On a problem of F.Smarandache

Yang Ming-shun
(Department of Mathematics, Weinan Teacher's College, Weinan 714000, China)
yangms64@sohu.com

Abstract In this paper, the generalized constructive set *S* is defined as: numbers formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \le m \le 9$. That is to say: (1) $d_1, d_2, \cdots, d_m$ belongs to *S*,(2) If a, b belong to *S*, then $\overline{ab}$ belongs to *S* too,(3) Only elements obtained by rules (1) and (2); we use the elementary methods to study the summation $\sum_{k=1}^{n} S_k$ and $\sum_{k=1}^{n} T_k$, where S_k denotes the summation of all *k* digits numbers in *S*, T_k denotes the summation of each digits of all *k* digits numbers in *S*, and obtain some interesting properties for it.

Keywords Generalized constructive set, summation, recurrence equation, characteristic equation.

1. Introduction and Results

The generalized constructive set *S* is defined as: numbers formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \le m \le 9$. That is to say:

(1) $d_1, d_2, \cdots, d_m$ belongs to *S*;

(2) If a, b belong to *S*, then $\overline{ab}$ belongs to *S* too;

(3) Only elements obtained by rules (1) and (2) applied a finite number of times belongs to *S*.

For example, the constructive set (of digits 1, 2) is: 1, 2, 11, 12, 21, 22,111, 112, 121, 122, 211,212, 221, 222, 1111, 1112, 1121, $\cdots$, And the constructive set (of digits 1, 2, 3) is: 1, 2, 3, 11, 12, 13, 21,22, 23, 31, 32, 33, 111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233,311, 312, 313, 321,322, 323, 331, 332, 333, 1111 , $\cdots$, In problem 6, 7 and 8 of reference [1], Professor F.Smarandache asked us to study the properties of this sequence. In [2], Gou Su had studied the convergent property of the series

$\sum_{n=1}^{+\infty} \frac{1}{a_n^{\alpha}}$ and proved that the series is convergent if $\alpha > \log m$, and divergent if $\alpha \le \log m$, where $\{a_n\}$ denotes the sequence of the constructive set *S*,
formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \le m \le 9$. In this paper, we shall use the elementary methods to study the summation $\sum_{k=1}^{n} S_k$ and $\sum_{k=1}^{n} T_k$, where S_k denotes the summation of all *k* digits numbers in *S*, T_k denotes the summation of each digits of all *k* digits numbers in *S*. That is, we shall prove the following:

Theorem 1. For the generalized constructive set *S* of digits $d_1, d_2, \cdots, d_m$ $(1 \le m \le 9)$,we have

$$\sum_{k=1}^{n} S_k = \frac{d_1 + d_2 + \cdots + d_m}{9}\left(10 \times \frac{(10m)^n - 1}{10m - 1} - \frac{m^n - 1}{m - 1}\right)$$

where S_k denotes the summation of all *k* digits numbers in *S*.

Taking *m* = 2, $d_1 = 1$ and $d_2 = 2$ in Theorem 1, we may immediately get

Corollary 1. For the generalized constructive set *S* of digits 1 and 2, we have

$$\sum_{k=1}^{n} S_k = \frac{1}{3}\left(10 \times \frac{(20)^n - 1}{19} - 2^n + 1\right)$$

Taking *m* = 3, $d_1 = 1$, $d_2 = 2$ and $d_3 = 3$ in Theorem 1, we may immediately get the following:

Corollary 2. For the generalized constructive set *S* of digits 1; 2 and 3, we have

$$\sum_{k=1}^{n} S_k = \frac{2}{3}\left(10 \times \frac{(30)^n - 1}{29} - \frac{3^n}{2} + \frac{1}{2}\right)$$

Theorem 2. For the generalized constructive set *S* of digits $d_1, d_2, \cdots, d_m$ $(1 \le m \le 9)$,we have

$$\sum_{k=1}^{n} T_k$$

Foundation project: Supported by the Education Department Foundation of shanxi Province（09jk432）.
Biography: Yang Mingshun (1964-), male , native of Weinan, Shannxi, an associate professor of Weinan Teacher's college ,engage in number theory

$$= (d_1 + d_2 + \cdots + d_m)\frac{nm^{n+1} - (n+1)m^n + 1}{(m-1)^2}$$ where T_k denotes the summation of each digits of all *k* digits numbers in *S*.

Taking $m = 2$; $d1 = 1$ and $d2 = 2$ in Theorem 2, we may immediately get the following:

Corollary 3. For the the generalized constructive set *S* of digits 1 and 2, we have

$$\sum_{k=1}^{n} T_k = 3n \cdot 2^{n+1} - 3(n+1)2^n + 3$$

Taking $m = 3$, $d_1 = 1$, $d_2 = 2$ and $d_3 = 3$ in Theorem 2, we may immediately get

Corollary 4. For the the generalized constructive set *S* of digits 1*;* 2 and 3, we have

$$\sum_{k=1}^{n} T_k = \frac{3}{2} n \cdot 3^{n+1} - \frac{3}{2}(n+1)3^n + \frac{3}{2}$$

2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1.

Let S_k denotes the summation of all *k* digits numbers in the generalized constructive set *S*.Note that for $k = 1,2,3,\cdots$, there are m^k numbers of *k* digits in *S*. So we have

$$S_k = 10^{k-1} m^{k-1}(d_1 + d_2 + \cdots + d_m) + mS_{k-1} \quad (1)$$

Meanwhile, we have

$$S_{k-1} = 10^{k-2} m^{k-2}(d_1 + d_2 + \cdots + d_m) + mS_{k-2}$$

(2)

Combining (1) and (2), we can get the following recurrence equation

$$S_k - 11m\, S_{k-1} + 10m^2 S_{k-2} = 0$$

Its characteristic equation

$$x^2 - 11mx + 10m^2 = 0$$

have two different real solution $x = m; 10m$:

So we let $S_k = Am^k + B(10m)^k$

Note that $S_0 = 0$, $S_1 = d_1 + d_2 + \cdots + d_m$

we can get

$$A = -\frac{d_1 + d_2 + \cdots + d_m}{9m}, B = \frac{d_1 + d_2 + \cdots + d_m}{9m}$$

So $$S_k = \frac{d_1 + d_2 + \cdots + d_m}{9m}\left((10m)^k - m^k\right)$$

Then

$$\sum_{k=1}^{n} S_k = \frac{d_1 + d_2 + \cdots + d_m}{9}\left(10 \times \frac{(10m)^n - 1}{10m - 1} - \frac{m^n - 1}{m - 1}\right)$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let T_k is denotes the summation of each digits of all *k*digits numbers in the generalized constructive set *S*.

Similarly, note that for $k = 1,2,3,\cdots$, there are m^k numbers of *k* digits in *S*, so we have

$$T_k = m^{k-1}(d_1 + d_2 + \cdots + d_m) + mT_{k-1}$$

(3)

Meanwhile, we have

$$T_{k-1} = m^{k-2}(d_1 + d_2 + \cdots + d_m) + mT_{k-2}$$

(4)

Combining (3) and (4), we can get the following recurrence equation

$$T_k - 2m\, T_{k-1} + m^2 T_{k-2} = 0$$

its characteristic equation $x^2 - 2mx + m^2 = 0$

have two solutions $x_1 = x_2 = m$

So we let $T_k = Am^k + kB(m)^k$

Note that $T_0 = 0$, $T_1 = d_1 + d_2 + \cdots + d_m$

We may immediately deduce that

$$A = 0, B = \frac{d_1 + d_2 + \cdots + d_m}{m}$$

So $$T_k = k(d_1 + d_2 + \cdots + d_m)m^{k-1}$$

Then

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(d_1 + d_2 + \cdots + d_m)m^{k-1}$$

$$= (d_1 + d_2 + \cdots + d_m)\sum_{k=1}^{n} km^{k-1}$$

$$= (d_1 + d_2 + \cdots + d_m)\left(\sum_{k=1}^{n} m^k\right)$$

$$= (d_1 + d_2 + \cdots + d_m)\frac{nm^{n+1} - (n+1)m^n + 1}{(m-1)^2}$$

This completes the proof of Theorem 2.

References（参考文献）

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House,1993.

[2] Gou Su, On the generalized constructive set, Research on Smarandache problems innumber theory, Hexis, 2005, 53-55.

[3] ZhangWenpeng, On an equation of Smarandache and its integer solutions, SmarandAche Notions (Book series), American Research Press, 13(2002), 176-178.

[4] Wenpeng Zhang, On Chebyshevs polynomials and Fibonacii numbers, The Fibonacii Quarterly, 40(2005), 420-428.

978-1-935068-01-3 © 2009 SciRes.

Part 5

Network Theory and Technology

第五部分

网络理论与技术