

Computation of the Multivariate Normal Integral over a Complex Subspace

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ABSTRACT

The computation of the multivariate normal integral over a Complex Subspace is a challenge, especially when the integration region is of a complex nature. Such integrals are met with, for example, in the generalized Neyman-Pearson criterion, conditional Bayesian problems of testing many hypotheses and so on. The Monte-Carlo methods could be used for their computation, but at increasing dimensionality of the integral the computation time increases unjustifiedly. Therefore a method of computation of such integrals by series after reduction of dimensionality to one without information loss is offered below. The calculation results are given.

Keywords: Multivariate Normal Integral; Random Variable; Probability; Moments; Series

1. Introduction

At testing many hypotheses with reference to the parameters of multivariate normal distribution, the problem of computation of multivariate normal integrals over a Complex Subspace of the following form arises [1]

$$p_{ij} = \int_{\Gamma_j} p(\mathbf{x}|H_i) d\mathbf{x}, j = 1, \dots, S, i \neq j, \quad (1)$$

where S is the number of tested hypotheses $H_i: \boldsymbol{\theta} = \boldsymbol{\theta}^i$, supposing that sample $\mathbf{x}^T = (x_1, \dots, x_n)$ was brought about by distribution

$$p(\mathbf{x}, \boldsymbol{\theta}^i) = p(x_1, \dots, x_n; \theta_1^i, \dots, \theta_m^i) \equiv p(\mathbf{x}|H_i), i = 1, \dots, S$$

where $\boldsymbol{\theta}^T = (\theta_1, \dots, \theta_m)$ is the vector of distribution parameters and Γ_i is the acceptance region of hypothesis H_i from sample space R^n ($\mathbf{x} \in R^n$), which has the following form

$$\Gamma_j = \left\{ \mathbf{x} : k_j^j p(\mathbf{x}|H_j) > \sum_{\ell=1, \ell \neq j}^S k_\ell^j p(\mathbf{x}|H_\ell) \right\}, \quad (2)$$

$$j = 1, \dots, S$$

where $0 \leq k_\ell^j < +\infty$, $\ell = 1, \dots, S$.

Such regions of hypotheses acceptance arise, for example, in the generalized Neyman-Pearson criterion, and also in conditional Bayesian problems of testing many hypotheses [2,3]. The dimensionality of these integrals often reaches several tens when practical problems are solved. For example, in ecological problems the number of controlled parameters, according to which the decision

is made, is quite often equal to several tens [4]; in the air defence problems, in particular, in the problems of tracking of flying objects using radar measurement information, the dimensionality of the problem is equal to the multiplication of the number of flying objects by the number of surveys made by the radar set [5] and so on. On the other hand, the time for solution of these problems is often limited and at times it plays a decisive role especially at solving the defence problems.

It is known that the complexity of realization and the obtained accuracy of numerical methods of computation of multidimensional integrals depend heavily on the dimensionality of these integrals and the complexity of the integration region configuration. In the considered case the integration regions are nonconvex and quite complex. Therefore it is difficult to realize the numerical methods and to provide the desired accuracy of calculation even when the dimensionality of integral is greater than or equal to three [6]. The methods of computation of the multivariate normal integral on the hyperrectangle offered in [7-12] are unsuitable for this case because of the complexity of the integration region.

Despite the convenience and the simplicity of computations, the Monte Carlo method is computer time consuming, especially at large dimensionality of integrals [3,13,14]. Therefore the method of approximate computation of integral (1) for a very short period of time is topical in many applications of mathematical statistics [15,16].

The aim of the present paper is the development of the method of computation of probability integral (1) with the desired accuracy in a minimum of time.

2. Problem Statement

Let us consider the case when the probability distribution density of the vector \mathbf{x} looks like

$$p(\mathbf{x}|H_i) = (2\pi)^{-n/2} |\mathbf{W}_i|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{a}^i)^T \mathbf{W}_i^{-1}(\mathbf{x}-\mathbf{a}^i)\right\}, \quad (3)$$

$$i = 1, \dots, S,$$

where

$$\mathbf{a}^{iT} = (a_1^i, \dots, a_n^i), \mathbf{W}_i = \begin{pmatrix} \sigma_1^{i2} & \rho_{12}^i & \dots & \rho_{1n}^i \\ \rho_{21}^i & \sigma_2^{i2} & \dots & \rho_{2n}^i \\ \dots & \dots & \dots & \dots \\ \rho_{n1}^i & \rho_{n2}^i & \dots & \sigma_n^{i2} \end{pmatrix}.$$

For probability distribution density (3), let us rewrite decision-making region (2) as

$$\Gamma_j = \left\{ \mathbf{x} : \sum_{\ell=1}^S C_\ell^j \exp(-y_\ell) < 0 \right\}, \quad (4)$$

where

$$C_\ell^j = k_\ell^j (2\pi)^{-n/2} |\mathbf{W}_\ell|^{-1/2}, \ell \neq j, \\ C_j^j = -k_j^j (2\pi)^{-n/2} |\mathbf{W}_j|^{-1/2},$$

$$y_\ell = \frac{1}{2}(\mathbf{x}-\mathbf{a}^\ell)^T \mathbf{W}_\ell^{-1}(\mathbf{x}-\mathbf{a}^\ell), \ell, j = 1, \dots, S. \quad (5)$$

Random variables $y_\ell, \ell = 1, \dots, S$, are squared forms of the normally distributed random vector, and, if hypothesis H_i is true, their mathematical expectations are equal to

$$E(y_\ell|H_i) = \frac{1}{2}(\mathbf{a}^i - \mathbf{a}^\ell)^T \mathbf{W}_\ell^{-1}(\mathbf{a}^i - \mathbf{a}^\ell) + \frac{1}{2} \text{trace}(\mathbf{W}_i \mathbf{W}_\ell^{-1}), \quad \ell, i = 1, \dots, S. \quad (6)$$

Therefore, if hypothesis H_i is true, the random variable y_ℓ has noncentral distribution χ^2 with the degree of freedom n and with the parameter of noncentrality equal to (6) [2,17,18].

It is obvious that, at $\ell = i$ and hypothesis H_i is true, the random variable y_i has the central χ^2 distribution with the degree of freedom n .

Let us write down (1) as follows

$$p_{ij} = \int_{\Gamma_j} p(\mathbf{x}|H_i) d\mathbf{x} = P\left(\sum_{\ell=1}^S C_\ell^j \exp(-y_\ell) < 0 | H_i\right) \quad (7)$$

The task consists in the computation of probability (7). The method of its analytical computation is not known so far. For its computation it is possible, for example, to use

a modified Monte-Carlo method (with the purpose of reducing the computation time) [3]. Though, at large S , it still takes a good deal of time. The method of computation of probability (7) if hypotheses are formulated with reference only to the mathematical expectation of normally distributed random vector is offered in [3]. This method is unsuitable here, as the random variable

$$\xi_j = \sum_{\ell=1}^S C_\ell^j \exp(-y_\ell), \quad (8)$$

which formulates integration region (4), in [3] is the weighted sum of log-normally distributed random quantities; C_ℓ^j and y_ℓ are determined by formulae (5). In our case, ξ_j is the weighted sum of the exponents of negative quadratic forms of the normally distributed random vector with correlated components.

Let us consider the case $S = 2$. In this case, regions (2) take the form

$$\Gamma_1 = \left\{ \mathbf{x} : p(\mathbf{x}|H_2) < k_1 p(\mathbf{x}|H_1) \right\},$$

$$\Gamma_2 = \left\{ \mathbf{x} : p(\mathbf{x}|H_1) < k_2 p(\mathbf{x}|H_2) \right\}.$$

With taking into account probability densities (3), for these regions we derive

$$\Gamma_1 = \left\{ \mathbf{x} : \mathbf{x}^T \mathbf{W}_1^{-1} \mathbf{x} - \mathbf{x}^T \mathbf{W}_2^{-1} \mathbf{x} + 2(\mathbf{a}^{2T} \mathbf{W}_2^{-1} - \mathbf{a}^{1T} \mathbf{W}_1^{-1}) \mathbf{x} < \lambda_{12} \right\},$$

$$\Gamma_2 = \left\{ \mathbf{x} : \mathbf{x}^T \mathbf{W}_2^{-1} \mathbf{x} - \mathbf{x}^T \mathbf{W}_1^{-1} \mathbf{x} + 2(\mathbf{a}^{1T} \mathbf{W}_1^{-1} - \mathbf{a}^{2T} \mathbf{W}_2^{-1}) \mathbf{x} < \lambda_{21} \right\},$$

where

$$\lambda_{12} = 2 \ln \left(k_1 \frac{|\mathbf{W}_1|^{1/2}}{|\mathbf{W}_2|^{1/2}} \right) + (\mathbf{a}^{2T} \mathbf{W}_2^{-1} \mathbf{a}^2 - \mathbf{a}^{1T} \mathbf{W}_1^{-1} \mathbf{a}^1),$$

$$\lambda_{21} = 2 \ln \left(k_2 \frac{|\mathbf{W}_2|^{1/2}}{|\mathbf{W}_1|^{1/2}} \right) + (\mathbf{a}^{1T} \mathbf{W}_1^{-1} \mathbf{a}^1 - \mathbf{a}^{2T} \mathbf{W}_2^{-1} \mathbf{a}^2).$$

Let us designate

$$\xi_{12} = \mathbf{x}^T \mathbf{W}_1^{-1} \mathbf{x} - \mathbf{x}^T \mathbf{W}_2^{-1} \mathbf{x} + 2(\mathbf{a}^{2T} \mathbf{W}_2^{-1} - \mathbf{a}^{1T} \mathbf{W}_1^{-1}) \mathbf{x},$$

$$\xi_{21} = \mathbf{x}^T \mathbf{W}_2^{-1} \mathbf{x} - \mathbf{x}^T \mathbf{W}_1^{-1} \mathbf{x} + 2(\mathbf{a}^{1T} \mathbf{W}_1^{-1} - \mathbf{a}^{2T} \mathbf{W}_2^{-1}) \mathbf{x}.$$

Then, finally, for the required regions, we shall obtain

$$\Gamma_1 = \left\{ \mathbf{x} : \xi_{12} < \lambda_{12} \right\},$$

$$\Gamma_2 = \left\{ \mathbf{x} : \xi_{21} < \lambda_{21} \right\}.$$

Each of random variables ξ_{12} and ξ_{21} is the sum of three random variables one of which is distributed by the normal law, and the two others are distributed by the χ^2 law. Therefore, the probability distribution laws of random variables ξ_{12} and ξ_{21} have not closed forms.

Thus, at $S = 2$, i.e. at testing two hypotheses with respect to all parameters of multivariate normal distri-

bution (in contradistinction to the case when hypotheses are formulated with respect to only the vector of mathematical expectation [3]), the principal complexity of the considered problem does not decrease.

3. Computation of Probability Integral (7) by Series

Let us use the expanded form of representation of the quadratic form in (8) [18,19]. Then

$$\xi_j = \sum_{\ell=1}^S C_{\ell}^j \exp \left\{ \frac{1}{2} \sum_{\ell_1=1}^n \sum_{\ell_2=1}^n \alpha_{\ell_1, \ell_2}^{\ell} \left(\frac{x_{\ell_1} - a_{\ell_1}^{\ell}}{\sigma_{\ell_1}^{\ell}} \right) \left(\frac{x_{\ell_2} - a_{\ell_2}^{\ell}}{\sigma_{\ell_2}^{\ell}} \right) \right\} \quad (9)$$

where $\alpha_{\ell_1, \ell_2}^{\ell}$ are the coefficients determined unambiguously by the elements of matrix W_{ℓ} (see formula (3)).

Let $p_j(z|H_i)$ be the conditional density of probability distribution of the random variable ξ_j . Then, for (7), we obtain

$$p_{ij} = \int_{-\infty}^0 p_j(z|H_i) dz. \quad (10)$$

Here the infinite interval $(-\infty, +\infty)$ is taken as the domain of definition of random variable ξ_j because of the signs of coefficients C_{ℓ}^j from (5).

As was mentioned above, the probability distribution law of the random variable ξ_j has not a closed form. Let us consider the opportunity of approximating this density by series. For this reason we need the moments of the random variable ξ_j [19-21]. Let us consider the problem of obtaining of these moments.

With this purpose let us calculate the initial moment of the r th order of random variable ξ_j provided that hypothesis H_i is true

$$\begin{aligned} \mu_r^{j,i} &= E \left[(\xi_j)^r | H_i \right] = E \left[\left(\sum_{\ell=1}^S C_{\ell}^j \exp(-y_{\ell}) \right)^r | H_i \right] \\ &= \sum_{\ell_1=1}^S \dots \sum_{\ell_r=1}^S C_{\ell_1}^j \dots C_{\ell_r}^j E \left[\exp \left(- (y_{\ell_1} + \dots + y_{\ell_r}) \right) | H_i \right], \quad (11) \\ r &= 1, 2, \dots \end{aligned}$$

Expression $y_{\ell_1} + \dots + y_{\ell_r}$ is the sum of correlated Quadratic Forms distributed by noncentral χ^2 probability distribution laws. Because of correlation, the property of reproducibility of the χ^2 distribution does not take place [2,18], and, consequently the mathematical expectation in (12) has not a closed form.

Let us use power series expansion of the exponent

$$\begin{aligned} \exp \left(- (y_{\ell_1} + \dots + y_{\ell_r}) \right) &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{1}{\nu!} (y_{\ell_1} + y_{\ell_2} + \dots + y_{\ell_r})^{\nu} \\ &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{1}{\nu!} \sum_{p_1 \in \{\ell_1, \dots, \ell_r\}; i=1, \dots, \nu} y_{p_1} \cdot y_{p_2} \dots y_{p_{\nu}}. \quad (12) \end{aligned}$$

Let us use the expanded representation of quadratic form (9) and be satisfied with the first M terms of expansion (12). Then expression for calculation of moments (11) can be represented as follows

$$\begin{aligned} \mu_r^{j,i} &\approx \sum_{\ell_1=1}^S \dots \sum_{\ell_r=1}^S C_{\ell_1}^j \dots C_{\ell_r}^j \left\{ 1 + \sum_{\nu=1}^M \left(-\frac{1}{2} \right)^{\nu} \right. \\ &\cdot \frac{1}{\nu!} \left[\sum_{p_1 \in \{\ell_1, \dots, \ell_r\}; i=1, \dots, \nu} \sum_{\ell_1=1}^n \sum_{\ell_2=1}^n \alpha_{\ell_1, \ell_2}^{p_1} \right. \\ &\left. \left. \dots \alpha_{\ell_1, \ell_2}^{p_{\nu}} E \left(\prod_{\xi=1}^{\Lambda} \left(\frac{x_{\xi} - a_{\xi}^{p_{\xi}}}{\sigma_{\xi}^{p_{\xi}}} \right)^{m_{\xi}} \right) | H_i \right] \right\}, \quad (13) \end{aligned}$$

where $\Lambda \in \{1, \dots, 2\nu\}$, $m_{\xi} \in \{0, 1, \dots, 2\nu\}$ and

$$\sum_{\xi=1}^{\Lambda} m_{\xi} = 2\nu.$$

Expression (13) contains product moments [17,19,20] of the 2ν ($\nu=1, \dots, M$) orders of normalized components of the correlated normally distributed random observation vector. Therefore, they are not equal to zero [18]. A lot of works are dedicated to the problem of computation of product moments [see, for example, 22-28].

In [22] the following problem was solved. Let x_1, x_2, \dots, x_n be random variables with mutually independent distributions, and let $X = \prod_{i=1}^n x_i$. There is found the

probability that X lies between A and B , i.e. $P\{A \leq X \leq B\}$, by using the central limit theorem in accordance with which the random variable

$\ln X = \sum_{i=1}^n \ln x_i$ is approximately distributed by the normal law. The better is this approximation the bigger is n .

The variance of the product of two random variables was studied by Barnett (1955) and Goodman (1960), in the case when they do not need to be independent. Sheldard (1952) studied the case when the distribution of

$\prod_{i=1}^k x_i$ was (approximately) logarithmic-normal. The author considered the case when x_1, x_2, \dots, x_n are random variables with mutually independent distributions. For

finding the probability that $X = \prod_{i=1}^n x_i$ lies between A

and B , i.e. $P\{A \leq X \leq B\}$, the central limit theorem is used to approach the probability distribution of the random variable $\ln X$ by normal distribution and this approach is better at increasing n . In work [25] no assumption is made about the distribution of $\prod_{i=1}^k x_i$. There is discussed the case when the K random variables,

x_1, x_2, \dots, x_k , ($K \geq 2$) are mutually independent, and the case when they do not have to be independent, and there are obtained their variance formulae. These results are generalizations of the results presented in [24].

In [26] are given exact formulae for the mathematical expectation of $(\bar{x}_i - \bar{X}_i)(\bar{x}_j - \bar{X}_j)(\bar{x}_k - \bar{X}_k)$ and

$$(\bar{x}_i - \bar{X}_i)(\bar{x}_j - \bar{X}_j)(\bar{x}_k - \bar{X}_k)(\bar{x}_h - \bar{X}_h),$$

($i \neq j \neq k \neq h$), where \bar{x}_i is the sample mean of the i th “character” in a sample of n elements from a population of N elements and \bar{X}_i is the corresponding population mean. Formulae for estimating these product moments from the sample were also given. These estimations are slightly biased. In [27] the unbiased estimate of the 4-variate product moment was obtained. Asymptotic results for the 3-variate and 4-variate product moments and their estimates were also obtained.

In [28] is derived a formula for the product moment $EX_1^{m_1} \dots X_p^{m_p}$, $m_1 \geq 1, \dots, m_p \geq 1$ in terms of the joint survival function when (X_1, \dots, X_p) is a non-negative random vector.

From the given review (of course incomplete, because this is not the aim of this paper) of the works dedicated to the study of product moments, it is seen that the problem considered here differs from them.

Theorem 3.1. *The initial moment of the r th order of random variable ξ_j determined by (9), provided that hypothesis H_i is true, can be calculated with any specified accuracy by the formula*

$$\begin{aligned} \mu_r^{j,i} &\approx \sum_{\ell_1=1}^S \dots \sum_{\ell_r=1}^S C_{\ell_1}^j \dots C_{\ell_r}^j \left\{ 1 + \sum_{\nu=1}^M \left(-\frac{1}{2} \right)^\nu \frac{1}{\nu!} \right. \\ &\times \left[\sum_{p_1 \in \{\ell_1, \dots, \ell_r\}; i=1, \dots, \nu} \sum_{t_1=1}^n \sum_{t_2=1}^n \alpha_{t_1, t_2}^{p_1} \dots \right. \\ &\alpha_{t_1, t_2}^{p_\nu} J^{i, \Lambda} \sum_{\tau_i \in \{0, 1, 2, \dots, 2\nu\}; i=1, \dots, \Lambda} d_{\tau_1, \dots, \tau_\Lambda}^{i, \Lambda} \\ &\left. \left. \times \prod_{\eta=1}^\Lambda \sum_{j=0}^{\tau_\eta} \binom{\tau_\eta}{j} \mu_{\tau_\eta-1}(\mu_i)^j \right] \right\}, r = 1, 2, 3, \dots \end{aligned} \tag{14}$$

where $J^{i, \Lambda} = \text{mod} \left\| \mathbf{K}^{i, \Lambda} (\boldsymbol{\beta}^{i, \Lambda})^{-1} \right\|$; $\boldsymbol{\beta}^{i, 2\nu}$ and $\mathbf{K}^{i, 2\nu}$ are the matrices of eigenvectors and eigenvalues of the inverse covariance matrix of normalized random variables

$$\frac{x_\tau - a_\tau^{p_\tau}}{\sigma_\tau^{p_\tau}}, \tau = 1, \dots, \Lambda; \quad d_{\tau_1, \dots, \tau_\Lambda}^{i, \Lambda}, \quad \tau_i \in \{0, 1, 2, \dots, 2\nu\},$$

$i = 1, \dots, \Lambda$, are the coefficients determined by the terms of matrices $\boldsymbol{\beta}^{i, 2\nu}$ and $\mathbf{K}^{i, 2\nu}$ and vector $\mathbf{b}^{i, 2\nu}$; μ_i^j and $\mu_{\tau_\eta-j}$ are the initial and central moments of the first and $\tau_\eta - j$ orders, respectively, defined by formulae (22), (23) and (24).

Proof. If hypothesis H_i is true, the values $\frac{x_\tau - a_\tau^{p_\tau}}{\sigma_\tau^{p_\tau}}$,

$\tau = 1, 2, \dots, \Lambda$, are correlated normally distributed random variables with the parameters

$$E \left(\frac{x_\tau - a_\tau^{p_\tau}}{\sigma_\tau^{p_\tau}} \middle| H_i \right) = \frac{a_\tau^i - a_\tau^{p_\tau}}{\sigma_\tau^{p_\tau}} = b_\tau^{i, p_\tau},$$

$$V \left(\frac{x_\tau - a_\tau^{p_\tau}}{\sigma_\tau^{p_\tau}} \middle| H_i \right) = \frac{(\sigma_\tau^i)^2}{(\sigma_\tau^{p_\tau})^2} = v_\tau^{i, p_\tau}, \tau = 1, \dots, \Lambda;$$

$$\begin{aligned} \text{cov} \left[\left(\frac{x_{\tau_1} - a_{\tau_1}^{p_{\tau_1}}}{\sigma_{\tau_1}^{p_{\tau_1}}} \right), \left(\frac{x_{\tau_2} - a_{\tau_2}^{p_{\tau_2}}}{\sigma_{\tau_2}^{p_{\tau_2}}} \right) \middle| H_i \right] &= \frac{\rho_{\tau_1, \tau_2}^i}{\sigma_{\tau_1}^{p_{\tau_1}} \cdot \sigma_{\tau_2}^{p_{\tau_2}}} \tag{15} \\ &= v_{\tau_1, \tau_2}^{i, p_{\tau_1}, p_{\tau_2}}, \tau_1, \tau_2 = 1, \dots, \Lambda. \end{aligned}$$

Thus, for calculation of moments (13), it is required to calculate the product moments of Λ -dimensional ($\Lambda \in \{1, \dots, 2\nu\}, \nu = 1, \dots, M$) normally distributed random vectors for which the components of the vectors of mathematical expectations and the covariance matrices are calculated by formulae (15).

Let us designate

$$\begin{aligned} \mathbf{b}^{i, \Lambda} &= (b_1^{i, p_1}, \dots, b_\Lambda^{i, p_\Lambda})_{1 \times \Lambda}^T, \\ \mathbf{V}^{i, \Lambda} &= \begin{pmatrix} v_{1,1}^{i, p_1, p_1} & v_{1,2}^{i, p_1, p_2} & \dots & v_{1,\Lambda}^{i, p_1, p_\Lambda} \\ v_{2,1}^{i, p_2, p_1} & v_{2,2}^{i, p_2, p_2} & \dots & v_{2,\Lambda}^{i, p_2, p_\Lambda} \\ \dots & \dots & \dots & \dots \\ v_{\Lambda,1}^{i, p_\Lambda, p_1} & v_{\Lambda,2}^{i, p_\Lambda, p_2} & \dots & v_{\Lambda,\Lambda}^{i, p_\Lambda, p_\Lambda} \end{pmatrix}_{\Lambda \times \Lambda}, \end{aligned} \tag{16}$$

and the corresponding random vector—by

$$\mathbf{y} = (y_1, \dots, y_\Lambda)^T, \text{ i.e.}$$

$$y_1 = \frac{x_1 - a_1^{p_1}}{\sigma_1^{p_1}}, \quad y_2 = \frac{x_2 - a_2^{p_2}}{\sigma_2^{p_2}}, \dots, \quad y_\Lambda = \frac{x_\Lambda - a_\Lambda^{p_\Lambda}}{\sigma_\Lambda^{p_\Lambda}}.$$

For calculation of conditional product moments of the 2ν -order, we have

$$\begin{aligned} E \left(y_1^{m_1} y_2^{m_2} \dots y_\Lambda^{m_\Lambda} \middle| H_i \right) &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} y_1^{m_1} y_2^{m_2} \dots y_\Lambda^{m_\Lambda} \\ &\times f(y_1, y_2, \dots, y_\Lambda | H_i) dy_1 dy_2 \dots dy_\Lambda, \end{aligned} \tag{17}$$

where $f(y_1, y_2, \dots, y_\Lambda)$ is the Λ -dimensional normal probability distribution density with the vector of mathematical expectations and the covariance matrix calculated by formulae (16).

It is known that the value of integral (17) is invariant to linear transformation of the components of vector \mathbf{x} [18] with the accuracy of Jacobian of Transformation. Let us designate the matrices of eigenvectors and eigenvalues of matrix $(\mathbf{V}^{i, \Lambda})^{-1}$ by $\boldsymbol{\beta}^{i, \Lambda} = \left\| \beta_{ij}^i \right\|_{\Lambda \times \Lambda}$ and

$\mathbf{K}^{i, \Lambda} = \left\| k_{ij}^i \right\|_{\Lambda \times \Lambda}$, respectively. It should be pointed out that $\mathbf{K}^{i, \Lambda}$ is a diagonal matrix. Then the components of Λ -

dimensional random vector

$$\mathbf{Z}^{i,\Lambda} = \boldsymbol{\beta}^{i,\Lambda} (\mathbf{K}^{i,\Lambda})^{-1} (\mathbf{y} - \mathbf{b}^{i,\Lambda}), \tag{18}$$

will be uncorrelated and will have standard normal distribution of probabilities [3,20].

From (18), we write

$$\mathbf{y} = \mathbf{K}^{i,\Lambda} (\boldsymbol{\beta}^{i,\Lambda})^{-1} \mathbf{Z}^{i,\Lambda} + \mathbf{b}^{i,\Lambda}.$$

Let us introduce the following designation

$$\boldsymbol{\gamma}^{i,\Lambda} = \|\boldsymbol{\gamma}_{\tau,t}^{i,\Lambda}\|_{\Lambda \times \Lambda} = \mathbf{K}^{i,\Lambda} (\boldsymbol{\beta}^{i,\Lambda})^{-1}.$$

Then, for the elements of the vector \mathbf{y} , we obtain the following expression

$$y_\tau = \sum_{t=1}^{\Lambda} \gamma_{\tau,t}^{i,\Lambda} z_t^{i,\Lambda} + b_\tau^{i,\Lambda}, \tau = 1, \dots, \Lambda. \tag{19}$$

Using transformation (19), for mathematical expectation (17), we obtain

$$E(y_1^{m_1} y_2^{m_2} \dots y_\Lambda^{m_\Lambda} | H_i) = \mathbf{J}^{i,\Lambda} \times E \left[\prod_{\tau=1}^{\Lambda} \left(\sum_{t=1}^{\Lambda} \gamma_{\tau,t}^{i,\Lambda} z_t^{i,\Lambda} + b_\tau^{i,\Lambda} \right)^{m_\tau} \middle| H_i \right], \tag{20}$$

where $\mathbf{J}^{i,\Lambda} = \text{mod} \left| \mathbf{K}^{i,\Lambda} (\boldsymbol{\beta}^{i,\Lambda})^{-1} \right|$ is the Jacobian of Transformation (18).

Let us raise to the powers the linear forms in the right-hand side of expression (20) and group the identical items. Then (20) can be written as

$$E(y_1^{m_1} y_2^{m_2} \dots y_\Lambda^{m_\Lambda} | H_i) = \mathbf{J}^{i,\Lambda} \times \sum_{\tau_i \in \{0,1,2,\dots,2\nu\}; i=1,\dots,\Lambda} d_{\tau_1,\dots,\tau_\Lambda}^{i,\Lambda} \prod_{\eta=1}^{\Lambda} E \left[(z_\eta^{i,\Lambda})^{\tau_\eta} \middle| H_i \right], \tag{21}$$

where, the coefficients of the identical items in (20) are designated by $d_{\tau_1,\dots,\tau_\Lambda}^{i,\Lambda}$, $\tau_i \in \{0,1,2,\dots,2\nu\}; i=1,\dots,\Lambda$; the items of the vector $\mathbf{Z}^{i,\Lambda}$ are determined as

$$z_\eta^{i,\Lambda} = \sum_{\ell=1}^{\Lambda} \left[\left(\sum_{\delta=1}^{\Lambda} \beta_{\eta,\delta}^{i,\Lambda} \mathbf{K}_{\delta,\ell}^i \right) (y_\ell - b_\ell^{i,\Lambda}) \right], \eta = 1, \dots, \Lambda.$$

It is known that [21,29]

$$E \left[(z_\eta^{i,\Lambda})^{\tau_\eta} \middle| H_i \right] = \mu'_{\tau_\eta} = \sum_{j=0}^{\tau_\eta} \binom{\tau_\eta}{j} \mu_{\tau_\eta-j} \left(\mu'_j \right)^j, \tag{22}$$

where μ'_{τ_η} and $\mu_{\tau_\eta-j}$ are the initial and central moments of τ_η and $\tau_\eta - j$ orders, respectively, of random variable $z_\eta^{i,\Lambda}$. After simple routine transformations, for the considered case we obtain

$$\mu_j = \begin{cases} \frac{(Q_{z,\eta}^{i,\Lambda})^j}{2^{j/2}} \frac{j!}{(j/2)!}, & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases} \tag{23}$$

Here

$$\begin{aligned} V(z_\eta^{i,\Lambda} | H_i) &= Q_{z,\eta}^{i,\Lambda} \\ &= \sum_{\ell_1=1}^{\Lambda} \sum_{\ell_2=1}^{\Lambda} C_{\eta,\ell_1}^{i,\Lambda} D_{\ell_2}^{i,p_{\ell_2}} \times \left(\rho_{\ell_1,\ell_2}^i + b_{\ell_1}^{i,p_{\ell_1}} b_{\ell_2}^{i,p_{\ell_2}} \right) \\ &\quad - \left(\sum_{\ell=1}^{\Lambda} C_{\eta,\ell}^{i,\Lambda} b_\ell^{i,p_\ell} - D_\ell^{i,p_\ell} \right)^2, \tag{24} \\ C_{\eta,\ell}^{i,\Lambda} &= \sum_{\delta=1}^{\Lambda} \beta_{\eta,\delta}^i \mathbf{K}_{\delta,\ell}^i, \\ D_\ell^{i,p_\ell} &= \left(\sum_{\delta=1}^{\Lambda} \beta_{\eta,\delta}^i \mathbf{K}_{\delta,\ell}^i \right) b_\ell^{i,p_\ell}. \end{aligned}$$

Taking advantage of ratios (21), (22), for computation of the moments (13), we obtain expression (14).

Probability integral (10) can be computed with the help of Edgewort's series [19-21] using formula (14) for computation of the initial moments of random variable (9). In particular, in the considered case, using well-known techniques of obtaining Edgewort's series [30], we have

$$\begin{aligned} \int_{-\infty}^0 p_j(z_i | H_i) dz &\approx \hat{\pi}_j(0 | H_i) = \Phi(z_i^*) \\ &+ \left\{ \frac{1}{3!} \frac{[\zeta_3]_j^i}{(\mu_2^{j,i})^{3/2}} \times (z_i^{*3} - 3z_i^*) + \frac{1}{4!} \frac{[\zeta_4]_j^i}{(\mu_2^{j,i})^2} (z_i^{*4} - 6z_i^{*2} + 3) \right. \\ &+ \frac{10}{6!} \left(\frac{[\zeta_3]_j^i}{(\mu_2^{j,i})^{3/2}} \right)^2 (z_i^{*6} - 15z_i^{*4} + 45z_i^{*2} - 15) \\ &+ \frac{1}{5!} \frac{[\zeta_5]_j^i}{(\mu_2^{j,i})^{5/2}} (z_i^{*5} - 10z_i^{*3} + 15z_i^*) - \frac{35}{7!} \frac{[\zeta_3]_j^i [\zeta_4]_j^i}{(\mu_2^{j,i})^{7/2}} \\ &\times (z_i^{*7} - 21z_i^{*5} + 105z_i^{*3} - 105z_i^*) - \frac{280}{9!} \left(\frac{[\zeta_3]_j^i}{(\mu_2^{j,i})^{3/2}} \right)^3 \\ &\times (z_i^{*9} - 36z_i^{*7} + 378z_i^{*5} - 1260z_i^{*3} + 945z_i^*) \\ &+ \frac{1}{6!} \frac{[\zeta_6]_j^i}{(\mu_2^{j,i})^3} (z_i^{*6} - 15z_i^{*4} + 45z_i^{*2} - 15) + \frac{35}{8!} \left(\frac{[\zeta_4]_j^i}{(\mu_2^{j,i})^2} \right)^2 \\ &\times (z_i^{*8} - 28z_i^{*6} + 210z_i^{*4} - 420z_i^{*2} + 105) \\ &+ \frac{2100}{10!} \frac{[\zeta_3]_j^i [\zeta_4]_j^i}{(\mu_2^{j,i})^5} \times (z_i^{*10} - 45z_i^{*8} + 630z_i^{*6} \\ &- 3150z_i^{*4} + 4725z_i^{*2} - 945) + \frac{23100}{12!} \left(\frac{[\zeta_3]_j^i}{(\mu_2^{j,i})^{3/2}} \right)^4 \\ &\times (z_i^{*12} - 66z_i^{*10} + 1485z_i^{*8} - 13860z_i^{*6} \\ &\left. + 51975z_i^{*4} - 62370z_i^{*2} + 10395) \right\} \alpha(z_i^*), \tag{25} \end{aligned}$$

where $z_1^* = (-\mu_1^{j,i}) / \sqrt{\mu_2^{j,i}}$; $[\zeta_k]_j^i$, ($k=1,2,\dots$) is the k th semi-invariant of the random variable ξ_j provided hypothesis H_i is true (the computation of semi-invariants is not difficult knowing all initial moments including $\mu_k^{j,i}$ (see, for example, [21])); $\mu_2^{j,i}$ is the second central moment; $\alpha(x)$ is standard normal density, i.e. $\alpha(x) = \exp(-x^2/2) / \sqrt{2\pi}$.

Satisfying the first seven terms in expansion (25), the absolute value of calculation error of the probability integral is calculated by the formula

$$\begin{aligned} |\Delta_i^j| \leq & \left| \left\{ \frac{1}{6!} \frac{[\zeta_6]_j^i}{(\mu_2^{j,i})^3} (z_1^{*6} - 15z_1^{*4} + 45z_1^{*2} - 15) \right. \right. \\ & + \frac{35}{8!} \left(\frac{[\zeta_4]_j^i}{(\mu_2^{j,i})^2} \right)^2 (z_1^{*8} - 28z_1^{*6} + 210z_1^{*4} - 420z_1^{*2} + 105) \\ & + \frac{2100}{10!} \frac{[\zeta_3^2]_j^i [\zeta_4]_j^i}{(\mu_2^{j,i})^5} (z_1^{*10} - 45z_1^{*8} + 630z_1^{*6} - 3150z_1^{*4} \\ & + 4725z_1^{*2} - 945) + \frac{23100}{12!} \left(\frac{[\zeta_3]_j^i}{(\mu_2^{j,i})^{3/2}} \right)^4 \\ & \left. \left. (z_1^{*12} - 66z_1^{*10} + 1485z_1^{*8} - 13860z_1^{*6} + 51975z_1^{*4} \right. \right. \\ & \left. \left. - 62370z_1^{*2} + 10395) \right\} \alpha(z_1^*) \right|. \end{aligned}$$

The variable $\xi_j = \sum_{\ell=1}^S C_\ell^j \exp(-y_\ell)$ is continuous and unambiguously defined for every value of x . Therefore, the random variable ξ_j is continuous. The characteristic function of the random variable ξ_j and its derivatives of any order exist, as the moments of any order of this random variable exist. At the same time, the characteristic function is uniformly continuous. Consequently, the distribution function of this random variable exists and is continuous [21].

Theorem 3.2. *The distribution function of random variable ξ_j exists and is uniquely determined by moments (14).*

Proof. For proving this theorem, it is necessary to show that all moments $\mu_r^{j,i}$, $r=1,2,\dots$ exist and the following condition takes place [19,21]

$$\limsup_{r \rightarrow \infty} \frac{(\mu_{2r}^{j,i})^{1/2r}}{2r} < \infty.$$

The fact that all moments exist is obvious from formula (14) as by using it, it is possible to calculate the moments of any order with any specified accuracy. The values of these moments exist and are finite.

When solving the practical problems coefficients k_ℓ^k

take on the values bounded above; correlation matrices W_ℓ are positively determined matrices the determinants of which differ from zero. Therefore, coefficients $|C_\ell^j|$ are bounded-above quantities.

There takes place

$$\begin{aligned} E \left[\exp(-y_{\ell_1} + \dots + y_{\ell_r}) \middle| H_i \right] &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{-(y_{\ell_1} + \dots + y_{\ell_r})} \\ &\times N(\mathbf{x} | \mathbf{a}^i, \mathbf{W}_i) d\mathbf{x}, \end{aligned}$$

where $y_{\ell_1} + \dots + y_{\ell_r}$ is the sum of quadratic forms of normally distributed n -dimensional vector x at different vectors of mathematical expectations and covariance matrices. Therefore, at changing components of the vector x from $-\infty$ up to $+\infty$, the quadratic form $y_{\ell_1} + \dots + y_{\ell_r}$ takes the values from 0 to $+\infty$, and the value of function $e^{-(y_{\ell_1} + \dots + y_{\ell_r})}$ varies from 1 to 0, respectively. Therefore [18]

$$\begin{aligned} E \left[\exp(-y_{\ell_1} + \dots + y_{\ell_r}) \middle| H_i \right] \\ \leq \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} N(\mathbf{x} | \mathbf{a}^i, \mathbf{W}_i) d\mathbf{x} = 1 \end{aligned}$$

Thus, taking into account (5) and (11) we can write down

$$\begin{aligned} |\mu_r^{j,i}| &\leq \left| \sum_{\ell_1=1}^S \dots \sum_{\ell_r=1}^S C_{\ell_1}^j \dots C_{\ell_r}^j \right|, \\ &\leq A^r (2\pi)^{-\frac{nr}{2}} \sum_{\ell_1=1}^S \dots \sum_{\ell_r=1}^S \prod_{v=1}^r p(H_{\ell_v}) |\mathbf{W}_{\ell_v}|^{-1} \end{aligned}$$

where A is the maximum by absolute value among coefficients k_ℓ^j .

Assume $r=1$, then we have

$$|\mu_1^{j,i}| \leq A(2\pi)^{-\frac{n}{2}} \sum_{\ell=1}^S p(H_\ell) |\mathbf{W}_\ell|^{-1}.$$

Let us designate $|\mathbf{W}_{\min}| = \min_{\{\ell\}} |\mathbf{W}_\ell|$. Then

$$\mu_1^{j,i} \leq A(2\pi)^{-\frac{n}{2}} |\mathbf{W}_{\min}|^{-1} \sum_{\ell=1}^S p(H_\ell) = A(2\pi)^{-\frac{n}{2}} |\mathbf{W}_{\min}|^{-1}.$$

If $A \leq (2\pi)^{\frac{n}{2}} |\mathbf{W}_{\min}|$, then $\mu_1^{j,i} \leq 1$ and it is not difficult to be convinced that $\mu_r^{j,i} \rightarrow 0$ at $r \rightarrow \infty$.

Let $A = C(2\pi)^{\frac{n}{2}} |\mathbf{W}_{\min}|$, where $C > 1$. Hence $\mu_r^{j,i} \leq C^r S^r$ and

$$\begin{aligned} \frac{(\mu_{2r}^{j,i})^{1/2r}}{2r} &\leq \frac{(C^{2r} S^{2r})^{1/2r}}{2r} = \frac{CS^{1/2r} (2r)^{1/2r}}{2r} \\ &= CS^{1/2r} (2r)^{\frac{2r-1}{2r}} = \frac{CS^{1/2r}}{(2r)^{\frac{2r-1}{2r}}} \rightarrow \frac{C}{2r} \rightarrow 0 \end{aligned}$$

at $r \rightarrow \infty$, which proves the theorem.

4. Computation Results

The accuracy of this algorithm depends heavily on M - the number of used items in expansion (12). In order to increase the accuracy of approximation of the exponent for given M and, in general, the reliability of computation in the tasks of hypotheses testing, it is expedient to perform first the normalization of initial data by formulae:

$$x'_i = (x - c_i) / (d_i - c_i),$$

$$\rho'_{ij} = \rho_{ij} / \sqrt{(d_i - c_i) \cdot (d_j - c_j)}, i, j = 1, \dots, n$$

where $c_i, d_i, i = 1, \dots, n$, are the minimum and maximum values of the i th parameter for the given set of the considered hypotheses, i.e. $c_i = \min_{\{j\}} \{a_i^j\}, d_i = \max_{\{j\}} \{a_i^j\}, i = 1, \dots, n, j = 1, \dots, S$ [3]. In this case, the values of the parameters of the algorithm $M = 15$ and seven items in expansion (25) provided the absolute error of computation of integral (1) that does not exceed 0.005 for computed examples (see below). This fact was established by modeling for the observation vector with non-correlated components. Unfortunately, by now the considered algorithm has been realized only for such a case [31].

The results of simulation showed that the time of execution of the task (decision-making and computation of the suitable value of the risk function) by using the Monte-Carlo method made up ≈ 1.2 sec, ≈ 4.5 sec, and ≈ 13 sec for the number of hypotheses $S = 3, S = 4$ and $S = 5$, respectively, the dimensionality of the observed vector being equal to $n = 8$ in all cases. The tested hypotheses and correlation matrix for the case $S = 5$ are given in tables of **Figure 1** and **Figure 2**, respectively. Figures are presented as the suitable forms of the task of hypotheses test of the statistical software in which the appropriate methods are realized [31]. For other values of S , there are chosen the suitable sub-sets of the tables of these Figures. In the first column of the table of **Figure 1** is given the vector of measurements and in the other columns are given hypothetical values of mathematical expectation of this vector.

Meanwhile, when using the method offered here, the computation time did not practically change and the results were obtained for the time noticeably less than 1 sec. In both cases probability integrals (7) were computed with the accuracy of ≤ 0.005 . In **Figures 3** and **4** are given the dependences of the integral computation time on the accuracy and number of tested hypotheses respectively.

At solving many practical problems, especially military problems [5,32], the dimensionality of the integrals

The screenshot shows a software window titled "Data input" with a menu bar (File, Edit, Tools, Help) and a toolbar. Below the toolbar, there is a language dropdown set to "English" and a title bar for the data table: "deci84. \$sf". The table has two input fields: "N = 8" and "m = 6". The main table contains 8 rows and 6 columns of numerical data.

j	x_j	a_{1j}	a_{2j}	a_{3j}	a_{4j}	a_{5j}
1	2.010000	1.010000	4.010000	7.010000	10.01000	13.00000
2	2.020000	1.020000	4.020000	7.020000	10.02000	13.00000
3	2.030000	1.030000	4.030000	7.030000	10.03000	13.00000
4	2.040000	1.040000	4.040000	7.040000	10.04000	13.00000
5	2.050000	1.050000	4.050000	7.050000	10.05000	13.00000
6	2.060000	1.060000	4.060000	7.060000	10.06000	13.00000
7	2.070000	1.070000	4.070000	7.070000	10.07000	13.00000
8	2.080000	1.080000	4.080000	7.080000	10.08000	13.00000

Figure 1. The form of entering the tested hypotheses and a measurement vector.

The screenshot shows a software window titled "Parameters" with a menu bar (File, Edit, Tools, Help) and a toolbar. Below the toolbar, there is a language dropdown set to "English" and a title bar for the covariance matrix: "##deci84. \$sa". The main table is titled "Covariance matrix" and contains 8 rows and 8 columns of numerical data.

j\i	1	2	3	4	5	6	7	8
1	1.79560	1.36360	1.93160	2.49960	3.06760	3.63560	4.20360	4.77160
2	1.36360	2.40320	1.97920	2.55520	3.13120	3.70720	4.28320	4.85920
3	1.93160	1.97920	3.02680	2.61080	3.19480	3.77880	4.36280	4.94680
4	2.49960	2.55520	2.61080	3.66640	3.25840	3.85040	4.44240	5.03440
5	3.06760	3.13120	3.19480	3.25840	4.32200	3.92200	4.52200	5.12200
6	3.63560	3.70720	3.77880	3.85040	3.92200	4.99360	4.60160	5.20960
7	4.20360	4.28320	4.36280	4.44240	4.52200	4.60160	5.68120	5.29720
8	4.77160	4.85920	4.94680	5.03440	5.12200	5.20960	5.29720	6.38480

Figure 2. The form of entering the covariance matrix.

like (1) often is equal to several tens and difference between the computation time necessary for the considered methods is significantly longer than in the above mentioned case [14], whereas the computation time for solving the defence problems are of great importance.

The theoretical investigation of the dependence of the accuracy of computation of integral (1) on M -the number of items in expansion (13) is a challenging task. Therefore, at program realization of the offered algorithm and, in general, algorithms of such a kind, it is worthwhile to

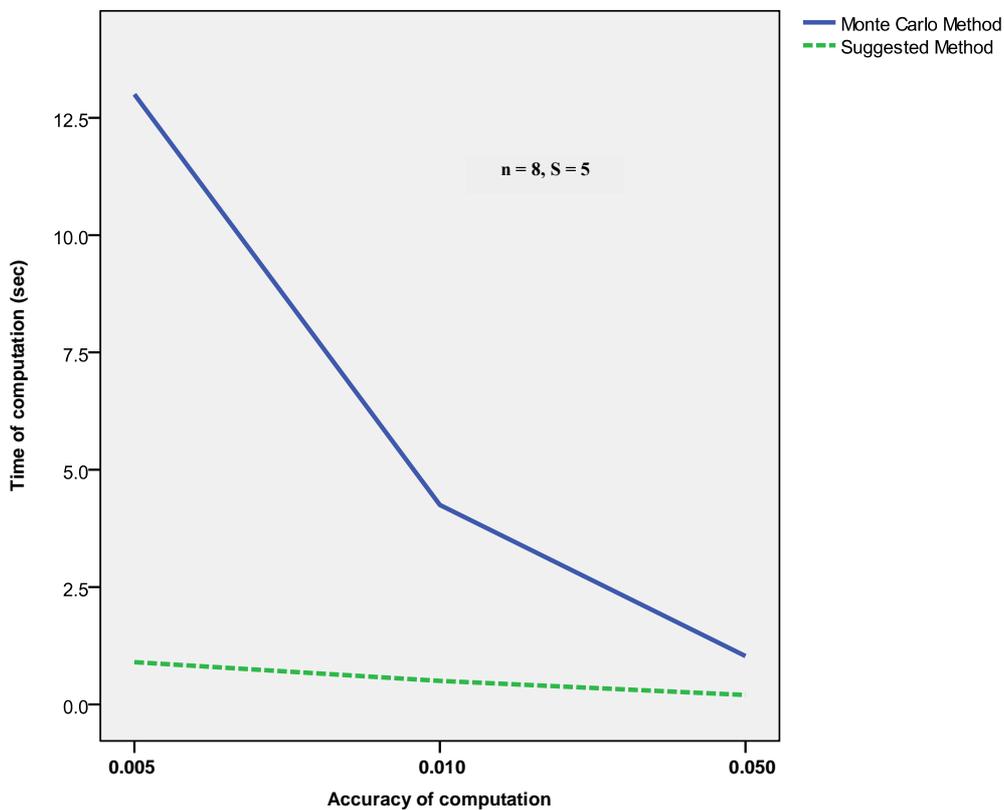


Figure 3. Dependence of the integral computation time on the accuracy.

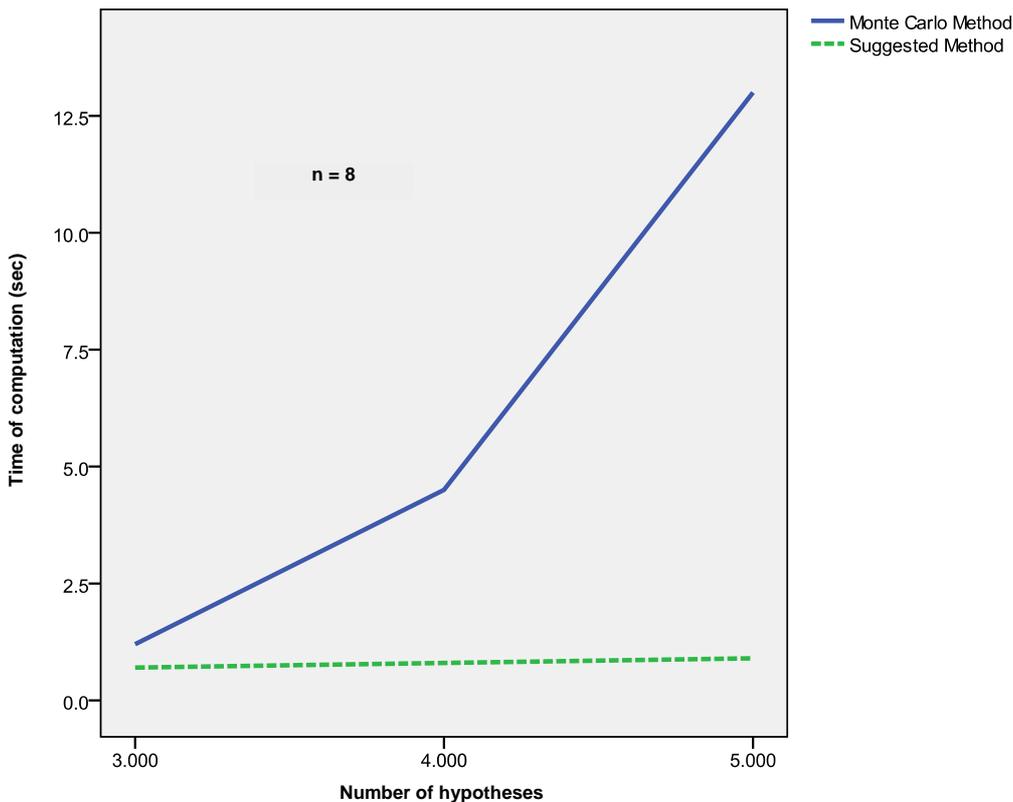


Figure 4. Dependence of the integral computation time on the number of hypotheses.

make parameter M and the number of items in expansion (25) external parameters of the program. This allows establishing their optimal values for each concrete case by experimentation depending on the desired accuracy of computation.

5. Conclusion

The method of computation of the probability integral from the multivariate normal density over the Complex Subspace by using series and the reduction of dimensionality of the multidimensional integral to one without losing the information was developed. The formulae for computation of product moments of normalized normally distributed random variables were also derived. The existence of the probability distribution law of the weighted sum of exponents of negative quadratic forms of the normally distributed random vector was justified. The opportunity of its unambiguous determination by the computed moments was proved.

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