. Using tensor product of cardinal cubic fractal splines, bicubic fractal splines introduced over rectangular domains with rectangular partition. These bicubic fractal splines are invariant in all scales due to underlying fixed point equation. L_∞-norm of the error of complete cubic fractal spline with respect to the original function $f \in C^p[\Omega]$, $p = 2,3$ or 4 has been deduced. The presence of scaling factors can be exploited in bivariate optimization problems with prescribed interpolation conditions. The effect of equal and non-equal scaling factors in complete bicubic splines is explained. The present work may play important role in smooth surface modelling in computer graphics and image processing applications.

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Appendix

In this section, the defining equations for the construction of a cubic spline FIF f_m in Subsection 2.2 are given. Using Property 3) of Definition 2.1 and (5), the polynomials associated with f_m'' is affine. So,

$$f_m''\left(L_n\left(t\right)\right)=\alpha_n f_m''\left(t\right)+\frac{c_n\left(t-t_0\right)}{t_N-t_0}+d_n, \quad (25)$$
$$n=1,2,\ldots,N.$$

By (1) and (25), $c_n=M_n-M_{n-1}-\alpha_n(M_N-M_0)$ and $d_n=M_{n-1}-\alpha_n M_0$. Substituting c_n and d_n in (25) and integrating it twice, we will have two constants of integration. Solving these constants by (1), the cubic fractal spline in terms of moments can be written as

$$\begin{aligned} f_m\left(L_n\left(t\right)\right)=a_n^2\Bigg\{&\alpha_n f_m\left(t\right)+\frac{\left(M_{m,n}-\alpha_n M_{m,N}\right)\left(t-t_0\right)^3}{6\left(t_N-t_0\right)}\\ &+\frac{\left(M_{m,n-1}-\alpha_n M_{m,0}\right)\left(t_N-t\right)^3}{6\left(t_N-t_0\right)}\\ &-\frac{\left(M_{m,n-1}-\alpha_n M_{m,0}\right)\left(t_N-t_0\right)\left(t_N-t\right)}{6}\\ &-\frac{\left(M_{m,n}-\alpha_n M_{m,N}\right)\left(t_N-t_0\right)\left(t-t_0\right)}{6}\\ &+\left(\frac{x_{n-1}}{a_n^2}-\alpha_n x_0\right)\frac{t_N-t}{t_N-t_0}+\left(\frac{x_n}{a_n^2}-\alpha_n x_N\right)\frac{t-t_0}{t_N-t_0}\Bigg\},\\ &n=1,2,\ldots,N. \end{aligned} \quad (26)$$

Set, $h_n=t_n-t_{n-1}; n=1,2,\ldots,N.$ Now, use the condition that $f_{m'}(t)$ is continuous at the knots $t_1,t_2,\ldots,t_{N-1}$ to give the following result.

$$\begin{aligned} &-a_{n+1}\alpha_{n+1}f_m'\left(t_0\right)-\frac{\alpha_n h_n+2\alpha_{n+1}h_{n+1}}{6}M_{m,0}\\ &+\frac{h_n}{6}M_{m,n-1}+\frac{h_n+h_{n+1}}{3}M_{m,n}+\frac{h_{n+1}}{6}M_{m,n+1}\\ &-\frac{2\alpha_n h_n+\alpha_{n+1}h_{n+1}}{6}M_{m,N}+a_n\alpha_n f_m'\left(t_N\right)\\ &=\frac{x_{n+1}-x_n}{h_{n+1}}-\frac{x_n-x_{n-1}}{h_n}-\left(a_{n+1}\alpha_{n+1}-a_n\alpha_n\right)\frac{x_N-x_0}{t_N-t_0},\\ &n=1,2,\ldots,N-1.; \end{aligned} \quad (27)$$

At the initial point t_0 of the interval I, we have the following relation for $f_m'(t_0)$.

$$\begin{aligned} &6\left(1-a_1\alpha_1\right)f_m'\left(t_0\right)+2\left(1-\alpha_1\right)h_1M_{m,0}+h_1M_{m,1}\\ &-\alpha_1h_1M_{m,N}=\frac{6}{h_1}\left[x_1-x_0-\alpha_1a_1^2\left(x_N-x_0\right)\right] \end{aligned} \quad (28)$$

Similarly, at the final point t_N of the interval I for $f_m'(t_N)$, we have

$$\begin{aligned} &-\alpha_N h_N M_{m,0}+h_N M_{m,N-1}+2\left(1-\alpha_N\right)h_N M_{m,N}\\ &-6\left(1-a_N\alpha_N\right)f_m'\left(t_N\right)=\frac{-6}{h_N}\left[x_N-x_{N-1}-\alpha_N a_N^2\left(x_N-x_0\right)\right] \end{aligned} \quad (29)$$

The moments $M_{m,n}; n=0,1,\ldots,N, f_m'(t_0)$ and $f_m'(t_N)$ are evaluated from the system of Equations (27)-(29). The existence of these parameters is guaranteed by the uniqueness of the attractor from the fixed point theorem.