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A Note on the Paper "Generalized ϕ-Contraction for a Pair of Mappings on Cone Metric Spaces"

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ABSTRACT

We note that Theorem 2.3 [1] is a consequence of the same theorem for one map.

Keywords: Generalized ϕ-Contraction; Weakly Compatible; Solid Cone

1. Introduction

Huang and Zhang [2] initiated fixed point theory in cone metric spaces. On the other hand, the authors [3] gave a lemma and showed that some fixed point generalizations are not real generalizations. In this note, we show that Theorem 2.3 [1] is so.

Following [2], let E be a real Banach space and θ be the zero vector in E, and $P \subseteq E$. P is called **cone** iff

1) P is closed, nonempty and $P \neq \{\theta\}$,

2) $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,

3) $P \cap (-P) = \{\theta\}$.

For a given cone P, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. $x \prec y$ (resp. $x \ll y$) stands for $x \preceq y$ and $x \neq y$ (resp. $y - x \in \operatorname{int}(P)$), where $\operatorname{int}(P)$ denotes the interior of P. In the paper we always assume that P is **solid**, *i.e.*, $\operatorname{int}(P) \neq \phi$. It is clear that $x \ll y$ leads to $x \preceq y$ but the reverse need not to be true.

The cone P is called **normal** if there exists a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the normal constant of P.

Definition 1.1 [2]. Let X be a nonempty set. A function $d: X \times X \to E$ is called cone metric iff

$$(M_1) \quad \theta \preceq d(x,y),$$

$$(M_2) \quad d(x,y) = d(y,x) = \theta \text{ iff } x = y,$$

$$(M_3) \quad d(x,y) = d(y,x),$$

$$(M_4) \quad d(x,y) \preceq d(x,z) + d(z,y),$$

for all $x, y, z \in X$, (X,d) is said to be a **cone metric space**.

In [3], the authors gave the following important lemma.

Lemma 1.1 Let X be a nonempty and $f: X \to X$. Then there exists a subset $Y \subseteq X$ such that $f(Y) = f(X)$ and $f: Y \to X$ is one-to-one.

Definition 1.2 [4]. Let (X,d) be a cone metric space and $f, g: X \to X$ be mappings. Then, $z \in X$ is called a **coincidence point** of f and g iff $f(z) = g(z)$.

Definition 1.3 [1]. Let (X,d) be a cone metric space. The mappings $f, g: X \to X$ are **weakly compatible** iff for every coincidence point $z \in X$ of f and g, $f(g(x)) = g(f(x))$.

Definition 1.4 (see [1]). Let P be a solid cone in a real Banach space E. A nondecreasing function $\phi: P \to P$ is called a **comparison function** iff

1) $\phi(\theta) = \theta$ and $\theta \prec \phi(x) \prec x$ for $x \in P - \{\theta\}$;

2) $x \in \operatorname{int}(P)$ implies $x - \phi(x) \in \operatorname{int}(P)$;

3) $\lim_{n\to\infty} \phi^n(x) = 0$ for all $x \in P - \{\theta\}$.

In [1], the authors established the following fixed point theorem.

Theorem 1.1 Let (X,d) be a cone metric space, P a solid cone and $f, g: X \to X$. Assume that (f,g) is a generalized ϕ-contraction; *i.e.*, $d(f(x), f(y)) \preceq \varphi(u)$ for all $x, y \in X$ and some u where

$$u \in \left\{ d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{d(g(x), f(y)) + d(g(y), f(x))}{2} \right\}.$$

Suppose that $f(X) \subseteq g(X)$, $f(X)$ or $g(X)$ is a complete subspace of X, and f and g are weakly compatible. Then the mappings *f* and *g* have a unique common fixed point in *X*.

2. Main Result

In Theorem 1.1, if we choose $g = I_X$ ($I_X :=$ the identity map on *X*), then we have the following theorem.

Theorem 2.1 Let (X, d) be a cone metric space, *P* a solid cone and $f: X \to X$. Assume that *f* is a generalized ϕ-contraction; *i.e.*, $d(f(x), f(y)) \preceq \phi(u)$ for all $x, y \in X$ and some *u* where

$$u \in \left\{ d(x,y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(x)) + d(y, f(y))}{2} \right\}.$$

Suppose that $f(X)$ or *X* is a complete subspace of *X*. Then the mapping *f* has a unique fixed point in *X*.

Now, we state and prove our main result in the following way.

Theorem 2.2 Theorem 1.1 is a consequence of Theorem 2.1.

Proof. By Lemma 1.1, there exists $Y \subseteq X$ such that $g(Y) = g(X)$ and $g: Y \to X$ is one-to-one. Define a map $h: g(Y) \to g(Y)$ by $h(g(x)) = f(x)$ for each $x \in g(Y)$. Since *g* is one-to-one on *Y*, then *h* is well-defined. $d(f(x), f(y)) \preceq \phi(u)$ for all $x, y \in X$ and some *u* where

$$u \in \left\{ d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{d(g(x), f(y)) + d(g(y), f(x))}{2} \right\}.$$

Since $f(X)$ or $g(Y) = g(X)$ is complete, by using Theorem 2.1, there exists $x_0 \in X$ such that $h(g(x_0)) = g(x_0) = f(x_0)$. Hence, *f* and *g* have a point of coincidence which is also unique. Since *f* and *g* are weakly compatible, then *f* and *g* have a unique common fixed point.

Remark 2.1 Since Theorem 1 [4] is a special case of Theorem 1.1, then it is a consequence of Theorem 2.1, too.

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