Intelligent Information Management, 2010, 2, 134-142
doi:10.4236/iim.2010.22016 Published Online February 2010 (http://www.scirp.org/journal/iim)

Existence and Uniqueness of the Optimal Control in Hilbert Spaces for a Class of Linear Systems

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Abstract

We analyze the existence and uniqueness of the optimal control for a class of exactly controllable linear systems. We are interested in the minimization of time, energy and final manifold in transfer problems. The state variables space *X* and, respectively, the control variables space *U*, are considered to be Hilbert spaces. The linear operator *T*(*t*) which defines the solution of the linear control system is a strong semigroup. Our analysis is based on some results from the theory of linear operators and functional analysis. The results obtained in this paper are based on the properties of linear operators and on some theorems from functional analysis.

Keywords: Existence and Uniqueness, Optimal Control, Controllable Linear Systems, Linear Operator

1. Introduction

A particular importance should be assigned to the analysis of the control of linear systems, since they represent the mathematical model for various dynamic phenomena. One of the fundamental problems is the functional optimization that defines the performance index of the dynamic product. Thus, under differential and algebraic restrictions, one determines the control corresponding to functional extremisation under consideration [1,2]. Variational calculation offers methods that are difficult to use in order to investigate the existence and uniqueness of optimal control. The method of determining the field of extremals (sweep method), that analyzes the existence of conjugated points across the optimal transfer trajectory (a sufficient optimum condition), proves to be a very efficient one in this context [3–5]. Through their resulting applications, time and energy minimization problems represent an important goal in system dynamics [6–11]. Recent results for controllable systems express the minimal energy through the controllability operator [11–13]. Also, stability conditions for systems whose energy tends to zero in infinite time are obtained in the literature [13,14]. By using linear operators in Hilbert spaces, in this study we shall analyze the existence and uniqueness of optimal control in transfer problems. The goal of this paper is to propose new methods for studying the optimal control for exactly controllable linear system. The minimization of time and energy in transfer problem is considered. The minimization of the energy can be seen as being a particular case of the linear regulator problem in automatics. Using the adjoint system, a necessary and sufficient condition for exact controllability is established, with application to the optimization of a broad class of Mayer-type functionals.

2. Minimum Time Control

2.1. Existence

2.1.1. Problem Formulation

We consider the linear system $(\Sigma_{A,B})$:

$$\frac{dx}{dt} = Ax(t) + Bu(t), \quad x(t_0) = x_0 \in H, \tag{1}$$

where H is a Hilbert space with inner product $<\cdot,\cdot>$ and norm $\|\cdot\|$.

$A: D(A) \subset H \to H$ is an unbounded operator on H which generates a strong semi group of operators on H, $(S(t))_{t\geq 0} = (e^{tA})_{t\geq 0}$.

$B: U \to H$ is a bounded linear operator on another Hilbert space U, for example $B \in \mathcal{L}(U,H)$.

$u: [0,\infty] \to H$ is a square integrable function representing the system control (1).

For any control function u, there exists a solution of (1) given by

$$x(t)=e^{tA}x_0+\int_0^t e^{(t-s)A}Bu(s)\,ds,\quad t\ge 0 \tag{2}$$

The control problem is the following one:

Given $t_0\in I$, $x(t_0)$, $z\in X$ and the constant $M>0$, let us determine $u\in U$ such that $\|u\|\le M$ and

$$\begin{aligned}&(i)\quad x(t_1)=z\\ &(ii)\quad t_1=\inf\left\{t\,\middle|\,x(t)=x\right\}\end{aligned} \tag{3}$$

Here, X and U are Hilbert spaces.

Theorem 1. The optimal time control exists for the above formulated problem.

Proof

Let $\{t_n\}$ be a decreasing monotonous sequence such that $t_n\to t_1$. Also, let us consider the sequence $\{u_n(\tau)\}$, with $u_n=u(t_n)\in U$. We have

$$x(t_n)=S(t_n)x(t_0)+\int_{t_0}^{t_n}S(t_n-\tau)Bu_n(\tau)\,d\tau, \tag{4}$$

which gives

$$\begin{aligned}x(t_n)=S(t_n)x(t_0)+\int_{t_0}^{t_1}S(t_n-\tau)Bu_n(\tau)\,d\tau+\\ \int_{t_1}^{t_n}S(t_n-\tau)Bu_n(\tau)\,d\tau\end{aligned} \tag{5}$$

$S(t)$ being a continuous linear operator, it is, therefore, bounded. It follows that we have:

$$S(t_n)x(t_0)\to S(t_1)x(t_0) \tag{6}$$

$$\int_{t_1}^{t_n}S(t_n-\tau)Bu_n(\tau)\,d\tau\to 0 \tag{7}$$

If we consider the set of all the admissible controls

$$\Omega=\left\{u\in U\,\middle|\,\|u\|\le M\right\} \tag{8}$$

then, Ω is a weakly compact closed convex set.

As $\Omega\subset U$ is weakly compact, any sequence $(u)\in\Omega$ possesses a weakly convergent subsequence to an element $u_1\in\Omega$. Thus, for $u_1\in\Omega$ and any $u^*\in\Omega^*$ (the dual of Ω), we have

$$\lim_{i\to\infty}\left\langle u_i,u^*\right\rangle=\left\langle u_1,u^*\right\rangle \tag{9}$$

Since $u_1\in\Omega$, it follows that

$$\|u_1\|\le M \tag{10}$$

Also, we have

$$\begin{aligned}&B^*:X\to U,\\ &S^*(t_n-\tau)x\in U.\end{aligned} \tag{11}$$

So,

$$B^*S^*(t_n-\tau)^*x\in U. \tag{12}$$

For every $x\in X$, let us evaluate the difference

$$\begin{aligned}F=\left\langle\int_{t_1}^{t_1}S(t_n-\tau)Bu_n(\tau)\,d\tau,x\right\rangle-\\ \left\langle\int_{t_1}^{t_n}S(t_1-\tau)Bu_1(\tau)\,d\tau,x\right\rangle\end{aligned} \tag{13}$$

By writing

$$\int_{t_1}^{t_1}S(t_k-t_0)B\,d\tau=\mathcal{L}_k,\quad k=1,n, \tag{14}$$

the Expression (13) becomes

$$\left\langle\mathcal{L}_n u_n,x\right\rangle-\left\langle\mathcal{L}_1 u_1,x\right\rangle=\left\langle u_n,\mathcal{L}_n^*x\right\rangle-\left\langle u_1,\mathcal{L}_1^*x\right\rangle \tag{15}$$

or

$$\begin{aligned}F=\left\langle u_n,\mathcal{L}_1^*x\right\rangle-\left\langle u_1,\mathcal{L}_1^*x\right\rangle-\left\langle u_n,\mathcal{L}_1^*x\right\rangle+\\ \left\langle u_n,\mathcal{L}_n^*x\right\rangle.\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}&F=\left\langle u_n-u_1,\mathcal{L}_1^*x\right\rangle+\left\langle u_n,\left(\mathcal{L}_n^*-\mathcal{L}_1^*\right)x\right\rangle=\\ &\int_{t_0}^{t_1}\left\langle u_n(\tau)-u_1(\tau),B^*S(t_1-\tau)^*x\right\rangle d\tau+\\ &\int_{t_0}^{t_1}\left\langle u_n(\tau),B^*S(t_n-\tau)^*x-B^*S(t_1-\tau)^*x\right\rangle d\tau\end{aligned} \tag{17}$$

By using the properties of the operator $S(t)$, one obtains

$$\begin{aligned}F=\int_{t_0}^{t_1}\left\langle u_n(\tau)-u_1(\tau),B^*S(t_1-\tau)^*x\right\rangle d\tau+\\ \int_{t_0}^{t_1}\left\langle u_n(\tau),B^*S(t_1-\tau)^*\left[S(t_n-t_1)^*x-x\right]\right\rangle d\tau.\end{aligned} \tag{18}$$

Because the sequence u_n un converges weakly to u_1, the first term in (18) tends to zero for $n\to\infty$ (see(9)). On the other hand, $S(t_1-\tau)^*$ is a strongly continuous semigroup and, hence, from its boundedness, it follows that there exists a constant $K>0$ such that

$$\left\| B^* S(t_1-\tau)^* \left[S(t_n-t_1)^* x - x \right] \right\| \le K \left\| S(t_n-t_1)^* x - x \right\| \tag{19}$$

So, it follows that the second term in (18) tends to zero as $n \to \infty$. The sequence $\{x(t_n)\} \in X$ is weakly convergent to $x(t_1) = z \in X$ if, for any $x \in X^* = X$, we have

$$\lim_{n\to\infty} \langle x(t_n), x \rangle = \langle x(t_1), x \rangle, \tag{20}$$

which gives

$$\langle z, x \rangle = \left\langle S(t_1)\, x(t_0) + \int_{t_0}^{t_1} S(t_1-\tau)\, B u_1(\tau)\, d\tau, x \right\rangle \tag{21}$$

From (21), one gets

$$S(t_1)\, x(t_0) + \int_{t_0}^{t_1} S(t_1-\tau)\, B u_1(\tau)\, d\tau = z \tag{22}$$

and, hence, u_1 is the optimal control.

An important result which is going to be used for proving the uniqueness belongs to A. Friedman:

Theorem 2 (bang-bang). Assuming that the set Ω is convex in a neighborhood of the origin and $u(t)$ is the control of optimal time in the problem formulated in (2.1.1), ([2, 10]) it follows that $u(t) \in \Omega$, for almost all $t \in [t_0, t_1]$.

2.2. Uniqueness of Time-Optimal Control

2.2.1. Rotund Space

Let β be the unity sphere in the Banach space U and let $\partial\beta$ be its boundary [11].

The space U is said to be rotund if the following equivalent conditions are satisfied:

a) if $\| x_1 + x_2 \| = \| x_1 \| + \| x_2 \|$, it follows that there exists a scalar $\lambda \ne 0$, such that $x_2 = \lambda x_1$;

b) each convex subset $K \subset U$ has at least one element that satisfies

$$\| u \| \le \| z \|, \quad u \in K, \quad z \in K;$$

c) for any bounded linear functional f on U there exists at least an element $x \in \beta$ such that

$$\langle x, f \rangle = f(x) = \| f \|;$$

d) each $x \in \partial\beta$ is a point of extreme of β.

Examples of rotund spaces**:**

1) Hilbert spaces.

2) Spaces l^p, L^p, $1 < p < \infty$ 1.

3) If the Banach spaces $U_1, U_2, \ldots, U_n$ are rotund, then the product space $U_1 \times U_2 \times \ldots \times U_n$ is rotund, too.

4) Uniform convex spaces are rotund, but the converse implication is not true.

2.2.2. Uniqueness

We assume that u_1 and u_2 are optimal, $u_i \in U$, $i = 1, 2$. Therefore,

$$S(t_1)\, x(t_0) + \int_{t_0}^{t_1} S(t_1-\tau)\, B u_1(\tau)\, d\tau = z \tag{23}$$

$$S(t_1)\, x(t_0) + \int_{t_0}^{t_1} S(t_1-\tau)\, B u_2(\tau)\, d\tau = z \tag{24}$$

By adding Equations (23) and (24), one obtains

$$S(t_1)\, x(t_0) + \int_{t_0}^{t_1} S(t_1-\tau)\, B \frac{1}{2}\big(u_1(\tau) + u_2(\tau)\big)\, d\tau = z \tag{25}$$

It follows that $u_1(\tau)$, $u_2(\tau)$, $1/2\big(u_1(\tau) + u_2(\tau)\big)$ are optimal. By using Theorem 2, we have

$$u_1(\tau), u_2(\tau), \frac{1}{2}\big(u_1(\tau) + u_2(\tau)\big) \in \partial\beta \Rightarrow$$

$$\| u_k \| = 1, \frac{1}{2} \| u_1 + u_2 \| = 1.$$

Since the condition (i) is satisfied, U is a rotund space and $u_1 = \lambda u_2$. Thus,

$$\| u_1 + u_2 \| = \big(1 + |\lambda|\big) \| u_2 \| = 2 \tag{26}$$

and, therefore

$$|\lambda| = 1 \Rightarrow u_1 = u_2 \tag{27}$$

which ends the proof of the uniqueness.

3. Minimum Energy Problem

3.1. Problem Formulation

Let $\Sigma_{A,B}$ (see(1)) be a controllable system with finite dimensional state space X [6,10,12, 14].

Let $I = [0, t_1]$, $x(0) = a \in X$, $b \in X$ be given. Let U be the Hilbert space $L^2(I, U)$. The minimum norm control problem can be formulated as follows: determine $u(t) \in U$ such that for some $t_1 \in I$,

1) $x(t_1) = b$,

2) $\|u\|$ is minimized on $[0, t_1]$, where $\|\cdot\|$ represents the norm on $L^2(I, U)$.

Let us fix $t_1 > 0$. Consider the linear operator

$$\mathcal{L}_n : L^2(0, t_1; U) \to H_1 \tag{28}$$

defined by

$$\mathcal{L}_n u =: \int_0^{t_1} S(t_1 - s) B u(s) ds \tag{29}$$

We have

$$x(t_1) = S(t_1) a + \mathcal{L}_{t_1} u = b. \tag{30}$$

Theorem 3. Let $\mathcal{L}_t : U \to H$ be a linear mapping between a Hilbert space U and a finite dimensional space H [11]. Then, there exists a finite dimensional space $M \subset U$ such that the restriction $\mathcal{L}_t^M$ of $\mathcal{L}_t$ to M is an injective mapping.

Proof

Let $\{e_i\}\ i = 1, \ldots, n$, be a basis for the range of $\mathcal{L}_t$ in H. Given any $u \in U$, $\mathcal{L}_t u$ can be written as

$$\mathcal{L}_t u = \sum_{i=1}^{n} \alpha_i e_i, \tag{31}$$

where each α_i can be expressed by

$$\alpha_i = \langle f_i, u \rangle \qquad f_i \in U^* = U. \tag{32}$$

Then,

$$\mathcal{L}_t u = \sum_{i=1}^{n} \langle f_i, u \rangle e_i . \tag{33}$$

f_i are linearly independent and generate a n dimensional subspace $M \subset U$.

From the properties of Hilbert spaces (the projection theorem), it follows that $U = M \oplus M^{\perp}$.

Let $u \in M^{\perp}$. Then, $\langle f_i, u \rangle = \alpha_i = 0$. Thus,

$$M^{\perp} \subset \{ u \in U \mid \mathcal{L}_t u = 0 \} = \ker \mathcal{L}_t . \tag{34}$$

Let $u \in \ker \mathcal{L}_t$. We get

$$\mathcal{L}_t u = 0 = \sum_{i=1}^{n} \langle f_i, u \rangle e_i . \tag{35}$$

Since e_i are independent, $\langle f_i, u \rangle = 0,\ i = 1, \ldots, n$ and, so, $M = \ker \mathcal{L}_t$. $\mathcal{L}_t$ maps M bijectively to the range of $\mathcal{L}_t$ and hence $\mathcal{L}_t^M$ is an injective mapping into H. Let

$$u = u_1 + u_2,\ u_1 \in M,\ u_2 \in M^{\perp},\ u_2 \neq 0. \tag{36}$$

Because $\mathcal{L}_t u_2 = 0$,

$$\mathcal{L}_t u = \mathcal{L}_t^M u_1 = x \in H \tag{37}$$

Since $\langle u_1, u_2 \rangle = 0$, we have

$$\| u \|^2 = \langle u_1 + u_2, u_1 + u_2 \rangle = \| u_1 \|^2 + \| u_2 \|^2 \tag{38}$$

Then, a unique minimum norm $\left(\mathcal{L}_t^M\right)^{-1} x$ exists and $u_1 \in \mathcal{L}_t^M x$ is the minimum norm element satisfying $\mathcal{L} u = x$.

Remark 1. The unique solution $u(t)$ of the equation $\mathcal{L} u = x$, with minimum norm control, is the projection of $u(t)$ onto the closed subspace $M = (f_1, \ldots\ldots, f_n)$.

Hence, from (30), there exists a control $u(\cdot) \in L^2(0, t_1; U)$ transferring a to b in time t_1 if and only if $(b - S(t_1) a) \in \operatorname{Im} \mathcal{L}_{t_1}$. The control which achieves this and minimizes the functional $E_{t_1} = \int_0^{t_1} \| u(s) \|^2 ds$, called the energy functional, is

$$u = \mathcal{L}_{t_1}^{-1} (b - S(t_1) a) \tag{39}$$

Define the linear operator

$$Q_t = \int_0^t S(r) B B^* S(r) dr, \quad t \geq 0 \tag{40}$$

We have the following results (see [10–12]):

Proposition 1.

1) The function Q_t, $t \geq 0$, is the unique solution of the Equation [1,13]

$$\frac{d}{dt} \langle Q_t x, x \rangle = 2 \langle Q_t^* x, x \rangle + \| B^* x \|^2 \qquad x \in D(A^*), Q_0 = I \tag{41}$$

where

$$D(A^*) = \{ y \in H \mid \exists C \in R^+, | \langle Ax, y \rangle_H | \leq C \| x \|_H, \forall x \in D(A) \}. \tag{42}$$

2) If A generates a stable semigroup, then

$$\lim_{t\to\infty} Q_t = Q \quad (43)$$

exists and is the unique solution of the equation

$$2\langle Q A^* x, x\rangle + \| B^* x \|^2 = 0, \quad x \in D(A^*), \quad (44)$$

The proof of Proposition 1 is given in [10,11].

The following theorem gives general results for the functionals $E_\eta(a,b)$, the minimal energy for transferring a to b in time t_1, and $E_\infty(0,b)$, where $a, b \in H$, $t_1 > 0$ (see [12]).

Theorem 4.

1) For arbitrary $t_1 > 0$ and $a, b \in H$

$$E_\eta(a,b) = \left\| \left(Q_\eta^{1/2}\right)^{-1} \left(S(t_1)a - b\right) \right\|^2 . \quad (45)$$

2) If $S(t)$ is stable and the system $\left(\Sigma_{A,B}\right)$ is null controllable in time $t_0 > 0$, then

$$E_\infty(0,b) = \left\| \left(Q^{1/2}\right)^{-1} b \right\|^2, b \in H \quad (46)$$

3) Moreover, there exists $C_{t_0} > 0$ such that

$$\left\| \left(Q^{1/2}\right)^{-1} b \right\|^2 \le E_{t_1}(0,b) \le C_{t_0} \cdot \left\| \left(Q^{1/2}\right)^{-1} b \right\|^2, b \in H, t_1 \ge t_0 \quad (47)$$

4. Numerical Methods for Minimal Norm Control

4.1. Presentation of the Numerical Methods

Let $\Sigma_{A,B}$ be a dynamic system with $U = R^m$, $X = R_n$, $I = R^1$ [9,10,13]. Given $x_0 = 0$, t_0, t_1, b, determine $u(t) \in U$ such that

1) $x(t_1) = b$,

2) $\| u_p \|$ is minimized for $p \in [1, \infty)$

Now,

$$e = b = \int_{t_0}^{t_1} S(t_1 - \tau) B u(\tau) d\tau = \int_{t_0}^{t_1} \Phi(t_1 - \tau) u(\tau) d\tau \quad (48)$$

In order to make Equation (48) true, $u(t)$ must be chosen on the interval $[t_0, t_1]$. Consider the i-th component e_i of the vector e:

$$e_i = \int_{t_0}^{t_1} \varphi_i(t_1 - \tau) u(\tau) d\tau = \langle \varphi_i, u \rangle = f(\varphi_i) \quad (49)$$

where φ_i is the row of the matrix Φ and f is the unique functional corresponding to the inner product (Rieszrepresentation theorem).

Let λ_i be an arbitrary scalar. Then,

$$\lambda_i e_i = \lambda_i f(\varphi_i) = f(\lambda_i \varphi_i), \quad (50)$$

Since R^n is a Hilbert space, the inner product

$$\sum_{i=1}^n \lambda_i e_i = \sum_{i=1}^n f(\lambda_i \varphi_i) = \langle \lambda, e \rangle \quad (51)$$

λ being a vector with arbitrary components $\lambda_1, \ldots, \lambda_n$. If Equation (51) is true for at least n different linearly independent, λ_i, $i = 1, \ldots, n$ then

Equation (48) is also true.

We have

$$\left| f\left(\sum_{i=1}^n \lambda_i \varphi_i \right) \right| = \left| \left\langle \sum_{i=1}^n \lambda_i \varphi_i, u \right\rangle \right| . \quad (52)$$

By Hölder's inequality,

$$\left| \left\langle \sum_{i=1}^n \lambda_i \varphi_i, u \right\rangle \right| \le \| u \|_p \left\| \sum_{i=1}^n \lambda_i \varphi_i \right\|_q , \quad \left(\frac{1}{p} + \frac{1}{q} = 1 \right). \quad (53)$$

From Equations (51) and (53),

$$\| u \|_p \ge \frac{\left| f\left(\sum_{i=1}^n \lambda_i \varphi_i \right) \right|}{\left\| \sum_{i=1}^n \lambda_i \varphi_i \right\|_q} = \frac{\sum_{i=1}^n \lambda_i e_i}{\left\| \sum_{i=1}^n \lambda_i \varphi_i \right\|_q} = \frac{\langle \lambda, e \rangle}{\left\| \sum_{i=1}^n \lambda_i \varphi_i \right\|_q} . \quad (54)$$

Every control driving the system to the point b must satisfy Equation (54), while for the optimal (minimum norm) control, Equation (54) must be satisfied with equality (Theorem 2). Numerically, one must search for a n vectors λ such that the right-hand side of Equation (54) takes its maximum.

4.2. A Simple Example

We consider a single output linear dynamic system $\Sigma_{A,B}$, where

$$U = R^1, \quad X = R^1, \quad I = R^1$$

Because $t_1 \in I$ fixed, then $S(t_1 - \tau)B = \Phi(\tau)$

$$e = b = \int_{t_0}^{t_1} \Phi(\tau) u(\tau) d\tau . \tag{55}$$

Therefore,

$$|e| = b = \left| \int_{t_0}^{t_1} \Phi(\tau) u(\tau) d\tau \right| \le \int_{t_0}^{t_1} |\Phi(\tau) u(\tau)| d\tau \tag{56}$$

and from Hölder's inequality,

$$\int_{t_0}^{t_1} |\Phi(\tau) u(\tau)| d\tau \le \|\Phi\| \|u\| , \tag{57}$$

where we have assumed that $\Phi(t) \in L^2(I)$. The minimum norm control, if exists, will satisfy

$$\|u\| = \frac{|e|}{\|\Phi\|} . \tag{58}$$

From (56) and from (57), we have

$$\operatorname{sign}(u(t)) = \operatorname{sign}(\Phi(t)), \quad \forall t \in [t_0, t_1] \tag{59}$$

and, respectively,

$$|\Phi(t)|^q = h |u(t)|^p , \quad \forall t \in [t_0, t_1], \tag{60}$$

h arbitrary constant.

The control u satisfies the relation

$$|u(t)| = h^{-1/p} |\Phi(t)|^{q/p} = k |\Phi(t)|^{q/p} \tag{61}$$

or

$$u(t) \operatorname{sign}(\mathrm{u(t)}) = k |\Phi(t)|^{q/p} . \tag{62}$$

From condition (59),

$$u(t) = k \operatorname{sign}(\Phi(t)) |\Phi(t)|^{q/p} . \tag{63}$$

By substituting Equation (63) into Equation (55), the constant k can be determined.

4.2.1. The Particular Case $p = q = 2$

In this case, the Equation (55) represents the inner product in $L^2(t_0, t_1; U)$,

$$b = \langle \Phi, u \rangle \tag{64}$$

The relation (63) becomes

$$u(t) = k \operatorname{sign}(\Phi(t)) |\Phi(t)| . \tag{65}$$

Then,

$$b = \int_{t_0}^{t_1} k \Phi^2(\tau) d\tau = k \|\Phi\|^2 . \tag{66}$$

Thus,

$$k = \frac{b}{\|\Phi\|^2} \tag{67}$$

and the optimal control is given by

$$u(t) = \frac{b \Phi(t)}{\|\Phi\|^2} . \tag{68}$$

5. Exact Controllability

5.1. Adjoint System

Let $S(t), t \in [0, t_1]$, be the fundamental solution of an homogeneous system associated to the linear control system $\Sigma_{A,B}$ (see(1)) [4,5,13] .

Thus, we have

$$\frac{dS(t)}{dt} = AS(t), \quad S(0) = I, t \in [0, t_1] \tag{69}$$

From

$$\frac{d}{dt}\left(S(t) S^{-1}(t)\right) = \frac{dI}{dt} = 0 . \tag{70}$$

one obtains

$$\frac{d}{dt} S^{-1}(t) = S^{-1}(t) A . \tag{71}$$

which implies that

$$\left(S^{-1}(t)\right)^* = -A^* \left(S^{-1}(t)\right)^* . \tag{72}$$

Since

$$\left(S^{-1}(t)\right)^* = \left(S^*(t)\right)^{-1} . \tag{73}$$

the system becomes

$$\frac{d}{dt}\left[S^*(t)^{-1}\right] = -A^* \left(S^*(t)\right)^{-1}, t \in [0, t_1] \tag{74}$$

It follows that $\left(S^*(t)\right)^{-1}, t \in [0, t_1]$, is the fundamental solution for the adjoint System (69).

5.2. A Class of Linear Controllable Systems

We consider the class of linear control systems in vectorial form [1,3,4,8,13,15]

$$\frac{dx}{dt} = Ax + Bu + C, \quad x(0) = x_0 \tag{75}$$

Let u_0 be any internal point of the closed bounded convex control domain U.

This domain contains the space E_m of variables $u = (u_1, \ldots, u_m)$. We take

$$\bar{u} = u - u^\circ \tag{76}$$

By this transformation, system (75) becomes

$$\frac{dx}{dt} = Ax + B\bar{u} + \left(Bu^\circ + C\right). \tag{77}$$

Thus, one transfers the origin of the coordinates of space E_m in u°. At the same time, the origin of the coordinates of space E_m is an internal point inside the domain U. Let us denote by $x^\circ(t)$ the solution of system (75) which corresponds to the control $u \equiv 0$.

This control is admissible, as the origin of the coordinates belongs to the domain U and satisfies the initial condition $x^\circ(0) = x_0$. As a result,

$$\frac{dx^\circ(t)}{dt} = Ax^\circ(t) + C. \tag{78}$$

Let $u_1(t), 0 \le t \le t_1$, be any control and $x_1(t)$ be the trajectory corresponding to system (75).

We have

$$\frac{dx_1(t)}{dt} = Ax_1(t) + Bu_1(t) + C, x_1(0) = x_0. \tag{79}$$

We take

$$\bar{x}(t) = x_1(t) - x^\circ(t) \tag{80}$$

Then, from (78) and (79), one obtains

$$\frac{d\bar{x}(t)}{dt} = A\bar{x}(t) + Bu_1(t).$$

Hence, for $u = u_1(t)$, the system (75) becomes

$$\Sigma_{A,B}: \begin{cases} \dot{x} = Ax + Bu \\ x(0) = 0 \end{cases} \quad x \in R^n, \tag{81}$$

Thus, the linear control system belongs to the class $\Sigma_{A,B}$ defined by (81).

5.3. Optimal Control Problems

For a given t_1, let us determine the optimal control $\tilde{u}$ that extremises the functional of final values

$$J = F\left(x(t_1)\right) = \sum_{i=1}^{n} c_i x_i(t_1) \tag{82}$$

and satisfies the differential constraints represented by the system (81).

The adjoint variable $y \in R$ satisfies equation

$$\dot{y} = -\frac{\partial H}{\partial x} \tag{83}$$

where H is the Hamiltonian associated to the optimal problem

$$H = \langle y, \dot{x} \rangle \tag{84}$$

Since the final conditions $x(t_1)$ are free and the final time has been indicated from the condition of transversality, one obtains $y(t_1) = y_{t_1}$.

It follows that the adjunct system becomes

$$\Sigma^*_{A,B}: \begin{cases} \dot{y} = -A^* y \\ y(t_1) = y_{t_1} \end{cases} \tag{85}$$

with solution

$$y(t) = S^*(t_1 - t) y_{t_1}, \quad y_{t_1} \in H. \tag{86}$$

Proposition 2.

For the class of optimum problems under consideration, the following identity holds true:

$$\left\langle x_{t_1}, y_{t_1} \right\rangle_H = \int_0^{t_1} \left\langle u, B^* y \right\rangle_U dt \tag{87}$$

Proof

Assuming that $u \in C^1\left(\left[0, t_1\right], U\right)$ and $y(t_1) \in D(A^*)$ it follows that $x, y \in C^1\left(\left[0, t_1\right], U\right)$.

Integrating by parts, since $x(0) = 0$, $y(t_1) = y_{t_1}$, $B: U \to H$, $B^*: H \to U$, we have

$$\begin{aligned} 0 &= \int_0^{t_1} \left\langle \dot{x} - Ax - Bu, y \right\rangle_H dt = \\ &\int_0^{t_1} \left\langle \dot{x}, y \right\rangle_H dt - \int_0^{t_1} \left\langle Ax, y \right\rangle_H dt - \\ &\int_0^{t_1} \left\langle u, B^* y \right\rangle_U dt = \left\langle x, y \right\rangle \Big|_0^{t_1} - \\ &\int_0^{t_1} \left\langle x, -A^* y \right\rangle_H dt - \int_0^{t_1} \left\langle x, A^* y \right\rangle_H dt - \\ &\int_0^{t_1} \left\langle u, B^* y \right\rangle_U dt \end{aligned} \tag{88}$$

One obtains

$$\left\langle x_{t_1}, y_{t_1} \right\rangle_H - \int_0^{t_1} \left\langle u, B^* y \right\rangle_U dt = 0 \tag{89}$$

The identity has been demonstrated.

This result can be extended for arbitrary $u \in L^2(0,t_1;U)$ and $y(t_1) \in H$.

An important result referring to the exact controllability of the linear system (81) is stated in

Theorem 5. The system $\Sigma_{A,B}$ is exactly controllable if only if the following condition is satisfied

$$\int_0^{t_1} \left\| B^* S^*(t) y_0 \right\|_U^2 dt \ge c \left\| y_0 \right\|_U^2, \quad \forall y_0 \in H \tag{90}$$

Proof

"$\Rightarrow$" We assume that $\Sigma_{A,B}$ is exactly controllable. Let $u \in L^2(0,t_1;U)$ and $y(t_1) \in H$.

We consider that the application

$$L_{t_1} : U \to x(T) \tag{91}$$

is well defined.

Let $\Lambda : H \to L^2(0,t_1;U)$ be the inverse of L_{t_1}. From Theorem 3 it follows that there exists a finite dimensional subspace $M \subset H$, $M^\perp = \ker L_{t_1}$ such that the restriction

$$\left(L_{t_1}\right)_M = \left(L_{t_1}\right)\Big|_{\left(\ker L_{t_1}\right)^\perp} \tag{92}$$

is injective.

Since $L_{t_1} u = x(t_1) \Rightarrow u = L_{t_1}^{-1}\left(x(t_1)\right) = \Lambda\left(x(t_1)\right)$ it follows that transfers 0 in $x(t_1)$ for the system $\Sigma_{A,B}$.

We choose $y(t_1) \in H$ and $x(t_1) = y(t_1)$, $u = \Lambda\left(x(t_1)\right)$. It follows that

$$\left\| y_{t_1} \right\|_H^2 = \left\langle y_{t_1}, y_{t_1} \right\rangle = \int_0^{t_1} \left\langle \Lambda(x_{t_1}), B^* y \right\rangle_U dt \tag{93}$$

For $\Lambda(x_{t_1}) = u \in L^2$, $B^* y \in L^2$ B, using Hölders's inequality, one obtains

$$\begin{aligned} \left| \int_0^{t_1} \left(\Lambda(x_{t_1})\right)\left(B^* y\right) dt \right| &\le \int_0^{t_1} \left| \Lambda(x_{t_1}) \left(B^* y\right) \right| dt \le \\ \left| \int_0^{t_1} \left| \Lambda(x_{t_1}) \right|^2 dt \right|^{1/2} &\cdot \left| \int_0^{t_1} \left| B^* y \right|^2 dt \right|^{1/2} \end{aligned} \tag{94}$$

From (93) and (94), for $x_{t_1} = y_{t_1}$, we have

$$\begin{aligned} \left\| y_{t_1} \right\|_H^2 &\le \left\| \Lambda(y_{t_1}) \right\| \left(\int_0^{t_1} \left| B^* y \right|_U^2 dt \right)^{1/2} \le \\ &\left\| \Lambda \right\| \left\| y_{t_1} \right\|_H \left(\int_0^{t_1} \left| B^* y \right|_U^2 dt \right)^{1/2} \end{aligned} \tag{95}$$

or

$$\int_0^{t_1} \left| B^* y \right|_U^2 dt \ge \frac{1}{\left\| \Lambda \right\|^2} \left\| y_{t_1} \right\|_H^2 \tag{96}$$

By changing $t \to t_1 - t \Rightarrow$ y_0 becomes y_{t_1} and $S^*(t)$ is transformed into $S^*(t_1 - t)$.

Therefore, we get one equivalent relation of controllable system in which y_{t_1} substitutes y_0.

$$\begin{aligned} \int_0^{t_1} \left| B^* S^*(t) y_0 \right|_U^2 dt &= \int_0^{t_1} \left| B^* S^*(t_1 - t) y_0 \right|_U^2 dt \ge \\ \frac{1}{\left\| \Lambda \right\|^2} \left\| y_{t_1} \right\|_H^2 &= c \left\| y_{t_1} \right\|_H^2 \end{aligned} \tag{97}$$

The direct implication has been demonstrated.

"$\Leftarrow$" We assume that condition (90) is fulfilled.

Then

$$\int_0^{t_1} \left| B^* y \right|_U^2 dt \ge c \left\| y_{t_1} \right\|_H^2 \tag{98}$$

For any $y_{t_1} \in H$ we take the set $\left\{ u(t) = B^* y(t) \right\}$ ($y(t)$ is solution of $\Sigma^*_{A,B}$).

We consider the solution $x(t)$ for $\Sigma_{A,B}$ which corresponds to the above mentioned control $u(t)$. Let us define the bounded operator

$$\Gamma : y_{t_1} \in H \to x(t_1) = L_{t_1}\left(B^* y(\cdot)\right) \in H \tag{99}$$

Since $u(t) = B^* y$, the identity (89) becomes

$$\left\langle \Gamma y_{t_1}, y_{t_1} \right\rangle_H = \int_0^{t_1} \left| B^* y(t) \right|_U^2 dt \ge c \left\| y_{t_1} \right\|_H^2 \tag{100}$$

Therefore, there exists a constant $c > 0$ for which (100) is satisfied. It follows that $\left\langle \Gamma y_{t_1}, y_{t_1} \right\rangle_H$ is positively defined.

This resumes the conclusion that the system $\Sigma_{A,B}$ is controllable.

Thus, since Γ is inversable, for any $x_{t_1} \in H$, state $y_{t_1} = \Gamma^{-1}(x_{t_1})$ is such that there exists a control $u(t) = B^* y$ which transfers 0 in y_{t_1}. The theorem has been demonstrated.

6. Conclusions

The main contribution in this paper consists in proving existence and uniqueness results for the optimal control in time and energy minimization problem for controllable linear systems.

A necessary and sufficient condition of exact controllability in optimal transfer problem is obtained. Also, a numerical method for evaluating the minimal energy is presented.

In Subsection 5.2 the nonhomogeneous linear control systems are transformed into linear homogeneous ones, with a null initial condition. For the control of such a class of systems, one needs to consider the adjoint system corresponding to the associated optimal transfer problem (Subsections 5.1 and 5.3).

The above theory can be used for solving various problems in spatial dynamics (rendez-vousspatial, satellite dynamics, space pursuing), [3,4,8,9]. Also, this theory can be successful applied to automatics, robotics and artificial intelligence problems [16–18] modelled by linear control systems.

7. References

[1] Q. W. Olbrot and L. Pandolfi, "Null controllability of a class of functional differential systems," International Journal of Control, Vol. 47, pp. 193–208, 1988.

[2] H. J. Sussmann, "Nonlinear controlability and optimal control," Marcel Dekker, New York, 1990.

[3] M. Popescu, "Variational transitory processes and nonlinear analysis in optimal control," Technical Education Bucharest, 2007.

[4] M. Popescu, "Sweep method in analysis optimal control for rendezvous problems," Journal of Applied Mathematics and Computing, Vol. 23, 1–2, pp.249–256, 2007.

[5] M. Popescu, "Optimal control in Hilbert space applied in orbital rendezvous problems," Advances in Mathematical Problems in Engineering Aerospace and Science; (ed. Sivasundaram), Cambridge Scientific Publishers 2, pp. 135–143, 2008.

[6] S. Chen and I. Lasiecka, "Feedback exact null controllability for unbounded control problems in Hilbert space," Journal of Optimization Theory and Application, Vol. 74, pp. 191–219, 1992.

[7] M. Popescu, "On minimum quadratic functional control of affine nonlinear control," Nonlinear Analysis, Vol. 56, pp. 1165–1173, 2004.

[8] M. Popescu, "Linear and nonlinear analysis for optimal pursuit in space," Advances in Mathematical Problems in Engineering Aerospace and Science; (ed. Sivasundaram), Cambridge Scientific Publishers 2, pp. 107–126, 2008.

[9] M. Popescu, "Optimal control in Hilbert space applied in orbital rendezvous problems," Advances in Mathematical Problems in Engineering Aerospace and Science; (ed. Sivasundaram), Cambridge Scientific Publishers 2, pp. 135–143, 2008.

[10] M. Popescu, "Minimum energy for controllable nonlinear and linear systems," Semineirre Theorie and Control Universite Savoie, France, 2009.

[11] M. Popescu, "Stability and stabilization dynamical systems," Technical Education Bucharest, 2009.

[12] G. Da Prato, A. J. Pritchard, and J. Zabczyk, "On minimum energy problems," SIAM J. Control and Optimization, Vol. 29, pp. 209–221, 1991.

[13] E. Priola and J. Zabczyk, "Null controllability with vanishing energy," SIAM Journal on Control and Optimization, Vol. 42, pp. 1013–1032, 2003.

[14] F. Gozzi and P. Loreti, "Regularity of the minimum time function and minimum energy problems: The linear case," SIAM Journal on Control and Optimization, Vol. 37, pp. 1195–1221, 1999.

[15] M. Popescu, "Fundamental solution for linear two-point boundary value problem," Journal of Applied Mathematics and Computing, Vol. 31, pp. 385–394, 2009.

[16] L. Frisoli, A. Borelli, and Montagner, *et al*., "Arm rehabilitation with a robotic exoskeleleton in Virtual Reality," Proceedings of IEEE ICORR'07, International Conference on Rehabilitation Robotics, 2007.

[17] P. Garrec, "Systemes mecaniques," in: Coiffet. P et Kheddar A., Teleoperation et telerobotique, Ch 2., Hermes, Paris, France, 2002.

[18] P. Garrec, F. Geffard, Y. Perrot (CEA List), G. Piolain, and A. G. Freudenreich (AREVA/NC La Hague), "Evaluation tests of the telerobotic system MT200-TAO in AREVANC/ La Hague hot-cells," ENC 2007, Brussels, Belgium, 2007.