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Bounds for the Zeros of a Polynomial with Restricted Coefficients

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ABSTRACT

In this paper we shall obtain some interesting extensions and generalizations of a well-known theorem due to Enestrom and Kakeya according to which all the zeros of a polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ satisfying the restriction $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$ lie in the closed unit disk.

Keywords: Polynomial; Bounds; Zeros

1. Introduction and Statement of Results

The following results which is due to Enestrom and Kakeya [1] is well known in the theory of the location of the zeros of polynomials.

THEOREM A. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0, \tag{1}$$

then $P(z)$ does not vanish in $|z| > 1$.

In the literature [2-5] there exist some extensions and generalization of Enestrom-Kakeya Theorem. Joyal, Labelle and Rahman [6] extended this theorem to polynomials whose coefficients are monotonic but not necessarily non-negative by proving the following result:

THEOREM B. Let

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0$$

then the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

of degree n has all its zeros in

$$|z| \leq \frac{1}{|a_n|}\left\{a_n - a_0 + |a_0|\right\} \tag{2}$$

Recently Aziz and Zarger [7] relaxed the hypothesis in several ways and among other things proved the following results:

THEOREM C. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be the polynomial of degree n, such that for some $k \geq 1$,

$$k a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0 \tag{3}$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq k \tag{4}$$

The aim of this paper is to prove some extensions of Enestrom-Kakeya Theorem (Theorem-A) by relaxing the hypothesis in various ways. Here we shall first prove the following generalization of Theorem C which is an interesting extension of Theorem A.

2. Main Results

THEOREM 1.1. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n. If for some positive numbers k and ρ with $k \geq 1$, and $0 < \rho \leq 1$

$$k a_n \geq a_{n-1} \geq \cdots \geq \rho a_0 \geq 0 \tag{5}$$

then all the zeros of $P(z)$ lie in the closed disk

$$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho) \tag{6}$$

If we take $k = \frac{a_{n-1}}{a_n} \geq 1$, in Theorem 1.1 we obtain the following result which is a generalization of Corollary 2 ([7]).

COROLLARY 1. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomials of degree n. If for some positive real number $\rho, 0 < \rho \le 1$

$$a_n \le a_{n-1} \ge a_{n-2} \ge \cdots \ge \rho a_0 > 0 \quad (7)$$

then all zeros of $P(z)$ lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \le \frac{a_{n-1}}{a_n} + \frac{2a_0}{a_n}(1-\rho)$$

REMARK 1. Theorem 1.1 is applicable to situations when Enestrom-Kakeya Theorem gives no information. To see this consider the polynomial.

$$P(z) = \alpha z^n + (\alpha - 1) z^{n-1} + \cdots + (\alpha - 1) z + \alpha,$$

with $\alpha > 1$ is a positive real number. Here Enestrom-Kakeya Theorem is not applicable to $P(z)$ where as Theorem 1.1 is applicable with $k = 1, \rho = \frac{n-1}{n}$ and according to our result, all the zeros of $P(z)$ lie in the disk.

$$|z| \le 1 + \frac{1}{\alpha}, \ \alpha > 1.$$

which is considerably better than the bound obtained by a classical result of Caushy ([4]) which states that all the zeros of $P(z)$ lie in

$$|z| \le 1 + A$$

where

$$A = \max_{1 \le j \le n} \left| \frac{a_{n-j}}{a_n} \right|,$$

Next, we present the following generalization of corollary 1 which includes Theorem 4 of [6] as a special case and considerably improves the bound obtained by Dewan and Bidkham ([8], Theorem1) for $t = 0$ and $0 \le k \le n-1$.

THEOREM 1.2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n. If for some positive number ρ, $0 < \rho \le 1$ and for some non-negative integer $\lambda, 0 \le \lambda \le n-1$

$$a_n \le a_{n-1} \le \cdots \le a_{\lambda+1} \le a_\lambda \ge a_{\lambda-1} \ge \cdots \ge a_1 \ge \rho a_{0,} \quad (8)$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \le \frac{1}{|a_n|} \left\{ 2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0 \right\} \quad (9)$$

Applying Theorem 1.2 to $P(tz)$, we get the following result:

COROLLARY 2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n. If for some positive numbers t and ρ with $0. < \rho \le 1$,

$$t^n a_n \le t^{n-1} a_{n-1} \le \cdots \le t^\lambda a_\lambda \ge \cdots \ge t a_1 \ge t \rho a_0$$

where λ, $0 \le \lambda \le n-1$ is a non negative integer then all the zeros of $P(z)$, lie in

$$\left| z + \frac{a_{n-1}}{a_n} - t \right| \le \frac{t}{|a_n|} \left\{ \left(\frac{2a_\lambda}{t^{n-\lambda}} - \frac{a_{n-1}}{t} \right) + \frac{1}{t^n} \left((2-\rho)|a_0| + a_0 \right) \right\}$$

If we assume $a_0 > 0$, in Theorem 1.2, we obtain.

COROLLARY 3. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n. If for some positive numbers $\rho, 0 < \rho < 1$ and for same non-negative integer λ, $0 \le \lambda \le n-1$

$$a_n \le a_{n-1} \le \cdots \le a_\lambda \ge \cdots \ge a_1 \ge \rho a_0 > 0$$

then all the zeros of $P(z)$ lie in,

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \le \frac{1}{a_n} \left(2a_\lambda - a_{n-1} + 2(1-\rho) a_0 \right) \quad (10)$$

Finally we present all following generalization of Theorem B due to Joyal, Labelle and Rahman which includes Theorem A as a special case.

THEOREM 1.3. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n, It for some positive number $\rho, 0 < \rho \le 1$ and for some non-negative integer $\lambda, 0 \le \lambda \le n-1$

$$a_n \le a_{n-1} \le \cdots \le a_\lambda \ge \cdots \ge a_1 \ge \rho a_0$$

then all the zeros of $P(z)$ lie in

$$|z| \le \frac{2a_\lambda - a_n + (2-\rho)|a_0| + \rho a_0}{|a_n|} \quad (11)$$

REMARK 2. For $\rho = 1$, Theorem 1.3 reduces to Theorem B.

3. Proofs of the Theorems

PROOF OF THEOREM 1.1. Consider

$$\begin{aligned} F(z) &= (1-z) P(z) = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_1 - a_0) z + a_0, \\ &= -a_n z^{n+1} + a_n z^n - k a_n z^n + (k a_n - a_{n-1}) z^n + \cdots + (a_1 - \rho a_0) z + (\rho - 1) a_0 z + a_0 \end{aligned}$$

then for $|z|>1$, we have

$$\begin{aligned}|F(z)| &= \left|-a_n z^{n+1} + a_n z^n - k a_n z^n + (k a_n - a_{n-1}) z^n + \cdots + (a_1 - \rho a_0) z + (\rho - 1) a_0 z + a_0\right| \\ &\geq |a_n||z|^n \left[|z+k-1| - \frac{1}{|a_n|}\left\{\left|(k a_n - a_{n-1}) + (a_{n-1} - a_{n-2})\frac{1}{z} + \cdots + (a_0 - p a_0) z(\rho - 1)\frac{a_0}{z^n} + \frac{a_0}{z^n}\right|\right\}\right] \\ &\geq |a_n||z|^n \left[|z+k-1| - \frac{1}{|a_n|}\left\{k a_n - \rho a_0 + (1-\rho)|a_0| + |a_0|\right\}\right] \\ &= |a_n||z|^n \left[|z+k-1| - \frac{1}{a_n}\left\{k a_n - \rho a_0 + a_0 + (1-\rho) a_0\right\}\right] > 0,\end{aligned}$$

if $|z+k-1| > \dfrac{k a_n + 2(1-\rho) a_0}{a_n}$

this shows that if $|z|>1$ then $|F(z)|>0$, if

$$|z+k-1| > k + 2(1-\rho)\frac{a_0}{a_n}$$

therefore all the zeros of $F(z)$, whose modulus is greater than 1 lie in the closed disk

$$|z+k-1| \leq k + 2(1-\rho)\frac{a_0}{a_n}$$

But those zeros of $F(z)$ whose modules is less than or equal to 1 already satisfy the Inequality (6).

Since all the zeros of $P(z)$ are also the zeros of $F(z)$. therefore it follows that all the zeros of $P(z)$ lie in the circle defined by (6). Which completes the proof of Theorem 1.1.

PROOF OF THEOREM 1.2. Consider

$$\begin{aligned}F(z) &= (1-z)P(z) \\ &= (1-z)\left(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_\lambda z^\lambda + \cdots + a_1 z + a_0\right), \\ &= -a_n z^{n+1} + (a_{n-\lambda} a_{n-1}) z^n + \cdots + (a_{\lambda-1} - a_\lambda) z^{\lambda+1} + (a_\lambda - z_{\lambda-1}) z^\lambda + \cdots + a_1 z - a_0 z + a_0\end{aligned}$$

therefore, for $|z|>1, \leq \lambda \leq n-1$, and $0<\rho<1$, we have

$$\begin{aligned}|F(z)| &\geq \left|a_n z^{n+1} - a_n z^n + a_{n-1} z^n\right| - \left|(a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda) z^{\lambda+1} + (a_\lambda - a_{\lambda-1}) z^\lambda\right. \\ &\quad \left. + \cdots + (a_1 - \rho a_0) z + (\rho a_0 - a_0) z + a_0\right| \\ &\geq |a_n||z|^n \left|z + \frac{a_{n-1}}{a_n} - 1\right| - |z|^n \left\{|a_{n-1} - a_{n-2}|\frac{1}{|z|} + \cdots + |a_{\lambda-1} - a_\lambda|\frac{1}{|z|^{n-\lambda-1}} + |a_{\lambda-1} - a_\lambda|\frac{1}{|z|^{n-\lambda}}\right. \\ &\quad \left. + \cdots + |a_1 - \rho a_0|\frac{1}{|z|^{n-1}} + |1-\rho||a_0|\frac{1}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\right\} \\ &> |a_n||z|^n \left[\left|z + \frac{a_{n-1}}{a_n} - 1\right| - \frac{1}{|a_n|}\left\{(a_{n-2} - a_{n-1}) + (a_{n-3} - a_{n-2}) + (a_\lambda - a_{\lambda+1})\right.\right. \\ &\quad \left.\left. + (a_\lambda - a_{\lambda-1}) + \cdots + (a_1 - \rho a_0) + (1-\rho)|a_0| + |a_0|\right\}\right] \\ &= |a_n||z|^n \left[\left|z + \frac{a_{n-1}}{a_n} - 1\right| - \frac{1}{|a_n|}\left\{2 a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0\right\}\right] > 0,\end{aligned}$$

if $\left|z + \dfrac{a_{n-1}}{a_n} - 1\right| > \dfrac{2 a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0}{a_n}$

Therefore all the zeros of $F(z)$ whose modulus is greater than 1 lie in the circle.

$$\left|z + \frac{a_{n-1}}{a_n} - 1\right| \le \frac{2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0}{a_n}$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the Inequality (9).

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, therefore it follows that all the zeros of $P(z)$ lie in the circle defined by (9). This completes the proof of Theorem 1.2.

PROOF OF THEOREM 1.3. Consider

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)\left(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_\lambda z^\lambda + \ldots + a_1 z + a_0\right) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{\lambda+1} a_\lambda) z^{\lambda+1} + (a_\lambda - a_{\lambda-1}) z^\lambda + \cdots + a_1 z - a_0 z + a_0 \end{aligned}$$

therefore, for $|z| > 1,\ 0 \le \lambda \le n-1$ and $0 < \rho \le 1$, we have

$$|F(z)| \ge |a_n z^{n+1}| - \left|(a_n - a_{n-1}) z^n + \cdots + (a_\lambda - a_{\lambda-1}) z^\lambda + a_1 z - a_0 z + a_0\right|$$

Proceeding similarly as in the proof of Theorem 1.2, we have

$$|F(z)| > \frac{|a_n||z|^n \left\{|z| - 2a_\lambda - a_n + (2-\rho)|a_0| - \rho a_0\right\}}{a_n} > 0,$$

$$\text{if } |z| > \frac{2a_\lambda - a_n + (2-\rho)|a_0| + \rho a_0}{|a_n|}$$

therefore all the zeros of $F(z)$ whose modules is greater than 1 lie in the circle

$$|z| \le \frac{(2a_\lambda - a_n) + (2-\rho)|a_0| + \rho a_0}{a_n}$$

But those zeros of $F(z)$ whose modulus is ≤ 1 already satisfy the (11). Since all the zeros of $P(z)$ are also the zero of $F(z)$, therefore it follows that all the zeros of $P(z)$ lie in circle defined by (11) and hence Theorem 1.3 is proved completed.

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