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Scale Invariant Theory of Gravitation in Einstein-Rosen Space-Time

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Abstract

In this paper, we have studied the perfect fluid distribution in the scale invariant theory of gravitation, when the space-time described by Einstein-Rosen metric with a time dependent gauge function. The cosmological equations for this space-time with gauge function are solved and some physical properties of the model are studied.

1. Introduction

Several new theories of gravitation have been formulated which are considered to be alternative to Einstein's theory of gravitation. In alternative theories of gravitation, scalar tensor theories proposed by Brans and Dicke [1], Nordvedt [2], Wagoner [3], Ross [4], Dunn [5] and Saez and Ballester [6] are most important among them. In the theory proposed by Brans and Dicke [1] there exists a variable gravitational parameter G. Another theory, which admits a variable G, is the scale covariant theory of Canuto *et al*. [7]. Dirac [8,9] rebuilt the Weyl's unified theory by introducing the notion of two metrics and an additional gauge function β. A scale invariant variation principle was proposed from which gravitational and electromagnetic field equations can be derived. It is concluded that an arbitrary gauge function is necessary in all scale-invariant theories.

It is found that the scale invariant theory of gravitation agrees with general relativity up to the accuracy of observations made up now. Dirac [8,9], Hoyle and Narlikar [10] and Canuto *et al*. [7] have studied several aspects of the scale invariant theory of gravitation. But Wesson's [11,12] formulation is so far the best one to describe all the interactions between matter and gravitation in scale free manner.

Mohanty and Mishra [13] have studied the feasibility of Bianchi type VIII and IX space-times with a time dependent gauge function and a matter field in the form of perfect fluid. In that paper, they have constructed a radiating model of the universe for the feasible Bianchi type VIII space-time. Mishra [14] has constructed the non- static plane symmetric Zeldovich fluid model in this

theory with a time dependent gauge function. Recently, Mishra [15] has constructed static plane symmetric Zeldovich fluid model in scale invariant theory.

Rao *et al*. [16,17] have discussed cylindrically symmetric scalar meson fields and Brans-Dicke scalar fields. It is found from the literature that the scale invariant theory of gravitation has not been studied so far in Einstein-Rosen space-time. Hence, in this paper, we have taken an attempt to study the cylindrically symmetric space-time in the scale invariant theory of gravitation. A cosmological model has been presented.

2. Field Equations

Wesson [11,12] formulated a scale invariant theory of gravitation using a gauge function $\beta\left(x^i\right)$, where, $x^i, i = 1, 2, 3, 4$ are coordinates in the four-dimensional space-time and the tensor field is identified with the metric tensor g_{ij}. This theory is both coordinate and scale invariant in nature. The field equations formulated by Wesson [11,12] for the combined scalar and tensor fields are:

$$G_{ij} + 2\frac{\beta_{;ij}}{\beta} - 4\frac{\beta_{,i}\beta_{,j}}{\beta^2} + \left(g^{ab}\frac{\beta_{,a}\beta_{,b}}{\beta^2} - 2g^{ab}\frac{\beta_{;ab}}{\beta} \right)g_{ij} = -\kappa T_{ij}$$

(1)

with

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij}$$

(2)

Here, G_{ij} is the conventional Einstein tensor involving

g_{ij}. Semicolon and comma respectively denote covariant differentiation with respect to g_{ij} and partial differentiation with respect to coordinates. The cosmological term Λg_{ij} of Einstein theory is transformed to $\Lambda_0 \beta^2 g_{ij}$ in scale invariant theory with a dimensionless constant Λ_0. T_{ij} is the energy momentum tensor of the matter field and $\kappa = \dfrac{8\pi G}{c^4}$.

The line element for Einstein- Rosen metric with a gauge function $\beta = \beta(ct)$ is.

$$ds_W^2 = \beta^2 ds_E^2 \tag{3}$$

with

$$ds_E^2 = e^{2A-2B}\left(c^2 dt^2 - dr^2\right) - r^2 e^{-2B} d\varphi^2 - e^{2B} dz^2 \tag{4}$$

where A and B are functions of t only, and c is the velocity of light. Here we intend to build cosmological models in this space-time with a perfect fluid having the energy momentum tensor of the form

$$T_{ij}^m = \left(p_m + \rho_m c^2\right) U_i U_j - p_m g_{ij} \tag{5}$$

together with $g_{ij} U^i U^j = 1$

where U^i is the four-velocity vector of the fluid; p_m and p_m are the proper isotropic pressure and energy density of the matter respectively.

The non – vanishing components of conventional Einstein's tensor (2) for the metric (4) can be obtained as

$$G_{11} = \frac{1}{c^2}\left[B_4^2 \right] \tag{6}$$

$$G_{14} = -\frac{1}{r}\left[A_4 \right] \tag{7}$$

$$G_{22} = \frac{1}{c^2}\left[A_{44} + B_4^2 \right] \tag{8}$$

$$G_{33} = \frac{1}{c^2}\left[A_{44} - 2B_{44} + B_4^2 \right] \tag{9}$$

$$G_{44} = \left[B_4^2 \right] \tag{10}$$

Here afterwards the suffix 4 after a field variable denotes exact differentiation with respect to time t only.

Using the comoving coordinate frame where $U^i = \delta_4^i$, the non-vanishing components of the field Equation (1) for the metric (3) can be written in the following explicit form:

$G_{11} =$

$$-\kappa p_m e^{2A-2B} - \frac{1}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(-A_4 + B_4\right) - \Lambda_0 \beta^2 c^2 e^{2A-2B} \right] \tag{11}$$

$$G_{14} = 0 \tag{12}$$

i.e. $A = k_1$, where k_1 is an integrating constant.

$G_{22} =$

$$-\kappa p_m e^{2A-2B} - \frac{1}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2A-2B} \right] \tag{13}$$

$G_{33} =$

$$-\kappa p_m e^{2A-2B} - \frac{1}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2A-2B} \right] \tag{14}$$

$G_{44} =$

$$-\kappa \rho_m c^4 e^{2A-2B} + \left[3\frac{\beta_4^2}{\beta^2} - 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2A-2B} \right] \tag{15}$$

Equation (12) reduces the above set of Equations (11)-(15) as

$G_{11} = G_{22} =$

$$-\kappa p_m e^{2k_1-2B} - \frac{1}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2k_1-2B} \right] \tag{16}$$

$$G_{33} = -\kappa p_m e^{2k_1-2B} - $$
$$\frac{1}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2k_1-2B} \right] \tag{17}$$

$$G_{44} = -\kappa \rho_m c^4 e^{2k_1-2B} + $$
$$\left[3\frac{\beta_4^2}{\beta^2} - 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2k_1-2B} \right] \tag{18}$$

Now, Equation (1) and Equations (16)-(18) (Wesson [12]) suggest the definitions of quantities p_v (vacuum pressure) and p_v (vacuum density) that involves neither the Einstein tensor of conventional theory nor the properties of conventional matter. These two quantities can be obtained as:

$$2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2k_1-2B} = \kappa p_v c^2 \tag{19}$$

$$2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} - 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2k_1-2B} = \kappa p_v c^2 \tag{20}$$

$$3\frac{\beta_4^2}{\beta^2} - 2\frac{\beta_4}{\beta}\left(B_4\right) - \Lambda_0 \beta^2 c^2 e^{2k_1-2B} = \kappa \rho_v c^4 \tag{21}$$

It is evident from Equations (19) and (20) that

$$B_4 = 0 \Rightarrow B = k_2 \quad \text{since} \quad \frac{\beta_4}{\beta} \neq 0 \tag{22}$$

where k_2 is an integrating constant. Using Equation (22) in Equations (19)-(21), the pressure and energy density for vacuum case can be obtained as

$$p_v = \frac{1}{\kappa c^2 e^{2k_1 - 2k_2}} \left[2\frac{\beta_{44}}{\beta} - \frac{\beta_4^2}{\beta^2} + \Lambda_0 \beta^2 c^2 e^{2k_1 - 2k_2} \right] \quad (23)$$

$$\rho_v = \frac{1}{\kappa c^4 e^{2k_1 - 2k_2}} \left[3\frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2 e^{2k_1 - 2k_2} \right] \quad (24)$$

Here p_v and p_v relate to the properties of vacuum only in conventional physics. The definition of above quantities is natural as regards to the scale invariant properties of the vacuum. The total pressure and energy density can be defined as

$$p_t \equiv p_m + p_v \quad (25)$$

$$\rho_t \equiv \rho_m + \rho_v \quad (26)$$

Using the aforesaid definitions of p_t and p_t, the field equations in scale invariant theory *i.e.* (16)-(18) can now be written by using the components of Einstein tensor (6)-(10) and the results obtained in Equations (22)-(24) as:

$$B_4^2 = -\kappa p_t c^2 e^{2k_1 - 2B} \quad (27)$$

$$-2B_{44} + B_4^2 = -\kappa p_t c^2 e^{2k_1 - 2B} \quad (28)$$

$$B_4^2 = -\kappa \rho_t c^4 e^{2k_1 - 2B} \quad (29)$$

3. Solution

From Equations (27) and (29), we obtained the equation of state

$$p_t = \rho_t c^2 \quad (30)$$

Using Equation (27) in Equation (28), we obtained

$$B = d_1 t + d_2 \quad (31)$$

where d_1 and d_2 are integrating constants.

Substituting Equation (31) in Equation (27) and Equation (29) respectively, the total pressure 'p_t' and energy density 'ρ_t' can be obtained as:

$$p_t = \rho_t c^2 = \frac{1}{\kappa c^2} \left[-\frac{d_1^2}{qe^{-2d_1 t}} \right] \quad (32)$$

where $q = e^{2k_1 - 2k_2}$ is a constant. The reality condition demands that $d_1^2 < 0$.

Using Equation (31) in Equations (23) and (24) respectively and taking $\beta = \frac{1}{ct}$, the pressure and energy density corresponding to vacuum case can be calculated as:

$$p_v = -\frac{1}{\kappa c^2 q} \left[\frac{\Lambda_0 q - 1}{t^2} \right] \quad (33)$$

$$\rho_v = \frac{1}{\kappa c^4 q} \left[\frac{\Lambda_0 q - 3}{t^2} \right] \quad (34)$$

In this case, when there is no matter and the gauge function β is a constant, one recovers the relation $c^2 \rho_v = c^4 \frac{\Lambda_{GR}}{8\pi G} = -p_v$ *i.e.* $c^2 \rho_v + p_v = 0$, which is the equation of state for vacuum. Here $\Lambda_{GR} = \Lambda_0 \beta^2$ = constant, is the cosmological constant in general relativity. Also p_v being dependent on the constants Λ_{GR}, c and G, is uniform in all directions and hence isotropic in nature. The cosmological model with this equation of state is rare in literature and is known as ρ – vacuum or false vacuum or degenerate vacuum model [18-21], the corresponding model in the static case is a well known de-Sitter model.

Now the matter pressure and density can be obtained as:

$$p_m = p_t - p_v = \frac{1}{\kappa c^2} \left[\frac{\Lambda_0 q - 1}{qt^2} - \frac{d_1^2}{e^{2k_1 - 2d_1 t - d_2}} \right] \quad (35)$$

$$\rho_m = \rho_t - \rho_v = \frac{1}{\kappa c^4} \left[\frac{3 - \Lambda_0 q}{qt^2} - \frac{d_1^2}{e^{2k_1 - 2d_1 t - d_2}} \right] \quad (36)$$

Now, we have $\rho_m \to \infty$ as $t \to 0$ and $\rho_m \to \infty$ as $t \to \infty$. Also when $t < 0$, $\rho_m \to$ constant. It is interesting to note that the model free from singularity.

So, the Einstein-Rosen cylindrically symmetric model in scale invariant theory of gravitation is given by the Equations (12), (31) and (32) and the metric in this case is

$$dS_W^2 =$$
$$\frac{1}{(ct)^2} \left[e^{2k_1 - 2d_1 t - 2d_2} \left(c^2 dt^2 - dr^2 \right) - r^2 e^{-2d_1 t - 2d_2} d\varphi^2 - e^{2d_1 t + 2d_2} dz^2 \right]$$
$$(37)$$

4. Some Physical Properties of the Model

The scalar expansion, $\theta = U_{;i}^i = 3\frac{Q_T}{Q}$ for the model given by Equation (37) takes the form

$$\theta = -\frac{1}{c} \left(d_1 e^{d_1 t + d_2 - k_1} \right) \quad (38)$$

Thus, we find $\theta \to -\frac{1}{c}(d_1 e^{d_2 - k_1})$ as $t \to 0$ and $\theta \to 0$ as $t \to \infty$.

If $c > 0$, $d_1 < 0$ and $d_2 > k_1$ the model represents expanding one for $t < t_1 (= \frac{k_1 - d_2}{d_1})$.

It is also observed that as $\frac{\rho_m}{\theta^2} \to$ constant as $t \to \infty$ and $\frac{\rho_m}{\theta^2} \to \theta$ as $t \to 0$. Thus the universe confirms the homogeneity nature of the space-time.

Following Raychaudhuri [22], the anisotropy $|\sigma|$ can be defined as

$$\sigma^2 = \left[\left(\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}} \right)^2 + \left(\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}} \right)^2 + \left(\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}} \right)^2 \right] \tag{39}$$

Consequently for the model (37), $\sigma^2 = \frac{8}{3}(d_1 t + d_2) \neq 0$.

So the shear scalar remains constant for $t \to 0$ and becomes indefinitely large for $t \to \infty$.

The ratio of anisotropy to expansion $\frac{\sigma^2}{\theta^2} = \frac{8c^2}{3}e^{2k_1 - 2d_2} \neq 0$ for $t = 0$. Thus there is a singularity of $t = 0$ for $2k_1 - 2d_2$ is not very large. Moreover, the model is isotropy for finite t and does not approach isotropy for large value of t.

It is observed that the vorticity 'w' vanishes which indicates that u^i is hypersurface orthogonal. As the acceleration $\dot{u}_i$ found to be zero, the matter particle follows geodesic path in this theory.

5. Conclusions

Every physical theory carries its own mathematical structure and the validity of the theory is usually studied through the exact solution of the mathematical structure. In this theory black holes do not appear to exist. If the existence of black holes in nature is confirmed, it will represent a great success of general theory of relativity. Since there is no concrete evidence at present for the existence of black holes, one can take a stand point that black holes represents a familiar concept of space time. Therefore the scale invariant theory involves gauge theories as it relates to gravitational theories with an added scalar field.

The significance of the present work deals with the modification of gravitational and geometrical aspects of Einstein's equations. These are 1) scale invariant theory of gravitation which describes the interaction between matter and gravitation in scale free manner; and 2) the gauge transformation, which represents a change of units of measurements and hence gives a general scaling of physical system. The nature of the cosmological model with modified gravity that would reproduce the kinematical history and evolution of perturbation of the universe is investigated.

Here, cylindrically symmetric static zeldovich fluid model is obtained in the presence of perfect fluid distribution in scale invariant theory of gravitation. As far as matter is concerned the model does not admit either big bang or big crunch during evolution till infinite future. The model appears to be a steady state.

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7. References

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