Applied Mathematics, 2012, 3, 1680-1685
http://dx.doi.org/10.4236/am.2012.311232 Published Online November 2012 (http://www.SciRP.org/journal/am)

Integral Inequalities of Hermite-Hadamard Type for Functions Whose 3rd Derivatives Are *s*-Convex

Ling Chun[1], Feng Qi[2]
[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China
[2]Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin, China
Email: chunling1980@qq.com, qifeng618@gmail.com

Received September 2, 2012; revised October 2, 2012; accepted October 10, 2012

ABSTRACT

In the paper, the authors find some new inequalities of Hermite-Hadamard type for functions whose third derivatives are *s*-convex and apply these inequalities to discover inequalities for special means.

Keywords: Integral Inequality; Hermite-Hadamard's Integral Inequality; *s*-Convex Function; Derivative; Mean

1. Introduction

The following definition is well known in the literature.

Definition 1.1. A function $f: I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

In [1,2], among others, the concepts of so-called quasi-convex and *s*-convex functions in the second sense was introduced as follows.

Definition 1.2 ([1]). A function $f: I \subseteq \mathbb{R} \to \mathbb{R}_0 = [0, \infty)$ is said to be quasi-convex if

$$f(\lambda x + (1-\lambda) y) \le \sup\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.3 ([2]). Let $s \in (0,1]$. A function $f: \mathbb{R}_0 \to \mathbb{R}_0$ is said to be *s*-convex in the second sense if

$$f(\lambda x + (1-\lambda) y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

If $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on $[a,b]$ with $a, b \in I$ and $a < b$, Then we have Hermite-Hardamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x \le \frac{f(a)+f(b)}{2}. \quad (1.1)$$

Hermite-Hadamard inequality (1.1) has been refined or generalized for convex, *s*-convex, and quasi-convex functions by a number of mathematicians. Some of them can be reformulated as follows.

Theorem 1.1 ([3, Theorems 2.2 and 2.3]). Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$.

(1) If $|f'(x)|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \quad (1.2)$$

(2) If the new mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$ for $p > 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{1-1/p}.$$

Theorem 1.2 ([4, Theorems 1 and 2]). Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$, and let $q \ge 1$. If $|f'(x)|^q$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \quad (1.3)$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \quad (1.4)$$

Theorem 1.3 ([5, Theorems 2.3 and 2.4]). Let

$f:I\subset\mathbb{R}\to\mathbb{R}$. be differentiable on I°, $a,b\in I$ with $a<b$, and let $p>1$. If $\left|f'(x)\right|^{p/(p-1)}$ is convex on $[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\times\left\{\left[\left|f'(a)\right|^{p/(p-1)}+3\left|f'(b)\right|^{p/(p-1)}\right]^{(p-1)/p}\right.$$
$$\left.+\left[3\left|f'(a)\right|^{p/(p-1)}+\left|f'(b)\right|^{p/(p-1)}\right]^{(p-1)/p}\right\}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).\tag{1.5}$$

Theorem 1.4 ([6, Theorems 1 and 3]). Let $f:I\subset\mathbb{R}_0\to\mathbb{R}$ be differentiable on I° and $a,b\in I$ with $a<b$.

(1) If $\left|f'(x)\right|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q\ge1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{2+(1/2)^s}{(s+1)(s+2)}\right]^{1/q}\times\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}.\tag{1.6}$$

(2) If $\left|f'(x)\right|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\le\frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{1/q}\cdot\left\{\left[\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}\\&\le\frac{b-a}{2}\left\{\left[\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}.\end{aligned}\tag{1.7}$$

Theorem 1.5 ([7, Theorem 2]). Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function on I° such that $f'''\in L([a,b])$ for $a,b\in I^\circ$ with $a,b\in I^\circ$. If $\left|f'''(x)\right|$ is quasi-convex on $[a,b]$, then

$$\begin{aligned}&\left|\int_a^b f(x)\mathrm{d}x-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\le\frac{(b-a)^4}{1152}\left[\max\left\{\left|f'''(a)\right|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|,\left|f'''(b)\right|\right\}\right]\end{aligned}$$

In recent years, some other kinds of Hermite-Hadamard type inequalities were created in, for example, [8-17], especially the monographs [18,19], and related references therein.

In this paper, we will find some new inequalities of Hermite-Hadamard type for functions whose third derivatives are s-convex and apply these inequalities to discover inequalities for special means.

2. A Lemma

For finding some new inequalities of Hermite-Hadamard type for functions whose third derivatives are s-convex, we need a simple lemma below.

Lemma 2.1. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a three times differentiable function on I° with $a,b\in I$ and $a<b$. If $f'''\in L[a,b]$, then

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]=\frac{(b-a)^3}{12}\int_0^1 t(1-t)(2t-1)f'''(ta+(1-t)b)\mathrm{d}t.\tag{2.1}$$

Proof. By integrating by part, we have

$$\begin{aligned}&\int_0^1 t(1-t)(2t-1)f'''(ta+(1-t)b)\mathrm{d}t\\&=\frac{1}{b-a}\int_0^1\left(-6t^2+6t-1\right)f''(ta+(1-t)b)\mathrm{d}t\\&=-\frac{\left[f'(b)-f'(a)\right]}{(b-a)^2}+\frac{1}{(b-a)^2}\int_o^1(-12t+6)f'(ta+(1-t)b)\mathrm{d}t\\&=-\frac{\left[f'(b)-f'(a)\right]}{(b-a)^2}-\frac{1}{(b-a)^3}\int_o^1(-12t+6)\mathrm{d}f(ta+(1-t)b)\\&=-\frac{\left[f'(b)-f'(a)\right]}{(b-a)^2}+\frac{6\left[f(a)+f(b)\right]}{(b-a)^3}-\frac{12}{(b-a)^3}\int_o^1 f(ta+(1-t)b)\mathrm{d}t\\&=-\frac{\left[f'(b)-f'(a)\right]}{(b-a)^2}+\frac{6\left[f(a)+f(b)\right]}{(b-a)^3}-\frac{12}{(b-a)^4}\int_a^b f(x)\mathrm{d}x\end{aligned}$$

The proof of Lemma 2.1 is complete.

3. Some New Hermite-Hadamard Type Inequalities

We now utilize Lemma 2.1, Hölder's inequality, and others to find some new inequalities of Hermite-Hadamard type for functions whose third derivatives are s-convex.

Theorem 3.1. Let $f : I \subseteq R_0 \to \mathbb{R}$ be a three times differentiable function on I° such that $f''' \in L[a,b]$ for $a,b \in I$ with $a<b$. If $|f'''|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \le \frac{(b-a)^3}{192}\left(\frac{2^{2-s}\left(s+6+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\right)^{1/q} \times\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}. \tag{3.1}$$

Proof. Since $|f'''|^q$ is s-convex on $[a,b]$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \\
&\le \frac{(b-a)^3}{12}\int_0^1 t(1-t)\left|(2t-1)\right|\left|f'''(ta+(1-t)b)\right|\mathrm{d}t \\
&\le \frac{(b-a)^3}{12}A_0^{1-\frac{1}{q}} \cdot\left[\int_0^1 t(1-t)\left|(2t-1)\right|\left|f'''(ta+(1-t)b)\right|^q\mathrm{d}t\right]^{1/q} \\
&\le \frac{(b-a)^3}{12}A_0^{1-\frac{1}{q}}\left\{\int_0^1 t(1-t)\left|(2t-1)\right|\left[t^s\left|f'''(a)\right|^q +(1-t)^s\left|f'''(b)\right|^q\right]\mathrm{d}t\right\}^{1/q},
\end{aligned}$$

where

$$A_0=\int_0^1 t(1-t)\left|(2t-1)\right|\mathrm{d}t=\frac{1}{16}$$

and

$$\begin{aligned}
A_s&=\int_0^1 t(1-t)\left|(2t-1)\right|t^s\mathrm{d}t \\
&=\int_0^1 t(1-t)\left|(2t-1)\right|(1-t)^s\mathrm{d}t \\
&=\frac{6+s+2^{s+2}s}{2^{s+2}(s+2)(s+3)(s+4)}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \\
&\le \frac{(b-a)^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(\frac{s+6+2^{s+2}s}{2^{s+2}(s+2)(s+3)(s+4)}\right)^{1/q} \times\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q} \\
&= \frac{(b-a)^3}{192}\left(\frac{2^{2-s}\left(s+6+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\right)^{1/q} \times\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}.
\end{aligned}$$

The proof of Theorem 3.1 is complete.

Corollary 3.1.1. Under conditions of Theorem 3.1,

1) if $s=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \le \frac{(b-a)^3}{192}\left(\frac{1}{2}\right)^{1/q}\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}; \tag{3.2}$$

2) if $q=s=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \le \frac{(b-a)^3}{384}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right].$$

Theorem 3.2. Let $f : I \subseteq R_0 \to \mathbb{R}$ be a three times differentiable function on I° such that $f''' \in L[a,b]$ for $a,b \in I$ with $a<b$. If $|f'''|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right| \le \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{2^{1-s}\left(s2^s+1\right)}{(s+1)(s+2)}\right)^{1/q} \cdot\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}, \tag{3.3}$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. Using Lemma 2.1, the s-convexity of $|f'''|^q$ on $[a,b]$, and Hölder's integral inequality yields

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{12}\int_0^1 t(1-t)\left|(2t-1)\right|\left|f'''(ta+(1-t)b)\right|\mathrm{d}t$$
$$\le\frac{(b-a)^3}{12}B^{1/p}\left[\int_0^1|2t-1|\left|f'''(ta+(1-t)b)\right|^q\,\mathrm{d}t\right]^{1/q}$$
$$\le\frac{(b-a)^3}{12}B^{1/p}$$
$$\cdot\left\{\int_0^1|2t-1|\left[t^s\left|f'''(a)\right|^q+(1-t)^s\left|f'''(b)\right|^q\right]\mathrm{d}t\right\}^{1/q},$$

where an easy calculation gives

$$B=\int_0^1 t^p(1-t)^p|2t-1|\,\mathrm{d}t=\frac{1}{2^{2p+1}(p+1)}\tag{3.4}$$

and

$$\int_0^1 t^s|2t-1|\mathrm{d}t=\int_0^1(1-t)^s|2t-1|\,\mathrm{d}t=\frac{s2^s+1}{2^s(s+1)(s+2)}.\tag{3.5}$$

Substituting Equations (3.4) and (3.5) into the above inequality results in the inequality (3.3). The proof of Theorem 3.2 is complete.

Corollary 3.2.1. Under conditions of Theorem 3.2, if $s=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{2}\right)^{1/q}\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}.$$

Theorem 3.3. Under conditions of Theorem 3.2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{24}\left(\frac{1}{(p+1)(p+3)}\right)^{1/p}\left(\frac{2}{(s+2)(s+3)}\right)^{1/q}$$
$$\times\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}.\tag{3.6}$$

Proof. Making use of Lemma 2.1, the s-convexity of $\left|f'''\right|^q$ on $[a,b]$, and Hölder's integral inequality leads to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{12}C^{1/p}\left[\int_0^1 t(1-t)\left|f'''(ta+(1-t)b)\right|^q\,\mathrm{d}t\right]^{1/q}$$
$$\le\frac{(b-a)^3}{12}C^{1/p}$$
$$\cdot\left\{\int_0^1 t(1-t)\left[t^s\left|f'''(a)\right|^q+(1-t)^s\left|f'''(b)\right|^q\right]\mathrm{d}t\right\}^{1/q},$$

where

$$C=\int_0^1 t(1-t)\left|(2t-1)\right|^p\mathrm{d}t=\frac{1}{2(p+1)(p+3)}\tag{3.7}$$

and

$$\int_0^1 t^{s+1}(1-t)\,\mathrm{d}t=\int_0^1 t(1-t)^{s+1}\,\mathrm{d}t=\frac{1}{(s+2)(s+3)}.\tag{3.8}$$

Substituting Equations (3.7) and (3.8) into the above inequality derives the inequality (3.6). The proof of Theorem 3.3 is complete.

Corollary 3.3.1. Under conditions of Theorem 3.3, if s = 1, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{24}\left(\frac{1}{(p+1)(p+3)}\right)^{1/p}\left(\frac{1}{6}\right)^{1/q}$$
$$\cdot\left[\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}.$$

Theorem 3.4. Under conditions of Theorem 3.2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{48}\left(\frac{5+2^{p+1}(p-1)+p}{(p+1)(p+2)(p+3)}\right)^{1/p}$$
$$\left(\frac{1}{2^s(s+1)(s+2)(s+3)}\right)^{1/q}$$
$$\times\min\left\{\left[\left(5+2^{s+1}(s-1)+s\right)\left|f'''(a)\right|^q\right.\right.$$
$$\left.+\left(2^{s+1}(s+1)^2+s+1\right)\left|f'''(b)\right|^q\right]^{1/q},$$
$$\left[\left(2^{s+1}(s+1)^2+s+1\right)\left|f'''(a)\right|^q\right.$$
$$\left.\left.+\left(5+2^{s+1}(s-1)+s\right)\left|f'''(b)\right|^q\right]^{1/q}\right\}$$

Proof. Since $\left|f'''\right|^q$ is s-convex on $[a,b]$, by Lemma

2.1 and Hölder's inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{12}\int_0^1 t(1-t)|2t-1|\left|f'''(ta+(1-t)b)\right|\mathrm{d}t$$
$$\le\frac{(b-a)^3}{12}D^{1/p}$$
$$\left\{\int_0^1 t(1-t)|2t-1|\left[t^s\left|f'''(a)\right|^q+(1-t)^s\left|f'''(b)\right|^q\right]\mathrm{d}t\right\}^{1/q}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{12}D^{1/p}$$
$$\cdot\left\{\int_0^1 t|2t-1|\left[t^s\left|f'''(a)\right|^q+(1-t)^s\left|f'''(b)\right|^q\right]\mathrm{d}t\right\}^{1/q}$$

where a straightforward computation gives

$$D=\int_0^1 t^P(1-t)|2t-1|\,\mathrm{d}t=\frac{5+2^{p+1}(p-1)+p}{2^{p+1}(p+1)(p+2)(p+3)},$$

$$\int_0^1 t(1-t)^p|2t-1|\,\mathrm{d}t=\frac{5+2^{p+1}(p-1)+p}{2^{p+1}(p+1)(p+2)(p+3)},$$

$$\int_0^1(1-t)^{s+1}|2t-1|\,\mathrm{d}t=\frac{2^{s+1}(s+1)+1}{2^{s+1}(s+2)(s+3)},$$

$$\int_0^1(t)^{s+1}|2t-1|\,\mathrm{d}t=\frac{2^{s+1}(s+1)+1}{2^{s+1}(s+2)(s+3)}.$$

Substituting these equalities into the above inequality brings out the inequality (3.10). The proof of Theorem 3.4 is complete.

Corollary 3.4.1. Under conditions of Theorem 3.4, if $s=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|$$
$$\le\frac{(b-a)^3}{48}\left(\frac{5+2^{p+1}(p-1)+p}{(p+1)(p+2)(p+3)}\right)^{1/p}\left(\frac{1}{8}\right)^{1/q}$$
$$\times\min\left\{\left[\left|f'''(a)\right|^q+3\left|f'''(b)\right|^q\right]^{1/q},\right.$$
$$\left.\left[3\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right]^{1/q}\right\}.$$

4. Applications to Special Means

For positive numbers $a>0$ and $b>0$, define

$$A(a,b)=\frac{a+b}{2}\tag{4.1}$$

and

$$L_r(a,b)=\begin{cases}\left[\dfrac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & r\ne-1,0;\\ \dfrac{b-a}{\ln b-\ln a}, & r=-1;\\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & r=0.\end{cases}\tag{4.2}$$

It is well known that A and L_r are respectively called the arithmetic and generalized logarithmic means of two positive number a and b.

Now we are in a position to construct some inequalities for special means A and L_r by applying the above established inequalities of Hermite-Hadamard type.

Let

$$f(x)=\frac{x^{s+3}}{(s+1)(s+2)(s+3)}\tag{4.3}$$

for $0<s\le1$ and $x>0$. Since $f'''(x)=x^s$ and $(\lambda x+(1-\lambda)y)^s\le\lambda^s x^s+(1-\lambda)^s y^s$ for $x,y>0$ and $\lambda\in[0,1]$, then $f'''(x)=x^s$ is s-convex function on $\mathbb{R}_0$ and

$$\frac{f(a)+f(b)}{2}=\frac{1}{(s+1)(s+2)(s+3)}A\left(a^{s+3},b^{s+3}\right),$$
$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x=\frac{1}{(s+1)(s+2)(s+3)}L_{s+3}^{s+3}\left(a^{s+4},b^{s+4}\right),$$
$$f'(b)-f'(a)=\frac{1}{12(s+1)}L_{s+1}^{s+1}\left(a^{s+2},b^{s+2}\right).$$

Applying the function (4.3) to Theorems 3.1 to 3.3 immediately leads to the following inequalities involving special means A and L_r.

Theorem 4.1. Let $b>a>0$, $0<s\le1$, and $q\ge1$. Then

$$\left|12A\left(a^{s+3},b^{s+3}\right)-12L_{s+3}^{s+3}\left(a^{s+4},b^{s+4}\right)\right.$$
$$\left.-(b-a)^2(s+2)(s+3)L_{s+1}^{s+1}\left(a^{s+2},b^{s+2}\right)\right|$$
$$\le\frac{(b-a)^3(s+1)}{16}\left[(s+2)(s+3)\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{2^{3-s}\left(s+6+2^{s+2}s\right)}{s+4}\right]^{1/q}$$
$$\times A^{1/q}\left(a^{sq},b^{sq}\right).$$

Theorem 4.2. For $b>a>0$, $0<s\le 1$, and $q>1$, we have

$$\begin{aligned}&\Big|12A\left(a^{s+3},b^{s+3}\right)-12L_{s+3}^{s+3}\left(a^{s+4},b^{s+4}\right)\\&-(b-a)^2(s+2)(s+3)L_{s+1}^{s+1}\left(a^{s+2},b^{s+2}\right)\Big|\\&\le\frac{(b-a)^3(s+3)}{8}\left(\frac{(s+1)(s+2)}{p+1}\right)^{1/p}\\&\times\left[2^{2-s}\left(s2^s+1\right)\right]^{1/q}A^{1/q}\left(a^{sq},b^{sq}\right).\end{aligned}\tag{4.4}$$

Theorem 4.3. For $b>a>0$, $0<s\le 1$, and $q>1$, we have

$$\begin{aligned}&\Big|12A\left(a^{s+3},b^{s+3}\right)-12L_{s+3}^{s+3}\left(a^{s+4},b^{s+4}\right)\\&-(b-a)^2(s+2)(s+3)L_{s+1}^{s+1}\left(a^{s+2},b^{s+2}\right)\Big|\\&\le 2(b-a)^3(s+1)\left[\frac{(s+2)(s+3)}{4(p+1)(p+3)}\right]^{1/p}A^{1/q}\left(a^{sq},b^{sq}\right).\end{aligned}$$

5. Acknowledgements

The first author was supported by Science Research Funding of Inner Mongolia University for Nationalities under Grant No. NMD1103.

REFERENCES

[1] S. S. Dragomir, J. Pecaric and L.-E. Persson, "Some Inequalities of Hadamard Type," *Soochow Journal of Mathematics*, Vol. 21, No. 3, 1995, pp. 335-341.

[2] H. Hudzik and L. Maligranda, "Some Remarks on *s*-Convex Functions," *Aequationes Mathematicae*, Vol. 48, No. 1, 1994, pp. 100-111.

[3] S. S. Dragomir and R. P. Agarwal, "Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula," *Applied Mathematics Letters*, Vol. 11, No. 5, 1998, pp. 91-95. doi:10.1016/S0893-9659(98)00086-X

[4] C. E. M. Pearce and J. Pečarić, "Inequalities for Differentiable Mappings with Application to Special Means and Quadrature Formulae," *Applied Mathematics Letters*, Vol. 13, No. 2, 2000, pp. 51-55. doi:10.1016/S0893-9659(99)00164-0

[5] U. S. Kirmaci, "Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Midpoint Formula," *Applied Mathematics and Computation*, Vol. 147, No. 1, 2004, pp. 137-146. doi:10.1016/S0096-3003(02)00657-4

[6] U. S. Kirmaci, M. K. Bakula, M. E. Ozdemir and J. Pecaric, "Hadamard-Type Inequalities for *s*-Convex Functions," *Applied Mathematics and Computation*, Vol. 193, No. 1, 2007, pp. 26-35. doi:10.1016/j.amc.2007.03.030

[7] M. Alomari and S. Hussain, "Two Inequalities of Simpson Type for Quasi-Convex Functions and Applications," *Applied Mathematics E-Notes*, Vol. 11, 2011, pp. 110-117.

[8] R.-F. Bai, F. Qi and B.-Y. Xi, "Hermite-Hadamard Type Inequalities for the *m*- and (*α*, *m*)-Logarithmically Convex Functions," *Filomat*, Vol. 27, No. 1, 2013, 1-7.

[9] S.-P. Bai, S.-H. Wang and F. Qi, "Some Hermite-Hadamard Type Inequalities for *n*-Time Differentiable (*α*, *m*)-Convex Functions," *Journal of Inequalities and Applications*, 2013, in Press.

[10] W.-D. Jiang, D.-W. Niu, Y. Hua and F. Qi, "Generalizations of Hermite-Hadamard Inequality to *n*-Time Differentiable Functions Which Are *s*-Convex in the Second Sense," *Analysis* (*Munich*), Vol. 32, No. 3, 2012, pp. 209-220. doi:10.1524/anly.2012.1161

[11] F. Qi, Z.-L. Wei and Q. Yang, "Generalizations and Refinements of Hermite-Hadamard's Inequality," *The Rocky Mountain Journal of Mathematics*, Vol. 35, No. 1, 2005, pp. 235-251. doi:10.1216/rmjm/1181069779

[12] S.-H. Wang, B.-Y. Xi and F. Qi, "On Hermite-Hadamard Type Inequalities for (*α*, *m*)-Convex Functions," *International Journal of Open Problems in Computer Science and Mathematics*, Vol. 5, No. 4, 2012, in Press.

[13] S.-H. Wang, B.-Y. Xi and F. Qi, "Some New Inequalities of Hermite-Hadamard Type for *n*-Time Differentiable Functions Which Are *m*-Convex," *Analysis* (*Munich*), Vol. 32, No. 3, 2012, pp. 247-262. doi:10.1524/anly.2012.1167

[14] B.-Y. Xi, R.-F. Bai and F. Qi, "Hermite-Hadamard Type Inequalities for the *m*- and (*α*; *m*)-Geometrically Convex Functions," *Aequationes Mathematicae*, 2012, in Press. doi:10.1007/s00010-011-0114-x

[15] B.-Y. Xi and F. Qi, "Some Hermite-Hadamard Type Inequalities for Differentiable Convex Functions and Applications," *Hacettepe Journal of Mathematics and Statistics*, Vol. 42, 2013, in Press.

[16] B.-Y. Xi and F. Qi, "Some Integral Inequalities of Hermite-Hadamard Type for Convex Functions with Applications to Means," *Journal of Function Spaces and Applications*, Vol. 2012, 2012, 14 pp. doi:10.1155/2012/980438

[17] T.-Y. Zhang, A.-P. Ji and F. Qi, "On Integral Inequalities of Hermite-Hadamard Type for *s*-Geometrically Convex Functions," *Abstract and Applied Analysis*, Vol. 2012, 2012, 15 pp. doi:10.1155/2012/560586

[18] S. S. Dragomir and C. E. M. Pearce, "Selected Topics on Hermite-Hadamard Type Inequalities and Applications," RGMIA Monographs, Victoria University, Melbourne, 2000.

[19] C. P. Niculescu and L.-E. Persson, "Convex Functions and Their Applications: A Contemporary Approach (CMS Books in Mathematics)," Springer-Verlag, New York, 2005.

