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Necessary Conditions for a Fixed Point of Maps in Non-Metric Spaces

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ABSTRACT

The main purpose of the present work is to introduce necessary conditions for a map on a non-metric space, defined by using a map on a metric space, to have a fixed point.

Keywords: Topological Space; Complete Metric Compact Space; Cauchy Sequence; Lipschitz Continuous Map; Contraction Map

1. Introduction

Let X denote a complete (or compact) metric space and also $f: X \to Y$ a continuous map of X onto Y, where Y is a bounded closed topological normal space with a countable base.

What must be the conditions, in the means of the meric space X, such that the continuous map $g: Y \to Y$ from Y onto Y will have a fixed point?

We suppose that (see [1-3]):

the continuous map $f: X \to Y$ (not one to one) and

the continuous map $g: Y \to Y$ are given and

the continuous inverse map of f, $f^{-1}: Y \to X$ exists.

$$\begin{array}{ccc} & f & \\ X & \to & Y \\ f^{-1} & \nwarrow & \downarrow \; g \\ & & Y \end{array}$$

We remind that Banach contraction principle for multivalued maps is valid and also the next Theorem, proved by H. Covitz and S. B. Nadler Jr. (see [4]).

Theorem 1. Let (X,d) be a complete metric space and $F: X \to B(X)$ a conraction map ($B(X)$ denotes the family of all nonempty closed bounded (compact) subsets of X). Then there exists $x \in X$ such that $x \in F(x)$.

2. Main Result

We consider now the next theorem:

Theorem 2. *Let X denote a complete (or compact) metric space X and also:*

$f: X \to Y$ a continuous map of X onto Y, where Y is a bounded closed topological normal space with a countable base.

We suppose also that the maps:

$g: Y \to Y$ is continuous and onto.

and

$f^{-1}: Y \to X$ exists and it is continuous.

If $x_0 \in X$ is a point from X and if we suppose also that $y_0 \in f(x_0)$.

Then if the rest terms of the sequence $\{y_i\}$ are received from $y_i \in g(y_{i-1}), i = 1,2,3,\cdots$ and the rest of the terms of the sequence $\{x_i\}$ are determined by $x_i \in f^{-1}(y_i), i = 1,2,3,\cdots$ and if also $\{x_i\}$ is a Cauchy sequence and therefore convergent to a fixed point x^* in X, then the sequence $\{y_i\}$ will be also convergent to a fixed point y^* in Y.

Proof. Let $x_0 \in X$ is a point from X and let us suppose also that $y_0 \in f(x_0) \subset Y$ and let the rest terms of the sequence $\{y_i\}$ are received from $y_i \in g(y_{i-1})$, $i = 1,2,3,\cdots$.

Let also the rest of the terms of the sequence $\{x_i\}$ are determined by $x_i \in f^{-1}(y_i)$, $i = 1,2,3,\cdots$.

If $\{x_i\}$ is a Cauchy sequence then for any $\varepsilon > 0$ there exists an integer N_ε, such that for all integers i and k, $i > N_\varepsilon$ and $k > N_\varepsilon$ will be satisfied the inequality

$$\|x_i - x_k\| < \varepsilon$$

and therefore the Cauchy sequence $\{x_i\}$ will be convergent with a fixed point x^* in X, and because X is complete (or compact), *i.e.*

$$\lim_{x\to\infty} x_i = x^*.$$

Since $x_i = f^{-1}(y_i)$ and $x_i = f^{-1}(y_k)$ and $f^{-1}(y)$ is a continuous map and $g(y)$ is continuous map onto the closed and bounded space Y, and also $y_i \in f(x_i)$ and $y_k \in f(x_k)$, therefore the sequence $\{y_i\}$ will be also convergent with a fixed point y^* in Y, such that $x^* = f^{-1}(y^*)$ and $y^* \in f(x^*)$, *i.e.*

$$\lim_{x\to\infty} y_i = y^*.$$

Q.E.D.

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