

From Normal vs Skew-Normal Portfolios: FSD and SSD Rules

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ABSTRACT

In this paper we study stochastic dominance rules of first and second order for univariate skew-normal random variables, the analysis being relevant in connection with the problem of portfolio choice in stock markets showing departure from the classical assumption of normality on returns. Besides that, our analysis is also relevant for markets where stocks returns are normally distributed: if standard derivatives are tradable and straddles, characterized by V-shaped pay-outs, are implementable at specific strike prices, then, portfolios including them, can exhibit exact skew-normality in their returns. We provide a set of simple conditions on the statistical parameters of the distributions which imply FSD and SSD and discuss some application of our criteria.

Keywords: Normal Distribution; Skew-Normal Distribution; Mixture Distributions; Stochastic Ordering; Stochastic Dominance; Portfolio Selection; Derivatives; Straddles

1. Introduction

Ranking portfolio return distributions is one of the key procedures which can be used by quantitative analysts to provide support to the decisional processes of portfolio managers. In this paper we discuss ranking criteria for skew-normal distributions based on stochastic dominance rules of first and second order; we refer the reader to Levy ([1]) for a detailed exposition of these concepts and many related results.

The economical motivation underpinning second order stochastic dominance (SSD) is particularly appealing: a representative investor with increasing and concave utility function will always prefer a portfolio w_1 to a portfolio w_2 if the returns of w_1 stochastically dominate at the second order the returns of w_2 . SSD encapsulates therefore the concept of risk-aversion. Investments theorists and practitioners have developed along the times several methods to compare risky investments prospects or portfolios of assets and choose among the feasible ones in some optimal way: the review paper Brandt ([2]) and the book Sharpe *et al.* ([3]) discuss many of these methodologies. Probably the Markowitz's mean-variance framework, Markowitz ([4]), remains the most famous approach, even if it has well documented intrinsic limitations. There have been various attempts to overcome some of these limitations: by taking

into account higher distributional moments as in Athayde *et al.* ([5]), by investigating bayesian versions of the Markowitz's idea as in Pastor ([6]) and Polson ([7]), or by proposing alternative or more general risk measures as in Konno and Yamazaki ([8]), Markowitz *et al.* ([9]) and Feiri *et al.* ([10]). A very original and influential contribution to the asset allocation problem has been given by Black and Litterman in Black and Litterman ([11]): there the authors nicely incorporate in their framework subjective beliefs or "views" on expected future returns of assets (see Meucci ([12]) for a survey and extensions). Skewnormally distributed returns, which we handle in this paper, have been already considered in portfolio theory by Adcock and Shutes ([13]) and Harvey *et al.* ([14]), furthermore Adcock ([15]) and Bacmann and Massi-Benedetti ([16]) contain interesting applications to hedge funds portfolios. However, to our best knowledge, the stochastic dominance approach for comparing skew-normally distributed returns is considered here for the first time. Notice that stochastic dominance rules aim to compare the whole distribution and not just a limited number of its moments. Finally we refer to Post ([17]) for applications of stochastic dominance rules to empirical data. This paper is organized as follows. In the next section we consider markets with skew-normal returns and portfolios on these markets; in Section 3 we show that even in markets with normal returns in the basic securities there can be portfolios exhibiting skew-normality in their returns if on the same market derivatives

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based strategies with V-shaped payouts can be implemented. In Section 4 we prove our stochastic dominance criteria and illustrate some possible applications.

2. Portfolios in Skew-Normal Markets

Following Azzalini ([18]) let us recall that a random vector $\mathbf{X} \in \mathbb{R}^n$ is said to follow the skew-normal distribution with location parameter $\mathbf{m} \in \mathbb{R}^n$, $(n \times n)$ -scale matrix Ω and shape parameter $\mathbf{a} \in \mathbb{R}^n$ if its density has the form:

$$f_{\mathbf{X}}(\mathbf{x}) = 2\varphi_n(\mathbf{x}; \mathbf{m}, \Omega) \Phi(\mathbf{a}' \boldsymbol{\omega}^{-1}(\mathbf{x} - \mathbf{m})) \quad (1)$$

where $\boldsymbol{\omega}$ is the diagonal matrix

$$\boldsymbol{\omega} = \text{diag}(\sqrt{\Omega_{11}}, \dots, \sqrt{\Omega_{nn}}), \quad \sqrt{\Omega_{ii}} \neq 0 \quad \text{for } i = 1, \dots, n,$$

$\varphi_n(\mathbf{y}; \mathbf{m}, \Omega)$ is the density of a $N_n(\mathbf{m}, \Omega)$ -random vector and $\Phi(\cdot)$ is the cumulative distribution function of a univariate standard normal. In this case we write $X \sim SN_n(\mathbf{m}, \Omega, \mathbf{a})$. In particular, in the univariate case, X has the skew-normal distribution of parameters $\mu, \alpha \in \mathbb{R}$ and $\sigma := \omega_1 > 0$ if for all x

$$P(X \leq x) = \frac{2}{\sigma} \int_{-\infty}^x \varphi\left(\frac{z - \mu}{\sigma}\right) \Phi\left(\alpha \frac{z - \mu}{\sigma}\right) dz,$$

where $\varphi(y) := \varphi_1(y, 0, 1)$. In this case we write $X \sim SN_1(\mu, \sigma^2, \alpha)$. Clearly for $\alpha = 0$ we have

$X \sim N_1(\mu, \sigma^2)$. For $\alpha \neq 0$ the distribution is skewed and indeed its mean, variance and skewness can be easily computed. They are respectively given by:

$$\begin{aligned} \mathbb{E}(X) &= \mu + \sqrt{\frac{2}{\pi}} \sigma \delta, \quad \text{var}(X) = \sigma^2 \left(1 - \frac{2}{\pi} \delta^2\right), \\ \text{skew}(X) &= \frac{4(4 - \pi)}{(2\pi)^{3/2}} (\sigma \delta)^3 \end{aligned}$$

with $\delta = \delta_\alpha := \alpha(1 + \alpha^2)^{-1/2} \in (-1, 1)$.

We refer the reader to Arellano and Azzalini ([19]) and Genton ([20]) for further properties and many interesting applications.

Let us now consider a market with n basic risky assets whose future returns, at time $T > 0$, are described by a random vector \mathbf{R} such that:

$$\mathbf{R} \sim SN_n(\mathbf{m}, \Omega, \mathbf{a}). \quad (2)$$

An investor, the decision maker, is facing with the problem of allocating her initial capital on the market by choosing a portfolio $\mathbf{w} = (w_1, \dots, w_n)$, with $\sum w_i = 1$ and holding it unchanged up to time T . The portfolio return is given by the random variable $R_{\mathbf{w}} = \mathbf{w}' \mathbf{R}$ with:

$$R_{\mathbf{w}} \sim SN_1(\mu_{\mathbf{w}}, \sigma_{\mathbf{w}}^2, \alpha_{\mathbf{w}}), \quad (3)$$

and parameters $\mu_{\mathbf{w}} = \mathbf{w}' \mathbf{m}$, $\sigma_{\mathbf{w}}^2 = \mathbf{w}' \Omega \mathbf{w}$, and

$$\alpha_{\mathbf{w}} = \frac{\sigma_{\mathbf{w}}^{-1} \mathbf{H}' \mathbf{a}}{\sqrt{1 + \mathbf{a}' (\bar{\Omega} - \mathbf{H} \sigma_{\mathbf{w}}^{-2} \mathbf{H}') \mathbf{a}}}, \quad (4)$$

where $\mathbf{H} = \boldsymbol{\omega}^{-1} \Omega \mathbf{w}$ and $\bar{\Omega} = \boldsymbol{\omega}^{-1} \Omega \boldsymbol{\omega}^{-1}$, see Azzalini ([18]). Henceforth it is clear that in order to compare the returns of two different portfolios \mathbf{w}_1 and \mathbf{w}_2 on this market we must have at our disposal criteria which allow us to compare two univariate skew-normal random variables. In Section 4 we prove simple criteria based on stochastic dominance, Levy ([1]). Although we only make use of standard techniques, to our best knowledge these results have never been discussed in the classical literature on the subject. An investor can use them to discard a portfolio or a family of portfolios whose returns are stochastically dominated by the return of another portfolio.

3. Portfolios Including Derivative Securities in Normal Markets

In this section we briefly show how the skew-normal return distributions can naturally appear even in markets with normal returns in the primary risky assets, if in the same markets are traded derivative securities which incorporate non-linear pay-offs. More specifically consider markets for which the shape parameters of the basic risky assets are all zero ($\mathbf{a} = 0$), that is

$$\mathbf{R} \sim SN_n(\mathbf{m}, \Omega, 0) = N_n(\mathbf{m}, \Omega) \quad (5)$$

and suppose that a straddle on one of the assets is available at a cost v_0 and strike price K . To simplify the exposition we fix $n = 2$ and denote by A and B the market basic risky securities; therefore their returns over the interval $[0, T]$ will follow the law:

$$(R_A, R_B) \sim N_2(\mathbf{m}, \Omega) \quad (6)$$

with $\mathbf{m}' = (\mu_A, \mu_B)$ and $\Omega_{12} = \rho \sigma_A \sigma_B$. Let (S_0^A, S_0^B) be their present spot prices, a straddle is written on the first asset and will pay to the holder $|S_T^A - K|$, S_T^A denoting the asset price at maturity date T . Consider an investor who wishes to allocate percentages of her capital both on the straddle and on shares of asset B , while investing the remaining part on a risk-free bond. It turns out that for a strike value K equal to $S_0^A(1 + \mu_A)$ the portfolios returns will display skew-normality. This condition on K appears to be very stringent, however in practice we could have $K \approx S_0^A(1 + \mu_A)$ with returns distribution close to skew-normality. Notice that the case $\mu_A = 0$ corresponds to a straddle traded ‘‘at the money’’ (this particular case, and with $\rho = 0$, has been discussed by Blasi ([21])). Indeed denoting by a the percentage of initial capital invested on the straddle on asset A , by b the percentage of initial capital invested on asset B , the return of the portfolio $\mathbf{w}_{a,b} = (a, b, 1 - a - b)$, at time

T will be

$$\begin{aligned}
 R(a,b) &= a \left[\frac{S_T^A - S_0^A(1 + \mu_A) - v_0}{v_0} \right] + b(R_B - \mu_B) \\
 &\quad + b\mu_B + (1 - a - b)r \\
 &= a \left(\frac{S_0^A}{v_0} |R_A - \mu_A| - 1 \right) + b(R_B - \mu_B) + b\mu_B + (1 - a - b)r \\
 &= a\gamma_0\sigma_A |Z_A| + b\sigma_B Z_B - a(1+r) + b(\mu_B - r) + r
 \end{aligned} \tag{7}$$

where $\gamma_0 \equiv S_0^A/v_0$, $R(a,b) \equiv R_{w_{a,b}}$, r is the bond yield and standardized assets returns have been denoted by Z_A, Z_B . We notice that a decomposition analogous to (7) continues to hold also for the multi-asset case ($n \geq 2$), that is for portfolios investing on a straddle on one of the assets, on shares of the remaining $n - 1$ risky assets and on the risky-free asset. The following result is a straight-forward generalization of a result by Henze ([22]) and can be proven by elementary methods:

Proposition 3.1 *Let X, Y be correlated $N_1(0, 1)$ random variables. Let ρ denote the correlation coefficient, $|\rho| < 1$, and a_1, a_3 and $a_2 \neq 0$ real numbers. Then the random variable*

$$Z = a_1 |X| + a_2 Y + a_3$$

is a mixture of two skew-normally distributed random variables. Specifically, denoting by g_Z the probability density of Z , it holds:

$$g_Z(t) = \frac{1}{2} g_{X_+}(t) + \frac{1}{2} g_{X_-}(t)$$

where

$$\begin{aligned}
 &X_{\pm} \sim \\
 &SN_1 \left(a_3, a_1^2(\rho) + a_{2,\pm}^2(\rho), \frac{a_1^2(\rho) + a_{2,\pm}^2(\rho) \pm a_{2,\pm}(\rho)a_2}{a_1(\rho)a_2} \right),
 \end{aligned}$$

with $a_1(\rho) \equiv a_1 \sqrt{1 - \rho^2}$ and $a_{2,\pm}(\rho) \equiv -(\rho a_1 \pm a_2)$.

In particular for $\rho = 0$ we have:

$$Z \sim SN_1 \left(a_3, a_1^2 + a_2^2, \frac{a_1}{a_2} \right).$$

For example it is readily seen that for $\rho = 0$ the above result implies:

$$R(a,b) \sim SN_1(\mu(a,b), \sigma^2(a,b), \alpha(a,b)) \tag{8}$$

with parameters:

$$\mu(a,b) = -a(1+r) + b(\mu_B - r) + r \tag{9}$$

$$\sigma^2(a,b) = \gamma_0^2 \sigma_A^2 a^2 + \sigma_B^2 b^2 \tag{10}$$

$$\alpha(a,b) = \frac{a}{b} \left(\frac{\gamma_0 \sigma_A}{\sigma_B} \right). \tag{11}$$

We remark that given γ_0 and the assets volatilities σ_A, σ_B the shape parameter depends only on the ratio of the chosen allocation percentages and vanishes for $a = 0$. Shorting the straddle or the second asset produces portfolios with negative shape parameter in the returns distribution and hence with negative skewness. On the other hand positive skewness increases unboundedly as the percentage b approaches zero. Furthermore, for the above portfolios returns distributions scale and shape parameters satisfy the relation

$\sigma^2(a,b) = (\alpha^2(a,b) + 1)b^2\sigma_B^2$, which implies that scale changes with shape. In the case $\rho \neq 0$ the returns

$R(a,b)$ will be described, accordingly to the previous proposition, by a mixture of two skew-normal distributions $SN_1(\mu(a,b), \sigma_{\pm}^2(a,b), \alpha_{\pm}(a,b))$, the identification of $\alpha_{\pm}(a,b)$ and $\sigma_{\pm}(a,b)$ being straightforward. We remark that while on one hand finite mixture of normal-distributions have been already considered in the financial literature as possible flexible tool for modelling assets returns in portfolio selection problems (Buckley *et al.* ([23])), on the other hand finite mixture of skew-normal distributions up to now have received attention mainly in applied statistical settings, see for instance Lee *et al.* ([24]), but not in connection with financial applications.

An immediate consequence of the example discussed above is the fact that even in a normal market an investor can face the problem of comparing portfolios having returns which are skew-normally distributed. Once again the decision maker needs criteria which can help her allocation choice process by discarding portfolios which have dominated returns. In the next section we shall provide some simple, yet rigorous, stochastic dominance criteria.

4. Stochastic Dominance Results

We start recalling some basic definitions: given two real-valued random variables X_1 and X_2 we say that X_1 stochastically dominates at *first* order X_2 (FSD), and we shortly write $X_1 \succeq_{FD} X_2$, if

$$P(X_2 \leq x) \geq P(X_1 \leq x) \tag{12}$$

for all x ; we say that X_1 stochastically dominates at *second* order X_2 (SSD), and we shortly write $X_1 \succeq_{SD} X_2$, if

$$\int_{-\infty}^y P(X_2 \leq x) dx \geq \int_{-\infty}^y P(X_1 \leq x) dx \tag{13}$$

for all y . More generally, a random variable X_1 stochastically dominates another random variable X_2 at

the order n -th if

$$\int_{-\infty}^{z_{n-1}} \dots \int_{-\infty}^{z_1} P(X_2 \leq x) dx dz_1 \dots z_{n-2} \geq \int_{-\infty}^{z_{n-1}} \dots \int_{-\infty}^{z_1} P(X_1 \leq x) dx dz_1 \dots z_{n-2}$$

for all values z_{n-1} . If the previous inequalities hold strictly we say that we have strict dominance. Obviously $(n - 1)$ -th order dominance implies n -th order dominance.

The classical stochastic ordering results for univariate normal random variables $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are the following (ref. Levy (2005)):

- 1) If $\mu_1 \geq \mu_2$ and $\sigma_1 = \sigma_2$ then $X_1 \succeq_{FD} X_2$;
- 2) If $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$ then $X_1 \succeq_{SD} X_2$.

In particular we have strict dominance for $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$ (or $\mu_1 > \mu_2$ and $\sigma_1 = \sigma_2$).

To provide a generalization of these results to the skew-normal case we need the following:

Theorem 4.1:

- 1) Let $X_1 \sim SN(\mu, \sigma^2, \alpha_1)$ and

$X_2 \sim SN(\mu, \sigma^2, \alpha_2)$. Suppose $\alpha_1 \geq \alpha_2$, then

$X_1 \succeq_{FD} X_2$.

- 2) Let $X_1 \sim SN(\mu_1, \sigma^2, \alpha)$ and

$X_2 \sim SN(\mu_2, \sigma^2, \alpha)$. Suppose $\mu_1 \geq \mu_2$ then

$X_1 \succeq_{FD} X_2$.

- 3) Let $X_1 \sim SN(\mu, \sigma_1^2, \alpha)$ and $X_2 \sim SN(\mu, \sigma_2^2, \alpha)$. Suppose $\sigma_1 \leq \sigma_2$ and $\alpha \leq 0$, then $X_1 \succeq_{SD} X_2$.

- 4) Let $X_1 \sim SN(\mu, \sigma_1^2, \alpha_1)$ and

$X_2 \sim SN(\mu, \sigma_2^2, \alpha_2)$. Suppose $\sigma_1 \leq \sigma_2$ and

$\sigma_1 \alpha_1 = \sigma_2 \alpha_2 \sqrt{\frac{1 + \alpha_1^2}{1 + \alpha_2^2}}$, then $X_1 \succeq_{SD} X_2$.

Proof.

1) Let $X \sim SN(\mu, \sigma^2, \alpha)$ and denote by $F_{\mu, \sigma, \alpha}(x)$ the corresponding distribution function. For each fixed (μ, σ) and x and arbitrary α we consider the function $\alpha \rightarrow h(\alpha) := F_{\mu, \sigma, \alpha}(x)$. We have:

$$\begin{aligned} h'(\alpha) &= \frac{2}{\sigma} \int_{-\infty}^x \varphi\left(\frac{y-\mu}{\sigma}\right) \frac{\partial}{\partial \alpha} \Phi\left(\alpha \frac{y-\mu}{\sigma}\right) dy \\ &= 2 \int_{-\infty}^x \frac{y-\mu}{\sigma} \varphi\left(\frac{y-\mu}{\sigma}\right) \varphi\left(\alpha \frac{y-\mu}{\sigma}\right) d\left(\frac{y-\mu}{\sigma}\right) \\ &= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} t \varphi\left(t \sqrt{1+\alpha^2}\right) dt \\ &= \frac{2}{\sqrt{2\pi}(1+\alpha^2)} \int_{-\infty}^{\frac{\sqrt{1+\alpha^2}}{\sigma}(x-\mu)} t \varphi(t) dt \\ &= -\frac{2}{\sqrt{2\pi}(1+\alpha^2)} \varphi\left(\frac{\sqrt{1+\alpha^2}}{\sigma}(x-\mu)\right), \end{aligned}$$

where we have used $\sqrt{2\pi}\varphi(\alpha t)\varphi(\beta t) = \varphi(\sqrt{\alpha^2 + \beta^2}t)$

and $\varphi'(t) = -t\varphi(t)$. Therefore $h(\alpha)$ is decreasing and $F_{\mu, \sigma, \alpha_1}(x) \leq F_{\mu, \sigma, \alpha_2}(x)$ for all x .

2) For each fixed (σ, α) and x and arbitrary μ we consider the function $\mu \rightarrow k(\mu) := F_{\mu, \sigma, \alpha}(x)$. We have:

$$\begin{aligned} k'(\mu) &= \int_{-\infty}^x \frac{2}{\sigma} \frac{\partial}{\partial \mu} \left[\varphi\left(\frac{y-\mu}{\sigma}\right) \Phi\left(\alpha \frac{y-\mu}{\sigma}\right) \right] dy \\ &= \frac{2}{\sigma} \int_{-\infty}^x \left(\frac{y-\mu}{\sigma} \right) \varphi\left(\frac{y-\mu}{\sigma}\right) \Phi\left(\alpha \frac{y-\mu}{\sigma}\right) d\left(\frac{y-\mu}{\sigma}\right) \\ &\quad - \frac{2\alpha}{\sigma} \int_{-\infty}^x \varphi\left(\frac{y-\mu}{\sigma}\right) \varphi\left(\alpha \frac{y-\mu}{\sigma}\right) d\left(\frac{y-\mu}{\sigma}\right) \\ &= -\frac{2}{\sigma} \int_{-\infty}^{\frac{x-\mu}{\sigma}} \varphi'(t) \Phi(\alpha t) dt - \frac{2\alpha}{\sigma} \int_{-\infty}^{\frac{x-\mu}{\sigma}} \varphi(t) \varphi(\alpha t) dt \\ &= -\frac{2}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right) \Phi\left(\alpha \frac{x-\mu}{\sigma}\right) \end{aligned}$$

by integrating by parts the first of the two integrals.

Therefore $k(\mu)$ is decreasing and

$F_{\mu_1, \sigma, \alpha}(x) \geq F_{\mu_2, \sigma, \alpha}(x)$ for all x .

3) For each fixed (μ, α) and y and arbitrary σ we consider the function $\sigma \rightarrow l(\sigma) := \int_{-\infty}^y F_{\mu, \sigma, \alpha}(x) dx$. We have:

$$\begin{aligned} l'(\sigma) &= \int_{-\infty}^y \frac{\partial}{\partial \sigma} \left[\int_{-\infty}^x \frac{2}{\sigma} \varphi\left(\frac{z-\mu}{\sigma}\right) \Phi\left(\alpha \frac{z-\mu}{\sigma}\right) dz \right] dx \\ &= 2 \int_{-\infty}^y \frac{\partial}{\partial \sigma} \left[\int_{-\infty}^{\frac{x-\mu}{\sigma}} \varphi(t) \Phi(\alpha t) dt \right] dx \\ &= -2 \int_{-\infty}^y \frac{x-\mu}{\sigma^2} \varphi\left(\frac{x-\mu}{\sigma}\right) \Phi\left(\alpha \frac{x-\mu}{\sigma}\right) dx \\ &= -2 \int_{-\infty}^{\frac{y-\mu}{\sigma}} t \varphi(t) \Phi(\alpha t) dt \end{aligned}$$

and by integrating by parts the last integral

$$\begin{aligned} l'(\sigma) &= 2\varphi\left(\frac{y-\mu}{\sigma}\right) \Phi\left(\alpha \frac{y-\mu}{\sigma}\right) \\ &\quad - \frac{2\alpha}{\sqrt{2\pi}(1+\alpha^2)} \Phi\left(\sqrt{1+\alpha^2}\left(\frac{y-\mu}{\sigma}\right)\right) \end{aligned}$$

which is non negative for all $\alpha \leq 0$.

Henceforth $l(\sigma)$ is increasing,

$\int_{-\infty}^y F_{\mu, \sigma_1, \alpha}(x) dx \leq \int_{-\infty}^y F_{\mu, \sigma_2, \alpha}(x) dx$ for all y , and we get the result.

- 4) Set $b := \frac{\sigma_1 \alpha_1}{\sqrt{1 + \alpha_1^2}} \in \mathbb{R}$ then $|b| < \sigma_1 \leq \sigma_2$ and the

second assumption on the parameters is equivalent to

$\alpha_i = \frac{b}{\sqrt{\sigma_i^2 - b^2}}$ for $i = 1, 2$. For each fixed μ and y

and arbitrary $\sigma > |b|$ we consider the function $\sigma \rightarrow g(\sigma) := \int_{-\infty}^y F_{\mu, \sigma, \alpha(\sigma)}(x) dx$, where $\alpha(\sigma) := \frac{b}{\sqrt{\sigma^2 - b^2}}$. We have:

$$\begin{aligned} g'(\sigma) &= \int_{-\infty}^y \left(\frac{\partial F_{\mu, \sigma, \alpha(\sigma)}}{\partial \alpha} \alpha'(\sigma) + \frac{\partial F_{\mu, \sigma, \alpha(\sigma)}}{\partial \sigma} \right) dx \\ &= \alpha'(\sigma) \int_{-\infty}^y \frac{\partial F_{\mu, \sigma, \alpha(\sigma)}}{\partial \alpha} dx + \int_{-\infty}^y \frac{\partial F_{\mu, \sigma, \alpha(\sigma)}}{\partial \sigma} dx \\ &= -\alpha'(\sigma) \frac{2\sigma}{\sqrt{2\pi} (1 + \alpha(\sigma)^2)^{3/2}} \\ &\quad \Phi \left(\sqrt{1 + \alpha(\sigma)^2} \frac{y - \mu}{\sigma} \right) + l'(\sigma) \end{aligned}$$

Now $\alpha'(\sigma) = -b\sigma / (\sigma^2 - b^2)^{3/2}$, hence:

$$\begin{aligned} g'(\sigma) &= \frac{2b}{\sqrt{2\pi}\sigma} \Phi \left(\sqrt{1 + \alpha(\sigma)^2} \frac{y - \mu}{\sigma} \right) \\ &\quad + 2\varphi \left(\frac{y - \mu}{\sigma} \right) \Phi \left(\alpha(\sigma) \frac{y - \mu}{\sigma} \right) \\ &\quad - \frac{2b}{\sqrt{2\pi}\sigma} \Phi \left(\sqrt{1 + \alpha(\sigma)^2} \frac{y - \mu}{\sigma} \right) \\ &= 2\varphi \left(\frac{y - \mu}{\sigma} \right) \Phi \left(\alpha(\sigma) \frac{y - \mu}{\sigma} \right) \end{aligned}$$

which is non negative. Henceforth $g(\sigma)$ is increasing for all y .

Remark: The argument by which the result (3) has been obtained fails for $\alpha > 0$. In such a case $l'(\sigma)$ is given by the difference of two positive quantities. The difference is positive for $y = \mu$ since $\sqrt{1 + \alpha^2} > \alpha$. On the contrary for values of y much larger than μ the difference is negative since the first quantity becomes much smaller than the second one. Therefore for positive α the sign of $l'(\sigma)$ is going to depend on the values of the variable y .

Corollary 4.2:

1) Let $X_1 \sim SN(\mu_1, \sigma^2, \alpha_1)$ and

$X_2 \sim SN(\mu_2, \sigma^2, \alpha_2)$. Suppose $\mu_1 \geq \mu_2$ and $\alpha_1 \geq \alpha_2$, then $X_1 \succeq_{FD} X_2$.

2) Let $X_1 \sim SN(\mu_1, \sigma_1^2, \alpha_1)$ and

$X_2 \sim SN(\mu_2, \sigma_2^2, \alpha_2)$. Suppose $\mu_1 \geq \mu_2$, $\sigma_1 \leq \sigma_2$ and $0 \geq \alpha_1 \geq \alpha_2$, then $X_1 \succeq_{SD} X_2$.

3) Let $X_1 \sim SN(\mu_1, \sigma_1^2, \alpha_1)$ and

$X_2 \sim SN(\mu_2, \sigma_2^2, \alpha_2)$. Suppose $\mu_1 \geq \mu_2$, $\sigma_1 \leq \sigma_2$ and

$\sigma_1 \alpha_1 = \sigma_2 \alpha_2 \sqrt{\frac{1 + \alpha_1^2}{1 + \alpha_2^2}}$, then $X_1 \succeq_{SD} X_2$.

Proof.

1) Let $Y \sim SN(\mu_1, \sigma, \alpha_2)$, then by lemma 2.1 (i) and (2) we have: $X_1 \succeq_{FD} Y \succeq_{FD} X_2$.

2) Let $Y \sim SN(\mu_1, \sigma_1^2, \alpha_2)$ and $Z \sim SN(\mu_1, \sigma_2^2, \alpha_2)$, then by lemma 2.1 (1), (2) and (3) we have:

$X_1 \succeq_{SD} Y \succeq_{SD} Z \succeq_{SD} X_2$.

3) Let $Y \sim SN(\mu_2, \sigma_1^2, \alpha_1)$, then by lemma 2.1 (2) and (4) we have: $X_1 \succeq_{SD} Y \succeq_{SD} X_2$.

Remark: a) The condition appearing in Corollary 4.2 (3) can be better understood by noticing that is equivalent to $\sigma_1 \delta_1 = \sigma_2 \delta_2$. Therefore the result can be restated in the following form: If $\mu_1 \geq \mu_2$, $\sigma_1 \leq \sigma_2$ and $skew(X_1) = skew(X_2)$ then $X_1 \succeq_{SD} X_2$.

b) Let $X_1 \sim SN \left(\mu, \sigma^2 \frac{\beta^2 (1 + \alpha^2)}{\alpha^2 (1 + \beta^2)}, \alpha \right)$ and

$X_2 \sim SN(\mu, \sigma^2, \beta)$ with $0 < \beta \leq \alpha$. Being $\delta_\beta / \delta_\alpha \leq 1$ we have $\sigma_1 \leq \sigma_2 \equiv \sigma$ and from Corollary 4.2 (3) it follows $X_1 \succeq_{SD} X_2$. Notice that

$$var(X_1) = var(X_2) + \sigma^2 \left[\frac{\delta_\beta^2}{\delta_\alpha^2} - 1 \right] \leq var(X_2).$$

Henceforth, a larger positive skewness parameter in a financial position gives rise to an improvement over a previous position, from a stochastic dominance viewpoint, if it is accompanied by a variance reduction which leaves unchanged the skewness of the position.

c) It is well known that $X_1 \succeq_{SD} X_2$ if and only if $\mathbb{E}(u(X_1)) \geq \mathbb{E}(u(X_2))$ for all non decreasing and concave functions u . Therefore, by choosing $u(x) = x$, conditions (2) of Corollary 4.2 imply $\mathbb{E}(X_1) \geq \mathbb{E}(X_2)$.

d) Consider the two-dimensional risky market discussed in Section 2 and the parametrized family of portfolios which invest on the straddle on asset A , on shares of asset B , putting all the remaining wealth on a riskfree bond. Assume $\rho = 0$ and $\mu_B > r$. Consider two portfolios $\mathbf{w}_1 = (a_1, b_1, 1 - a_1 - b_1)$ and $\mathbf{w}_2 = (a_2, b_2, 1 - a_2 - b_2)$ and the corresponding returns $R_1 \equiv R_{\mathbf{w}_1}$ and $R_2 \equiv R_{\mathbf{w}_2}$. Suppose 1) $0 > a_1 > a_2, 2)$

$b_1 > b_2 + \frac{a_1 - a_2}{\mu_B - r} (1 + r)$, and 3)

$b_1 < \sqrt{b_2^2 + \frac{\gamma_0^2 \sigma_A^2}{\sigma_B^2} (a_2^2 - a_1^2)}$, then from (2) we get

$$\begin{aligned} \mu(a_1, b_1) &= -a_1(1+r) + b_1(\mu_B - r) + r \\ &> -a_2(1+r) + b_2(\mu_B - r) + r = \mu(a_2, a_2) \end{aligned}$$

In addition from (3) we obtain $\sigma(a_1, b_1) < \sigma(a_2, b_2)$. Finally from (1) we have $b_1 > b_2$ and therefore $0 > \alpha(a_1, b_1) > \alpha(a_2, b_2)$. Then by Corollary 4.2 (2) we get $R_1 \succeq_{SD} R_2$.

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