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L(0,1)-Labelling of Cactus Graphs

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ABSTRACT

An $L(0,1)$-labelling of a graph G is an assignment of nonnegative integers to the vertices of G such that the difference between the labels assigned to any two adjacent vertices is at least zero and the difference between the labels assigned to any two vertices which are at distance two is at least one. The span of an $L(0,1)$-labelling is the maximum label number assigned to any vertex of G. The $L(0,1)$-labelling number of a graph G, denoted by $\lambda_{0,1}(G)$, is the least integer k such that G has an $L(0,1)$-labelling of span k. This labelling has an application to a computer code assignment problem. The task is to assign integer control codes to a network of computer stations with distance restrictions. A cactus graph is a connected graph in which every block is either an edge or a cycle. In this paper, we label the vertices of a cactus graph by $L(0,1)$-labelling and have shown that, $\Delta-1 \le \lambda_{0,1}(G) \le \Delta$ for a cactus graph, where Δ is the degree of the graph G.

Keywords: Graph Labelling; Code Assignment; *L*(0,1)-Labelling; Cactus Graph

1. Introduction

Cactus graph is a connected graph in which every block is a cycle or an edge, in other words, no edge belongs to more than one cycle. Cactus graphs have been extensively studied and used as models for many real-world problems. This graph is one of the most useful discrete mathematical structures for modelling problems arising in the real-world. It has many applications in various fields, like computer scheduling, radio communication system, etc. Cactus graphs have been studied from both theoretical and algorithmic points of view. This graph is a subclass of planar graph and superclass of tree.

An $L(0,1)$-labelling of a graph $G=(V,E)$ is a function of f from its vertex set V to the set of nonnegative integers such that $|f(x)-f(y)| \ge 0$ if $d(x,y)=1$ and $|f(x)-f(y)| \ge 1$ if $d(x,y)=2$, where $d(x,y)$ is the distance between the vertices x and y, *i.e.*, the number of edges between x and y. The *span* of an $L(0,1)$-labelling f of G is $\max\{f(v): v \in V\}$. The $L(0,1)$-labelling $\lambda_{0,1}(G)$ of G is the smallest k such that G has a $L(0,1)$-labelling of span k.

An interesting graph-labelling problem comes from the radio frequency assignment problem, as well as code assignment in computer networks. One version of the radio channel assignment problem [1] is to assign integer channels to a network of transmitters with distance restrictions, such that the several labels of interference between nearby transmitters are avoided and the span of the label used is minimized. A variation of the problem is code assignment in computer networks, *i.e.*, to assign integer control codes to a network of computer stations with distance restrictions.

Bertossi and Bonuccelli [2] introduced a kind of code assignment to avoid hidden terminal interference; this is as follows. Since some modern computer networks consist of including mobile computers or computer displaced in wide areas, they need to use broadcast communication media such as busses (only in local area networks) or radio frequencies. The computer network which communicates by radio frequencies is called Packet Radio Network. It consists of computer stations (computers and transceivers), in which the transceivers broadcast outgoing message packets and listen for incoming message packets. Unconstrained transmission in broadcast media may lead to collision on interference, *i.e.*, there is the time-overlap of two or more incoming message packets received at the destination station. This results in damaged useless packets at the destination. Collided message packets must be retransmitted. That increases the time delay of the transmission, and hence lowers the system throughput. Several protocols have been devised to reduce or eliminate the collisions. They form the medium access control sublayer. For example, under Code Division Multiple Access protocol, the collision-free property is guaranteed by the use of proper assignment of orthogonal control codes to stations and the spread of spectrum communication techniques (e.g., hopping over different time slots or frequency bands).

We represent the network by a graph, such that all stations are vertices and two vertices are adjacent if the corresponding stations can hear each other. Hence, two stations are at distance two, if they are outside the hearing range of each other but can be received by the same destination station. There are two types of collisions-interferences: direct collision, due to transmission of adjacent station, and hidden terminal collision, when stations at distance two transmit to the same receiving station at the same time.

To avoid hidden terminal interference, we assign a control codes to each station in the software as follows. For one station, to avoid hidden terminal interference from its adjacent stations (which hear each other) sending packets to it, we require distinct codes for its immediate adjacent station, *i.e.*, $d_2 = 1$. Here we suppose that there is a little direct interference in the system, *i.e.*, direct interference is so week that we can ignore it. Apparently in the model of [2] there are some special hardware designs, which can avoid direct interference in the system. Hence, we allow the same code for two adjacent stations (which can hear each other), meaning $d_1 = 0$. Therefore, we have the $L(0,1)$-labelling case.

It is important to note that the $L(0,1)$-labelling problem is just a special case of ordinary graph labelling. Each feasible $L(0,1)$-labelling of a graph $G=(V,E)$ yields a feasible labelling of the graph $G=(V,E)$, where E' contains edge (u,v) whenever u and v are distance two apart in G. Conversely, a labelling of G' becomes a feasible labelling of G by calling the labels $0,1,\cdots,\chi'(G)-1$, where χ' represents the maximum colour number of the graph.

In this paper, we label the vertices of a cactus graph G by $L(0,1)$-labelling and it is shown that $\Delta-1 \le \lambda_{0,1}(G) \le \Delta$, where Δ is the degree of the graph G, *i.e.*, $\Delta = \max\{\deg(v_i)$: $v_i \in V$, $\deg(v_i))$ is the degree of the vertex $v_i)\}$.

2. Review of Previous Works

Some results are available on $L(h,k)$-labelling problem. Here we discuss some particular cases. When $h=0$ and $k=1$ then we get $L(0,1)$-labelling problem. Several results are known for $L(0,1)$-labelling of graphs, but, to the best of our knowledge no result is known for cactus graph. In this section, the known result for general graphs and some related graphs of cactus graph are presented.

The upper bound for $\lambda_{0,1}(G)$ of any graph G is $\lambda_{0,1}(G) \le \Delta^2-\Delta$ [3], where Δ is the degree of the graph.

The problem is simple for paths P_n of n vertices. It can easily be verified that $\lambda_{0,1}(P_1)=\lambda_{0,1}(P_2)=0$, $\lambda_{0,1}(P_n)=1$ for $n \ge 3$ [4].

When the first and the last vertices of P_n are merged then P_n becomes C_{n-1}. In [2], Bertossi and Bonuccelli showed that $\lambda_{0,1}(C_n)$ is equal to 1 if *n* is multiple of 4 and 2 otherwise.

For complete graph K_n, it is easy to check that $\lambda_{0,1}(K_n)=n-1$.

The wheel W_n, is obtained by joining C_n and K_1, *i.e.*, $W_n = C_n + K_1$. It is also easy to check that $\lambda_{0,1}(W_n)=n$.

Bertossi and Bonuccelli [2] investigated the $L(0,1)$-labelling problem on complete binary trees, proving that 3 labels suffice. An optimum labelling as follows can be found. Assign first labels 0, 1 and 2, respectively, to the root, its left child and its right child. Then, consider the nodes by increasing levels: if a node has been assigned label c, then assign the remaining labels to its grandchildren, but giving different to brother grandchildren. The above procedure can be generalized to find an optimum $L(1,1)$-labelling for complete ($\Delta-1$)-ary trees, requiring span Δ. It is straight forward to see that when $\Delta=3$ and $\Delta=2$ this result gives the $\lambda_{0,1}$-number for complete binary trees and paths respectively.

It is shown in [5] that for any tree T, $\lambda_{1,1}(T)$ is equal to Δ, implying that $\lambda_{h,h}(T)=h\Delta$. An optimal $L(1,1)$-labelling can be also determined by exploiting the algorithm provided in [6] for optimally $L(1,\cdots,1)$-labelling trees.

Bodlaender *et al.* [7] compute upper bounds for graphs of treewidth bounded by t proving that $\lambda_{0,1}(G) \le t\Delta - t$. They give also approximation algorithms for the $L(0,1)$-labelling running in $O(tn\Delta)$ time.

In [2], the NP-completeness result for the decision version of the $L(0,1)$-labelling problem is derived when the graph is planar by means of a reduction from 3-VERTEX COLORING of straight-line planar graph. An exhausted survey on $L(h,k)$-labelling is available in [8].

In [7], an approximation algorithm is designed for $L(0,1)$-labelling a permutation graph in $O(n\Delta)$ time; it guarantees the bound $\lambda_{0,1}(G) \le 2\Delta-2$.

The *n-dimensional hypercube* Q_n is an n-regular graph with 2^n nodes. Then $\lambda_{0,1}(Q_n) \le 2^{\lceil logn \rceil}$ and there exists a labelling scheme using such a number of labels. This labelling is optimal when $n=2^k$ for some k and it is a 2-approximation otherwise [9]. For a bipartite graph, $\lambda_{0,1}(G) \geq \frac{\Delta^2}{4}$ [7]. Later this lower bound has been improved by a constant factor of $\frac{1}{4}$ [10]. A study on $L(d,1)$-labelling of cartesian product of a cycle and path is done by Chiang and Yan [11].

When $h=2$ and $k=1$ then we get $L(2,1)$-labelling problem. This problem was introduced by Grrigs and Yeh [12,13] in connection with the problem of assigning frequencies in a multihop radio network. Some results of $L(2,1)$-labelling problem are given below.

Kral and Skrekovski [14] improve the upper bound for any graph G, $\lambda(G) \leq \Delta^2 + \Delta - 1$. The best known result till date is $\lambda(G) \leq \Delta^2 + \Delta - 2$ due to Goncalves [15].

Heuvel and Mc Guinness showed that $\lambda(G) \leq 2\Delta + 35$ [16] for planar graphs. Molloy and Salavatipour [17] reduced this upper bound to $5\Delta/3 + 90$. Wang and Lih [18] proved that if G is a planar graph of girth (girth is defined to be the length of a shortest cycle in G) at least 5, then $\lambda(G) = \Delta + 21$.

In [19], we have showed that the upper and the lower bounds for λ of a cactus graph G is $\Delta + 1 \leq \lambda(G) \leq \Delta + 3$.

Adams *et al.* [20], give different bounds for certain generalized petersen graphs. A study on $L(d,1)$-labelling of cartesian product of a cycle and a path is done by Chiang and Yan [11].

For further studies on the $L(2,1)$-labelling, see [21-30].

When $h=1$ and $k=1$ then we get another special case which is called $L(1,1)$-labelling problem. Some results of $L(1,1)$-labelling problem are given below.

For path, $\lambda_{1,1}(P_2) = 1$ and $\lambda_{1,1}(P_n) = 2$ for each $n \geq 3$, and $\lambda_{1,1}(C_n)$ is 2 if n is a multiple of 3 and it is 3 otherwise [31].

3. The *L*(0,1)-Labelling of Induce Sub-Graphs of Cactus Graphs

Let $G = (V, E)$ be a given graph and subset U of V. The *induced subgraph* by U, denoted by $G[U]$, is the graph given by $G[U] = (U, E')$, where $E' = \{(u,v) : u, v \in U \text{ and } (u,v) \in E\}$. Some induced subgraphs of cactus graph are shown in **Figure 1**.

The cactus graphs have many interesting subgraphs, those are illustrated below. An edge is nothing but P, so $\lambda_{0,1}(an\ edge) = 0$. The star graph $K_{1,\Delta}$ is a subgraph of cactus graph, therefore, one can conclude the following result.

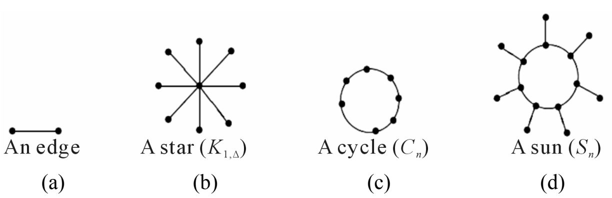

Figure 1. Some induce subgraphs of cactus graph.

Lemma 1. *For any star graph* $K_{1,\Delta}$,

$$\lambda_{0,1}(K_{1,\Delta}) = \Delta - 1. \quad (1)$$

4. *L*(0,1)-Labelling of Cycles

In [2], Bertossi and Bonuccelli have labeled C_n by $L(0,1)$-labelling and they have obtained the following result. Here we have given a constructive prove of this result.

4.1. *L*(0,1)-Labelling of One Cycle

Lemma 2. [2] *For any cycle* C_n *of length* n,

$$\lambda_{0,1}(C_n) = \begin{cases} 1, & \text{when } n \text{ is multiple of } 4, \\ 2, & \text{otherwise.} \end{cases} \quad (2)$$

Proof. Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of the cycle C_n. We classify C_n into five groups, viz., C_3, C_{4k}, C_{4k+1}, C_{4k+2} and C_{4k+3}.Then the $L(0,1)$-labelling of the vertices of a cycle are as follows.

Case 1. Let $n=3$.

$$f(v_i) = \begin{cases} 0, & \text{if } i = 0; \\ 1, & \text{if } i = 1; \\ 2, & \text{if } i = 2. \end{cases}$$

Case 2. Let $n = 4k \equiv 0 \pmod 4$, *i.e.*, C_{4k}.

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 4; \\ 0, & \text{if } i \equiv 1 \pmod 4; \\ 1, & \text{if } i \equiv 2 \pmod 4; \\ 1, & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

Case 3. Let $n = 4k+1 \equiv 1 \pmod 4$, *i.e.*, C_{4k+1}.

The label of first $4k$ vertices $v_0, v_1, \cdots, v_{n-2} = v_{4k-1}$ are same as in Case 2. For the last vertex v_{n-1}, f is define as

$$f(v_{n-1}) = 2.$$

Case 4. Let $n = 4k+2 \equiv 2 \pmod 4$, *i.e.*, C_{4k+2}.

Here the label of first $4k+1$ vertices $v_0, v_1, \cdots, v_{n-2} = v_{4k}$ are same as in Case 3. For the last vertex v_{4k+2}, f is define as

$$f(v_{n-1}) = 2.$$

Case 5. Let $n = 4k+3 \equiv 3 \pmod 4$, *i.e.*, C_{4k+3}.

The label of first $4k$ vertices $v_0, v_1, \cdots, v_{n-4} = v_{4k-1}$ are same as in Case 2. For the last three vertices v_{n-3}, v_{n-2}, v_{n-1}, f is define as

$$f(v_i)=\begin{cases}0, & \text{if } i=n-3;\\ 0, & \text{if } i=n-2;\\ 1, & \text{if } i=n-1.\end{cases}$$

Thus, from all above cases, we conclude that

$$\lambda_{0,1}(C_n)=\begin{cases}1, & \text{when } n \text{ is multiple of 4,}\\ 2, & \text{otherwise.}\end{cases} \qquad \square$$

4.2. *L*(0,1)-Labelling of Two Cycles

Lemma 3. *Let G be a graph which contains two cycles and they have a common cutvertex. If* Δ *be the degree of G, then*

$$\lambda_{0,1}(G)=\begin{cases}\Delta, & \text{when two cycles are of length 3;}\\ \Delta-1, & \text{otherwise.}\end{cases} \tag{3}$$

Proof. Let G contains two cycles C_n and C_m of lengths n and m respectively. Let v_0 be the cutverx and Δ be the degree of v_0. Let $v_0,v_1,\cdots,v_{n-1}$ and $v_0,v_1,\cdots,v_{n-1}$ be the vertices of C_n and C_m respecvely. The labelling procedure of v_i's of C_n of same as given in Lemma 2. Now we label the cycle C_m as follows.

Case 1. Let $n=3$ and $m=3$.

The label of the cutvertex v_0 is 0, *i.e.*, $f(v_0)=0$. The label of other vertices of C_m are as follows:

$$f(v_i')=\begin{cases}3, & \text{if } i=1;\\ 4, & \text{if } i=2.\end{cases}$$

Case 2. For $n=4k\equiv 0$ (mod 4) and $m=3$.

$$f(v_i')=\begin{cases}2, & \text{if } i=1;\\ 3, & \text{if } i=2.\end{cases}$$

Case 3. For $n=4k+1\equiv 1$ (mod 4) and $m=3$.

$$f(v_i')=\begin{cases}1, & \text{if } i=1;\\ 3, & \text{if } i=2.\end{cases}$$

Case 4. For $n=4k+2\equiv 2$ (mod 4) and $m=3$.

The label of the vertices of C_m are same as given in Case 3 of that lemma.

Case 5. For $n=4k+3\equiv 3$ (mod 4) and $m=3$.

In this case, we label the of C_m as given in Case 3.

Case 6. For $n=4k\equiv 0$ (mod 4) and $m=4k$.

Here we label the adjacent vertices of v_0 by $f(v_1')=2$ and $f(v_{m-1}')=3$. Now we label the other vertices $v_2',v_3',\cdots,v_{m-3}'$ of C_m as follows.

$$f(v_i)=\begin{cases}0, & \text{if } i\equiv 0 \pmod 4;\\ 0, & \text{if } i\equiv 1 \pmod 4;\\ 1, & \text{if } i\equiv 2 \pmod 4;\\ 1, & \text{if } i\equiv 3 \pmod 4.\end{cases}$$

The above f is redefine for the vertex v_{m-2}' as

$$f(v_{m-2}')=1.$$

In particular when $m=4$, then we label the vertices of C_m as follows.

The label of the cutvertex v_0 and two adjacent vertices v_1' and v_3' are same as above. And we label the remaining vertex v_2' by $f(v_2')=1$.

Case 7. For $n=4k\equiv 0$ (mod 4) and $m=4k+1\equiv 1$ (mod 4).

Here the label of two vertices v_1', v_{m-1}' of C_m are same as given in the above case. Now we label the vertices $v_2',v_3',\cdots,v_{m-4}'$ as

$$f(v_i)=\begin{cases}0, & \text{if } i\equiv 0 \pmod 4;\\ 0, & \text{if } i\equiv 1 \pmod 4;\\ 1, & \text{if } i\equiv 2 \pmod 4;\\ 1, & \text{if } i\equiv 3 \pmod 4.\end{cases}$$

For the vertices v_{m-3}' and v_{m-2}', f is defined as $f(v_{m-3}')=1$ and $f(v_{m-2}')=1$.

In particular when $m=5$, then we label the vertices of C_m as follows.

The label of the cutvertex v_0 and two adjacent vertices of v_0 of C_m are same as above. Now we label the remaining vertices of C_5 as

$$f(v_i')=\begin{cases}1, & \text{if } i=2;\\ 1, & \text{if } i=3.\end{cases}$$

Case 8. For $n=4k\equiv 0$ (mod 4) and $m=4k+2\equiv 2$ (mod 4).

The label of $v_1',v_2',\cdots,v_{m-3}',v_{m-1}'$ of C_m are same as in above case. For the vertex v_{m-2}', we label it as

$$f(v_{m-2}')=2.$$

When $m=6$, then the label of the vertices v_0, v_1', v_2', v_3' and v_5' are same as in the above case, and $f(v_4')=2$.

Case 9. For $n=4k\equiv 0$ (mod 4) and $m=4k+3\equiv 3$ (mod 4).

We label the vertices of C_n and the vertices v_1', v_2', $\cdots$, v_{m-3}' and v_{m-1}' of C_m according to the Case 8. For the vertex v_{m-2}', f is

$$f(v_{m-2}')=2.$$

In particular, when $m=7$, the the label of the vertices of C_7 are same as the label of the vertices of C_6 as in the above case except the vertex v_5'. The label of the vertex v_5' is $f(v_5')=2$.

Case 10. For $n=4k+1\equiv 1$ (mod 4) and $m=4k+1\equiv 1$ (mod 4).

We label first $m-4$ vertices $v_0,v_1',v_2',\cdots,v_{m-5}'$ of C_m as

$$f(v_0)=0 \text{ and } f(v_i')=\begin{cases}1, \text{ if } i\equiv 0 \pmod 4;\\ 1, \text{ if } i\equiv 1 \pmod 4;\\ 0, \text{ if } i\equiv 2 \pmod 4;\\ 0, \text{ if } i\equiv 3 \pmod 4.\end{cases}$$

For the last four vertices v'_{m-4}, v'_{m-3}, v'_{m-2} and v'_{m-1}, the above f is define as

$$f(v_i')=\begin{cases}1, \text{ if } i=m-4;\\ 1, \text{ if } i=m-3;\\ 2, \text{ if } i=m-2;\\ 3, \text{ if } i=m-1.\end{cases}$$

In particular when $m=5$, the label of the vertices v_1', v_2', v_3' and v_4' are same as the label of the vertices $v'_{m-4},\cdots,v'_{m-1}$ shown above.

Case 11. For $n=4k+1\equiv 1 \pmod 4$ and $m=4k+2\equiv 2 \pmod 4$.

We label first $m-5$ vertices $v_0,v_1',v_2',\cdots,v'_{m-6}$ of C_m as per Case 10. And for the last five vertices, f is define as

$$f(v_i')=\begin{cases}1, & \text{if } i=m-5;\\ 1, & \text{if } i=m-4;\\ 2, & \text{if } i=m-3;\\ 2, & \text{if } i=m-2;\\ 3, & \text{if } i=m-1.\end{cases}$$

In particular when $m=6$, then the label of the vertices $v_1',\cdots,v_5'$ are same as the label of the last five vertices of C_m of the above case.

Case 12. For $n=4k+1\equiv 1 \pmod 4$ and $m=4k+3\equiv 3 \pmod 4$.

Now we label first $m-2$ vertices of C_m as

$$f(v_0)=0 \text{ and } f(v_i')=\begin{cases}1, \text{ if } i\equiv 0 \pmod 4;\\ 1, \text{ if } i\equiv 1 \pmod 4;\\ 0, \text{ if } i\equiv 2 \pmod 4;\\ 0, \text{ if } i\equiv 3 \pmod 4.\end{cases}$$

For the last two vertices v'_{m-2} and v'_{m-1}, the f is

$$f(v_i')=\begin{cases}1, \text{ if } i=m-2;\\ 3, \text{ if } i=m-1.\end{cases}$$

Case 13. For $n=4k+2\equiv 2 \pmod 4$ and $m=4k+2\equiv 2 \pmod 4$.

Here we label the other vertices of C_m as in Case 11.

Case 14. For $n=4k+2\equiv 2 \pmod 4$ and $m=4k+3\equiv 3 \pmod 4$.

In this case, the label of C_m are same as in Case 12.

Case 15. For $n=4k+3\equiv 3 \pmod 4$ and $m=4k+3\equiv 3 \pmod 4$.

We label the vertices of C_m as per Case 12.

Thus from the above cases, it follow that

$$\lambda_{0,1}(G)=\begin{cases}\Delta, \text{ when two cycles are of length 3;}\\ \Delta-1, \text{ otherwise.}\end{cases}$$ □

4.3. *L*(0,1)-Labelling of Three Cycles

Lemma 4. *Let G be a graph, contains three cycles and they have a common cutvertex* v_0. *If* Δ *be the degree of* v_0, *then,*

$$\lambda_{0,1}(G)=\begin{cases}\Delta, \text{ when three cycles are of length 3;}\\ \Delta-1, \text{ otherwise.}\end{cases} \quad (4)$$

Proof. Let C_n, C_m and C_l be three cycles join by a common cutvertex v_0, of lengths n, m and l respectively. Let Δ be the degree of the cutvertex, *i.e.*, $\Delta=6$. Let v_0, v_1, $\cdots$, v_{n-1}; v_0, v_1', $\cdots$, v'_{m-1}; v_0, v_1'', $\cdots$, v''_{l-1} be the vertices of the cycles respectively.

Now we label the graph as follows.

Case 1. Let $n=3$, $m=3$ and $l=3$.

In Case 1 of Lemma 3, we label a graph which contains two cycles of length three and they have a common cutvertex v_0. According to the previous lemma we label the vertices of the third cycle of length 3 as follows:

$$f(v_0)=0 \text{ and } f(v_i'')=\begin{cases}5, \text{ if } i=1;\\ 6, \text{ if } i=2.\end{cases}$$

Case 2. For $n=4k+i$, $m=3$, $l=3$, where $i=0,1,2,3$.

All the subcases of this case, the label of two vertices v_1'' and v_2'' of the cycle C_l are

$$f(v_i'')=\begin{cases}4, \text{ if } i=1;\\ 5, \text{ if } i=2.\end{cases}$$

and the label of other two cycles are of different types, they are discussed below.

When $n=4k\equiv 0 \pmod 4$ and $m=3$.

Here the label of the vertices of the cycles C_n and C_m, joined with a common cutvertex v_0 are same as in Case 2 of Lemma 3.

When $n=4k+1\equiv 1 \pmod 4$ and $m=3$.

The label of the vertices of C_n and C_m are same as in Case 3 of Lemma 3.

When $n=4k+2\equiv 2 \pmod 4$ and $m=3$.

The label of the vertices of C_n and C_m are same as in Case 4 of Lemma 3.

When $n=4k+3\equiv 3 \pmod 4$ and $m=3$.

We label the vertices of C_n and C_m as in Case 5 of Lemma 3.

Case 3. For $n=4k+i$, $m=4k+i$, $l=3$, where $i=0,1,2,3$.

In all subcases of this case, the label of three vertices of C_l are

$$f(v_0)=0 \text{ and } f(v_i'')=\begin{cases}4, & \text{if } i=1;\\ 5, & \text{if } i=2.\end{cases}$$

And the label of the vertices of first two cycles C_n and C_m are same as in Case 6, Case 7, ..., Case 15, respectively of Lemma 3.

Case 4. For $n=4k+i$, $n=4k+i$, $l=4k\equiv 0 \pmod 4$, where $i=0,1,2,3$.

For all subcases of this case, we label the vertices of the third cycle C_l as same as the labelling of the vertices of C_m in Case 6 of Lemma 3 except two vertices v_1'' and v_{l-1}'' (the adjacent vertices of v_0). Then we label these vertices as $f(v_1'')=4$ and $f(v_{l-1}'')=5$.

And we label the first two cycles C_n and C_m (joined by a common cutvertex v_0), as same as in Case 6, Case 7, ..., Case 15 of Lemma 3.

Case 5. For $n=4k+i$, $m=4k+i$, $l=4k+1\equiv 1 \pmod 4$, where $i=0,1,2,3$.

In all subcases of this case, we label the vertices of the third cycle C_l using the same process to labelling the vertices of C_m in Case 7 of Lemma 3 except two vertices v_1'' and v_{l-1}'' (the adjacent vertices of v_0). Now we label the adjacent vertices of v_0 of C_l as $f(v_1'')=4$ and $f(v_{l-1}'')=5$.

And we label the first two cycles C_n and C_m (joined by a common cutvertex v_0), as same as in Case 10, Case 11, ..., Case 15 of Lemma 3.

Case 6. For $n=4k+i$, $m=4k+i$, $l=4k+2\equiv 2 \pmod 4$, where $i=2,3$.

In all subcases of this case, we label the vertices of the third cycle C_l using the same process of labelling of the vertices of C_m in Case 7 of Lemma 3, except two vertices v_1'' and v_{l-1}''. The label of v_1'' and v_{l-1}'' are $f(v_1'')=4$ and $f(v_{l-1}'')=5$.

And the label of the vertices of the cycles C_n and C_m are same as in Case 13, Case 14 and Case 15 respectively of Lemma 3.

Case 7. For $n=4k+3\equiv 3 \pmod 4$, $m=4k+3\equiv 3 \pmod 4$, $l=4k+3\equiv 3 \pmod 4$.

The label of the vertices of C_n and C_m are same as in Case 15 of Lemma 3. Here $f(v_0))=0$. Then we label the vertices of C_l using the same process of C_m in Case 9 of Lemma 3 except the vertices v_1'' and v_{l-1}''. We label these vertices as $f(v_1'')=4$ and $f(v_{l-1}'')=5$. Thus from all above cases, it follow that

$$\lambda_{0,1}(G)=\begin{cases}\Delta, & \text{when three cycles are of length 3;}\\ \Delta-1, & \text{otherwise.}\end{cases} \quad \square$$

4.4. *L*(0,1)-Labelling of Four Cycles

Using th results from Lemma 3 and Lemma 4 we can write the following statement.

Lemma 5. *Let G be a graph which contains four cycles of any length and they have a common cutvertex. Then,*

$$\lambda_{0,1}(G)=\begin{cases}\Delta, & \text{when four cycles are of length 3;}\\ \Delta-1, & \text{otherwise;}\end{cases} \tag{5}$$

where Δ be the degree of the cutvertex.

Corollary 1. *Let G be a graph which contains finite number of cycles and they have a common cutvertex. If the vertices of the cycles* (*except the cutvertex*) *contain one or more edges then*,

$$\lambda_{0,1}(G)=\begin{cases}\Delta, & \text{when four cycles are of length 3;}\\ \Delta-1, & \text{otherwise.}\end{cases} \tag{6}$$

4.5. *L*(0,1)-Labelling of Finite Number of Cycles

Let G be a graph which contains n number of cycles of length 3. Sometimes a cycle of length three is called *triangle*. A triangle is a subgraph of a cactus graph. Also, a triangle shaped star, (*i.e.*, all the triangles that have a common cutvertex) is a subgraph of a cactus graph. Now, we consider a triangle shaped star for $L(0,1)$-labelling. Let $T_0,T_1,\cdots,T_{n-1}$ be the n triangles meet at a common cutvertex v_0 and we denote this graph by G, which is equivalent to $\bigcup_{v_0} T_i$. The number of vertices and edges of G are $2n+1$ and $3n$ respectively. Again the graph G may also contains n number of cycles of finite length.

Then from Lemmas 3-5 we conclude the general form of these lemmas which is given below.

Lemma 6. *Let the graph G contains n number of cycles of any length and they joined at a cutvertex*, *then*

$$\lambda_{0,1}(G)=\begin{cases}\Delta, & \text{when all cycles are of length 3;}\\ \Delta-1, & \text{otherwise.}\end{cases} \tag{7}$$

where Δ be the degree of the cutvertex.

Proof. At first we prove that when G contains n number of cycles of length 3 then the value of $\lambda_{0,1}$ is Δ, where Δ be the degree of the graph. Let $T_0,T_1,\cdots,T_{n-1}$ be the n number of triangles joined with a common cutvertex v_0 (shown in **Figure 2**).

Let v_0, v_{i1} and v_{i2} be the vertices of T_i, where $\Delta=2n$. We label v_0 by 0.

Then according to the previous lemmas the labels of

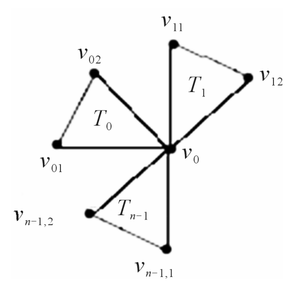

Figure 2. A graph contains *n* numbers of triangles.

T_i's are as follows.

For $i = 0,1,2\cdots,n-1$,

$$f\left(v_{ij}\right) = \begin{cases} 2i+1, & \text{if } j = 1; \\ 2i+2, & \text{if } j = 1. \end{cases}$$

Now, the label of the vertex $v_{n-1,2}$ of T_{n-1} is $f\left(v_{n-1,2}\right) = 2(n-1)+2 = 2n = \Delta$.

Therefore, $\lambda_{0,1}(G) = \Delta$, when G contains n number of cycles of length 3.

Again, if we consider a graph G contains n number of C_4 and m number of C_3, then, $\lambda_{0,1}(G) = \Delta - 1$, where Δ be the degree of cutvertex, then the general form can be proved by mathematical induction, that is, when a graph G contains finite number of cycles of any length then $\lambda_{0,1}(G) = \Delta - 1$.

Let G contains n number of C_4's $R_0, R_1, \cdots, R_{n-1}$ and m number of C_3's $T_0, T_1, \cdots, T_{m-1}$. Let v_0 be the common vertex and degree of v_0 is $\Delta = 2(m+n)$. Again let v_0, v_{i1}, v_{i2}, v_{i3} be the vertices of R_i and v_0, v'_{j1}, v'_{j2} be the vertices of T_j. We label v_0 as 0. Then we label the other vertices of R_i's as follows.

For $i = 0,1,\cdots,n-1$

$$f\left(v_{ij}\right) = \begin{cases} 2i, & \text{if } j = 1; \\ 1, & \text{if } j = 2; \\ 2i+1, & \text{if } j = 3. \end{cases}$$

and then the label of the vertices of T_j's are given by.

For $j = 0, 1, \cdots, n-1$,

$$f\left(v'_{jk}\right) = \begin{cases} 2n+2j, & \text{if } k = 1; \\ 2n+2j+1, & \text{if } k = 2. \end{cases}$$

Now the label of third vertex of T_{m-1} is

$$\begin{aligned} f\left(v'_{j2}\right) &= 2n + 2(m-1) + 1 = 2n + 2m - 1 \\ &= 2(n+m) - 1 = \Delta - 1. \end{aligned}$$

Therefore, $\lambda_{0,1}(G) = \Delta - 1$.

The general form can be proved by mathematical induction.

Hence the result. □

Lemma 7. *If a graph G contains finite number of cycles of any length and finite number of edges and they have a common cutvertex of degree* Δ, *then*

$$\lambda_{0,1}(G) = \Delta - 1. \tag{8}$$

Proof. Suppose that the lemma is true for k number of cycles of any length and q number of edges. Now we have to prove that if we add a cycle of any length to the cutvertex then the value of $\lambda_{0,1}$ for the new graph will be same, *i.e.*, the value of $\lambda_{0,1}$ will preserve for $k+1$ number of cycles of any length.

Now the graph G contains k number of cycles of any length and q number of edges joined with a common cutvertex v_0. Then the degree of v_0 is $\Delta = 2k+q$. In the previous lemma we proved that when a graph contains finite number of cycles of any length then,

$$\lambda_{0,1}(G) = \begin{cases} \Delta, & \text{when all cycles are of length 3;} \\ \Delta - 1, & \text{otherwise.} \end{cases}$$

At first we prove that if G contains $(k+1)$ number of cycles of length 3 and q number of edges then the value of $\lambda_{0,1}$ for that graph remains same. When all cycles are of length 3 then according to the Lemma 5 the label of two vertices of k th cycle of length 3 are as

$$f\left(v_i^k\right) = \begin{cases} 2k-1, & \text{if } i = 1; \\ 2k, & \text{if } i = 2; \end{cases}$$

where v_0, v_1^k and v_2^k are three vertices of k th cycle. Let v_0, v_{00}; v_0, v_{01}; $\cdots$; v_0, $v_{0,q-1}$ are the vertices of q edges. Here the label of the cutvertex v_0 is 0. Then we label the other vertices of the edges as follows

$$f\left(v_{0j}\right) = \begin{cases} 0, & \text{for } j = 0; \\ 2k+j-1, & \text{for } j = 1,2,\cdots,q-1. \end{cases}$$

Now we add another cycle of length 3 to the cutvertex v_0. Then the degree of v_0 is $2k+q+2$. Let v_0, v_1^{k+1} and v_2^{k+1} be the vertices of $(k+1)$th cycle. We label the two vertices v_1^{k+1} and v_2^{k+1} as

$$f\left(v_i^{k+1}\right) = \begin{cases} 2k+q, & \text{if } i = 1; \\ 2k+q+1, & \text{if } i = 2. \end{cases}$$

Here we see that the label of third vertex of $(k+1)$ th cycle is $f\left(v_2^{k+1}\right) = 2k+q+1 = \Delta - 1$ as $\Delta = 2k+q+2$. That is, the value of $\lambda_{0,1}$ of the graph which contains $(k+1)$ number of cycles of length 3 and q number of edges is same.

Similarly, we can prove that the value of $\lambda_{0,1}$ will preserve for the graph which contains $(k+1)$ number of cycles and q number of edges.

Hence the result. □

5. *L*(0,1)-Labelling of Sun

Let us consider the sun S_{2n} of $2n$ vertices. This graph is obtained by adding an edge to each vertex of a cycle C_n. So C_n is a subgraph of S_{2n}. But, what is the value of $\lambda_{0,1}(S_{2n})$?

Lemma 8. *For any sun* S_{2n},

$$\lambda_{0,1}(S_{2n}) = 2 = \Delta - 1. \tag{9}$$

Proof. Let S_{2n} be constructed from C_n by adding an edge to each vertex. To label this graph we consider five cases.

Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of C_n and v_i is adjacent to v_{i+1} and v_{i-1}. To complete S_{2n}, we add an edge (v_i, v'_i) to the vertex v_i, *i.e.*, v'_i's are the pendent vertices. The labelling procedure of C_n is same as given in Lemma 2. Now we label the pendent vertices as follows.

Case 1. Let $n=3$.

The label of v_i''s are as

$$f\left(v_i'\right)=\begin{cases}0, \text{ if } i=0;\\ 1, \text{ if } i=1;\\ 2, \text{ if } i=2.\end{cases}$$

Case 2. Let $n=4k\equiv 0 \pmod 4$.

The label of $v_0', v_1', \cdots, v_{n-1}'$ are assigned as $f\left(v_i'\right)=2$ for $i=0,1,\cdots,n-1$.

Case 3. Let $n=4k+1\equiv 1 \pmod 4$.

The label of the vertices v_i''s are given by

$f\left(v_0'\right)=1$; $f\left(v_i'\right)=2$ for $i=0,1,\cdots,n-3$ and $n-1$ and $f\left(v_{n-2}'\right)=0$.

Case 4. Let $n=4k+2\equiv 2 \pmod 4$.

The label of the vertices v_i''s are given by

$f\left(v_0'\right)=1$, for $i=0,n-1$;

$f\left(v_i'\right)=0$ for $i=n-2,n-3$;

$f\left(v_i'\right)=2$, for $i=2,3,\cdots,n-4$.

Case 5. Let $n=4k+3\equiv 3 \pmod 4$.

In this case the label of the pendent vertices v_i''s are as follows:

$f\left(v_i'\right)=1$, for $i=0,n-2,n-1$;

$f\left(v_{n-3}'\right)=0$; $f\left(v_i'\right)=2$, for $i=1,2,\cdots,n-4$.

Hence $\lambda_{0,1}\left(S_{2n}\right)=2=\Delta-1$. □

Lemma 9. *Let G be a graph obtained from S_{2n} by adding an edge to each of the pendent vertex of S_{2n}, then*

$$\lambda_{0,1}\left(G\right)=\lambda_{0,1}\left(S_{2n}\right)=\Delta-1=2. \qquad (10)$$

Proof. Let the graph is obtained by adding an edge $\left(v_i', v_i''\right)$ to each of the pendent vertices v_i' s. So in the new graph v_i'' are the pendent vertices.

Case 1. Let $n=3$.

$f\left(v_0''\right)=1$, $f\left(v_i''\right)=1$, for $i=1,2$.

Case 2. Let $n\equiv 0 \pmod 4$.

$f\left(v_i''\right)=1$, for $i=0,1,2,\cdots,n-1$.

Case 3. Let $n\equiv 1 \pmod 4$.

$f\left(v_0''\right)=1$, $f\left(v_{n-1}''\right)=0$ and $f\left(v_i''\right)=2$, for $i=1,2,\cdots,n-2$.

Case 4. Let $n\equiv 2 \pmod 4$.

$$f\left(v_i'\right)=\begin{cases}1, \text{ if } i=0,n-1;\\ 0, \text{ if } i=n-3,n-2.\end{cases}$$

and $f\left(v_i''\right)=2$, for $i=1,2,\cdots,n-4$.

Case 5. Let $n\equiv 3 \pmod 4$.

$f\left(v_i''\right)=1$, for $i=0,n-3,n-2,n-1$, and $f\left(v_i''\right)=2$, for $i=1,2,\cdots,n-4$.

Thus, from all above cases we have

$$\lambda_{0,1}\left(G\right)=2=\Delta-1. \qquad \square$$

Lemma 10. *If the graph G contains a cycle of any length and each vertex of the cycle has another cycle of any length, then*

$$\Delta-1\le\lambda_{0,1}\left(G\right)\le\Delta, \qquad (11)$$

where Δ is the degree of G.

Proof. At first we prove that if the graph G_1 contains a cycle C_n of length n and each vertex of C_n contains another cycle of length 3, or length 3 and length 4, or length 4, then $\lambda_{0,1}\left(G_1\right)=\Delta-1$ or Δ. Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of C_n and v_{01}, v_{02}; v_{11}, v_{12}; $\cdots$; $v_{n-1,1}, v_{n-1,2}$ are the vertices of all C_3's which are joined with each vertex of C_n. If the vertex v_{n-1} of C_n contains a cycle of length 4 then let the vertices of C_4 be $v_{n-1,1}$, $v_{n-1,2}$ and $v_{n-1,3}$. Again if the vertex v_{n-3} contains a cycle of length 4 then let the vertices of C_4 be $v_{n-3,1}$, $v_{n-3,2}$ and $v_{n-3,3}$. Therefore, all the vertices of C_n are cutvertices.

Now we label the graph as follows.

Case 1. For $n=3$.

Now, we label the vertices of other cycles joined with the vertices of C_n according to the following rule

$$f\left(v_{ij}\right)=\begin{cases}3, \text{ if } j=1;\\ 4, \text{ if } j=2, \text{ for } i=0,1,2.\end{cases}$$

If there are three cycles of length 4 then we label the vertices as follows:

$$f\left(v_{0j}\right)=\begin{cases}0, & \text{if } j=1;\\ 1, & \text{if } j=2;\\ 3, & \text{if } j=3;\end{cases}$$

$$f\left(v_{1j}\right)=\begin{cases}2, & \text{if } j=1;\\ 0, & \text{if } j=2;\\ 3, & \text{if } j=3;\end{cases}$$

and $f\left(v_{2j}\right)=\begin{cases}2, & \text{if } j=1;\\ 0, & \text{if } j=2;\\ 3, & \text{if } j=3.\end{cases}$

Case 2. Let $n=4k\equiv 0 \pmod 4$.

For $i=0,1,2,3$,

$$f\left(v_{ij}\right)=\begin{cases}2, \text{ if } j=1;\\ 3, \text{ if } j=2.\end{cases}$$

If all other cycles are of length 4 then the label of the last vertex of the last cycle is 3.

Case 3. Let $n=4k+1\equiv 1 \pmod 4$.

$$f\left(v_{0j}\right)=\begin{cases}1, & \text{if } j=1;\\ 3, & \text{if } j=2;\end{cases}$$

$$f\left(v_{n-2,j}\right)=\begin{cases}0, & \text{if } j=1;\\ 3, & \text{if } j=2;\end{cases}$$

$$f\left(v_{n-1,j}\right)=\begin{cases}3, & \text{if } j=1;\\ 4, & \text{if } j=2;\end{cases}$$

and $f\left(v_{ij}\right)=\begin{cases}2, & \text{if } j=1;\\ 3, & \text{if } j=2, \text{ for } i=1,2,\cdots,n-3.\end{cases}$

If the ($n-1$)th vertex v_{n-1} of C_n contains a cycle of length 4 then we label the vertices of C_4 as

$$f\left(v_{n-1,j}\right)=\begin{cases}2, & \text{if } j=1;\\ 0, & \text{if } j=2;\\ 3, & \text{if } j=3.\end{cases}$$

Case 4. Let $n=4k+2\equiv 2 \pmod 4$.

Now the label of all C_3 's are as follows:

for $i=0,n-1$, $f\left(v_{ij}\right)=\begin{cases}1, & \text{if } j=1;\\ 3, & \text{if } j=2.\end{cases}$

for $i=1,2,\cdots,n-4$, $f\left(v_{ij}\right)=\begin{cases}2, & \text{if } j=1;\\ 3, & \text{if } j=2.\end{cases}$

and for $i=n-3,n-2$, $f\left(v_{ij}\right)=\begin{cases}0, & \text{if } j=1;\\ 3, & \text{if } j=2.\end{cases}$

If C_n contain combined C_3 's and C_4 's then the minimum span is 3.

Case 5. Let $n=4k+3\equiv 3 \pmod 4$.

For $i=1,2,\cdots,n-4$, $f\left(v_{ij}\right)=\begin{cases}2, & \text{if } j=1;\\ 3, & \text{if } j=2;\end{cases}$

$$f\left(v_{n-3,j}\right)=\begin{cases}3, & \text{if } j=1;\\ 4, & \text{if } j=2;\end{cases}$$

and for $i=0,n-2,n-1$, $f\left(v_{ij}\right)=\begin{cases}1, & \text{if } j=1;\\ 3, & \text{if } j=2.\end{cases}$

If the vertex v_{n-3} contains a cycle of length 4 then the label of the vertices adjacent to v_{n-3} are

$$f\left(v_{n-3,j}\right)=\begin{cases}0, & \text{if } j=1;\\ 1, & \text{if } j=2;\\ 3, & \text{if } j=3.\end{cases}$$

From all the above cases, we see that 3 or 4 are used to label G, which is equal to $\Delta-1$ or Δ.

Therefore, $\Delta-1\le\lambda_{0,1}(G)\le\Delta$.

The proves of the other cases are similar. □

Corollary 2. *Let G be a graph which contains a cycle of length* n. *If each vertex of the cycle contains two or more cycles of length more than* 2, *then*

$$\lambda_{0,1}(G)=\Delta-1, \tag{12}$$

where Δ be the degree of *G*.

6. *L*(0,1)-Labelling of Caterpillar Graph

Now, we label another important subclass of cactus graphs called caterpillar graph.

Definition 1. *A caterpillar* C *is a tree where all vertices of degree* ≥ 3 *lie on a path*, *called the backbone of* C. *The hairlength of a caterpillar graph* C *is the maximum distance of a non-backbone vertex to the backbone.*

Lemma 11. *If G be a caterpillar graph and* Δ *be its degree*, *then*

$$\lambda_{0,1}(G)=\Delta-1. \tag{13}$$

Proof. Let P_n be a path of length n of the caterpillar graph and $v_0,v_1,\cdots,v_{n-1}$ be the vertices of P_n. We label the vertices of P_n according to the following rule.

$$f\left(v_i\right)=\begin{cases}0, & \text{if } i\equiv 0 \pmod 4;\\ 0, & \text{if } i\equiv 1 \pmod 4;\\ 1, & \text{if } i\equiv 2 \pmod 4;\\ 1, & \text{if } i\equiv 3 \pmod 4.\end{cases}$$

Let us assume that v_k be any vertex of P_n and v_{k-1}, v_{k+1} are the adjacent vertices of v_k. As we label of the vertices of P_n by 0 or 1 so without loss of generality let us consider that the label of v_k, v_{k-1} and v_{k+1} are 0, 0 and 1 respectively. Again let us consider that l number of paths $P_m^{(j)}$; $j=0,1,\cdots,l-1$, of same lengths are joined to the vertex v_k. Let $v_i^{(j)}$; $i=1,2,\cdots,m-1$ $j=0,1,\cdots,l-1$ are the vertices of l paths other than v_k. Now we label the vertices of these paths as in the following method:

$f\left(v_i^{(j)}\right)=2+j$ for $i=1,2$ and $j=0,1,\cdots,l-1$.

And we label other vertices of $P_m^{(j)}$'s as per the rule to label the vertices of P_n.

Now, $f\left(v_i^{(l-1)}\right)=2+(l-1)=l+1=\Delta-1$.

The result will be same when finite number of paths of different lengths are joined to one or more vertices of the path of the caterpillar graph.

So, $\lambda_{0,1}(G)=\Delta-1$. □

7. *L*(0,1)-Labelling of Lobster

Another subclass of cactus graphs is the lobster graph. The definition of lobster graph is given below.

Definition 2. *A lobster is a tree having a path* (*of maximum length*) *from which every vertex has distance at most* k, *where* k *is an integer.*

The maximum distance of the vertex from the path is called the diameter of the lobster graph. There are many types of lobsters given in literature like diameter 2, diameter 4, diameter 5, etc.

Lemma 12. *Let G be a lobster graph*. *If* Δ *be the degree of the lobster graph*, *then*

$$\lambda_{0,1}(G)=\Delta-1. \tag{14}$$

Proof. Let P_n be a path of length n of the lobster graph and $v_0, v_1, \cdots, v_{n-1}$ be the vertices of P_n. First we label the vertices of P_n according to Lemma 11.

Then we label the other vertices of that graph. Let us denote the other vertices of the graph by v_i'. Here v_i' are adjacent to v_i and v_{i+1}, $0 \le i \le n-2$. The label of the vertices of P_n are either 0 or 1. Then we label the vertices v_i' by 2, 3 and so on [it depends upon ($\Delta - 1$)].

Thus the $\lambda_{0,1}$-value of a lobster is $\Delta - 1$.□

We know from [2] that 3 labels are sufficent to label a complete binary tree by $L(0,1)$-labelling. Now we have to prove that for any tree the value of $\lambda_{0,1}$ is $\Delta - 1$, where Δ is the degree of the tree.

Lemma 13. *For any tree* T *of degree* Δ,

$$\lambda_{0,1}(T) = \Delta - 1. \tag{15}$$

Proof. Let T be a tree with degree Δ. We first label the root of the tree by 0. Now we know from the definition of $L(0,1)$-labelling that the label difference between any two adjacent vertices is at least 0 and the label difference between any two vertices which are at distance two is at least 1. Now we label the children of the root from left to right by 0, 1, 2, $\cdots$, $\Delta_{root} - 1$.

Let us consider the i th vertex of the tree. Assume that the label of the parent of i is known. Then the allowable label for the children of i are 0, 1, 2, $\cdots$ except the label of the parent of i. Now, we label the children of i by 0, 1, 2, $\cdots$, $\Delta_i - 2$, $\Delta_i - 1$, where Δ_i is the degree of the vertex i, except the label of the parent of i. This process is valid for any vertex i of the tree. Thus the maximum label used to label the entire tree by $L(0,1)$-labelling is $\max\{\Delta_i - 1 : i \in V\}$, which is exactly equal to $\Delta - 1$.

Hence $\lambda_{0,1}(T) = \Delta - 1$.□

The $L(0,1)$-labelling of all subgraphs of cactus graphs and their combinations are discussed in the previous lemmas. From these results we conclude that the $\lambda_{0,1}$-value of any cactus graph can not be more than Δ. Hence we have the following theorem.

Theorem 1. *If* Δ *is the degree of a cactus graph* G, *then*

$$\Delta - 1 \le \lambda_{0,1}(G) \le \Delta. \tag{16}$$

The graph of **Figure 3** is an example of a cactus graph, contains all possible subgraphs and its $L(0,1)$-labelling.

8. Conclusion

The bounds of $L(0,1)$-labelling of a cactus graph and various subclass viz., cycle, sun, star, caterpillar, lobster and tree are investigated. The bounds of $\lambda_{0,1}(G)$ for these graphs are $\lambda_{0,1}(C_n) = 1$ or 2, for sun, star, caterpillar, lobster and tree it is $\Delta - 1$. For the cactus graph the bound is $\Delta - 1 \le \lambda_{0,1}(G) \le \Delta$, where Δ is the

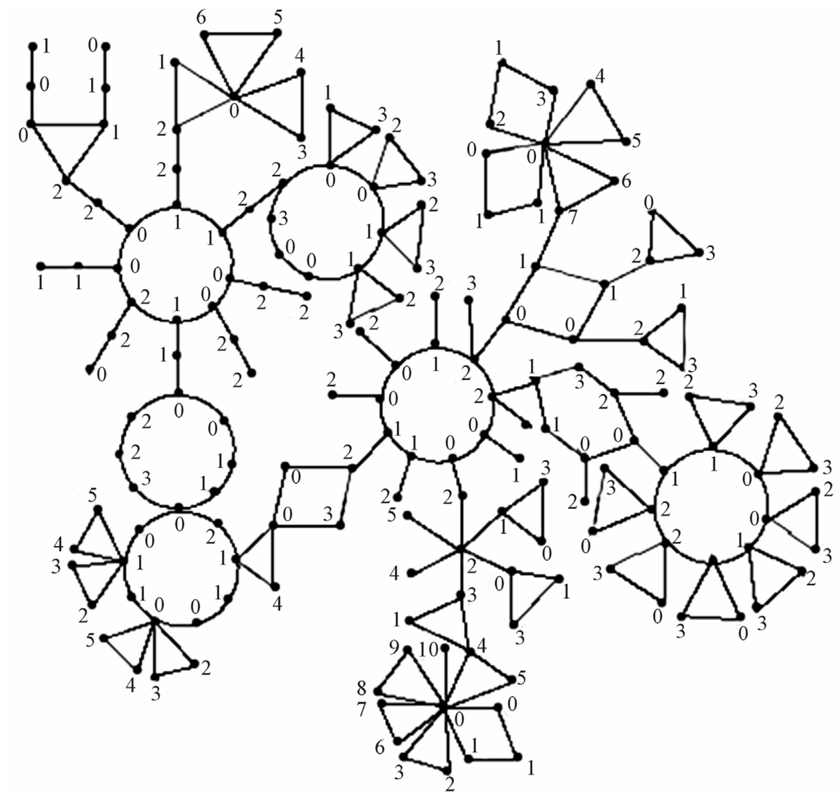

Figure 3. *L*(0,1)-labelling of a cactus graph.

maximum degree of the cactus graph G. Currently we are engaged to find the bounds for $L(h,k)$-labelling for different values of h, k on cactus graphs.

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