The C. Neumann System Related to the Second-Order Matrix Eigenvalue Problem

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Abstract: This paper discusses the second-order matrix eigenvalue problem by means of the nonlinearization of the Lax pairs，then the relation between the potential and the eigenfunctions is set up based on the neumann constraint which a new finite-dimensional Hamilton system and the involutive solutions of the evolution equations are abstained.

Keywords: eigenvalue problem; integrable system; involutive representation

1 Introduction

To seek a new completely integrable system which is associated with the development of non-linear equations is an interesting issue in the international mathematical physics Union. In this paper, we obtain a new finite-dimensional completely integrable system by using the nonlinear eigenvalue problem.

2 Lax Representation and the Evolution Equation Hierarchy Related to Eigenvalue Problem

We consider the second order matrix eigenvalue problem:

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_x=\begin{pmatrix}\lambda-q & \lambda\\ \lambda+r & q-\lambda\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=M\varphi, u=\begin{pmatrix}r\\ q\end{pmatrix} \tag{2.1}$$

Let $W=\sum_{j=0}^{\infty}\begin{pmatrix}W_{11} & W_{12}\\ W_{21} & W_{22}\end{pmatrix}\lambda^{-j}$, from $W_x=[M,W]$, we can get the following result:

$$w_{11}=-qa_{j-1}-\frac{1}{2}a_{j-1x}+\lambda a_{j-1},$$

$$w_{12}=\lambda a_{j-1}$$

$$w_{21}=-qa_{j-1}-\frac{1}{2}b_{j-1x}+\lambda a_{j-1},$$

$$w_{22}=qa_{j-1}+\frac{1}{2}a_{j-1x}-\lambda a_{j-1}$$

Definition:

$$K=\begin{pmatrix}\partial r+r\partial & q\partial-\frac{1}{2}\partial^2\\ \partial q+\frac{1}{2}\partial^2 & 0\end{pmatrix},$$

$$J=\begin{pmatrix}-2\partial & \partial\\ \partial & \frac{1}{2}\partial\end{pmatrix} \tag{2.2}$$

We have: $KG_{j-1}=JG_j$, $j=0,1,2,3,\ldots$,

$$G_j=\begin{pmatrix}a_j\\ b_j\end{pmatrix} \tag{2.3}$$

Note: Operators K, J are double *Hamilton* operator[1], it is that K, J have the Properties of antisymmetry, bilinear, non-degeneracy, and satisfy the *Jacobi* equation.

The definition of second-order matrix as follows:

$$W_m=\sum_{j=0}^{m}\begin{pmatrix}-qq_{j-1}-\frac{1}{2}a_{j-1x}+\lambda a_{j-1} & \lambda a_{j-1}\\ -qq_{j-1}-\frac{1}{2}b_{j-1x}+\lambda a_{j-1} & qq_{j-1}+\frac{1}{2}a_{j-1x}-\lambda a_{j-1}\end{pmatrix}\lambda^{m-j}$$

We have the following proposition:

Proposition 2.1: The Evolution Equation Hierarchy related to eigenvalue problem is ($u=(r,q)^T$):

$$u_{tm}=\begin{pmatrix}r_{tm}\\ q_{tm}\end{pmatrix}\quad m=1,2\cdots \tag{2.4}$$

It is equivalent to *Lax* Equation:

$$M_{tm}=(W_m)_x+W_mM-MW_m \tag{2.5}$$

In other words, Equation (2.4) is consistency condition of spectrum-preserving for the following two linear equations:

$$(\varphi_{xt_m} = \varphi_{t_m x}, \lambda_{t_m} = 0)$$

$$\begin{cases} \varphi_x = M\varphi \\ \varphi_{tm} = W_m\varphi \end{cases} \qquad (2.6)$$

Let $G_{-1} = 0$, then

$$G_{-1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix},$$

from

$$KG_{m-1} = JG_m,\ G_0 = \begin{pmatrix} 2q - r \\ 4q + 2r \end{pmatrix}, \qquad (2.7)$$

Then we have the evolution equation:

$$u_{t0} = \begin{pmatrix} r_{t0} \\ q_{t0} \end{pmatrix} = \begin{pmatrix} 4r_x \\ 4q_x \end{pmatrix} \qquad (2.8)$$

$$u_{t1} = \begin{pmatrix} r_{t1} \\ q_{t1} \end{pmatrix} = \begin{pmatrix} 4qr_x + 4rq_x - 3rr_x - r_{xx} - 2q_{xx} \\ 4qq_x + (qr)_x + q_{xx} - \frac{1}{2}r_{xx} \end{pmatrix} \qquad (2.9)$$

Proposition 2.2: Let $\varphi = (\varphi_1, \varphi_2)^T$, if the trace of the second-order matrices M over the reals is 0, so

$$\int_\Omega (\varphi_1, \varphi_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{M} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} dx = 0,$$

The Functional derivation is denoted by Symbol.

Proposition 2.3: If φ_1, φ_2 is the characteristic function that the eigenvalue problem (2.1) relates to λ, so the functional gradient as follows:

$$grad\ \lambda = \begin{pmatrix} \frac{\delta\lambda}{\delta v} \\ \frac{\delta\lambda}{\delta u} \end{pmatrix} = \begin{pmatrix} \lambda\varphi_1^2 \\ -2\lambda\varphi_1\varphi_2 \end{pmatrix},$$

and $Kgrad\lambda = \lambda Jgrad\lambda$.

3 The Hamilton Equation and Its Complete Integrability under the Neumann Constraint

We suppose that $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ is N different eigenvalues of Equation (2.5). $\varphi_{1j}, \varphi_{2j}$ are characteristic function of λ_j ($j = 1, 2, \cdots, N$), so

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_x = M(u, \lambda_j) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \qquad (3.1)$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_{t_m} = w_m(u, \lambda_j) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \qquad (3.2)$$

Let $\Lambda = diag(\lambda_1, \lambda_2, ..\lambda_N)$,

$\varphi_{1j} = (\varphi_{11}, \ldots, \varphi_{1N})^T, \varphi_{2j} = (\varphi_{21}, \ldots \varphi_{2N})^T$

Let $Kgrad\lambda = \lambda Jgrad\lambda$, we can get the result:

$$K\begin{pmatrix} \langle \Lambda^k \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^k \varphi_1, \varphi_2 \rangle \end{pmatrix} = J\begin{pmatrix} \langle \Lambda^{k+1} \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^{k+1} \varphi_1, \varphi_2 \rangle \end{pmatrix} \qquad (3.3)$$

From (2.7) and (3.2), we have

$$G_j = \begin{pmatrix} \langle \Lambda^{j+1} \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^{j+1} \varphi_1, \varphi_2 \rangle \end{pmatrix} \quad j = -1, 0, 1, 2, \cdots$$

$$\Lambda = diag(\lambda_1, ..\lambda_N);$$

At the same time, we have

$$\begin{cases} a_{j-1} \\ b_{j-1} \end{cases} = \begin{pmatrix} \langle \Lambda^j \varphi_1, \varphi_1 \rangle \\ \langle \Lambda^j \varphi_1, \varphi_2 \rangle \end{pmatrix},\ j = 1, 2, \cdots \qquad (3.4)$$

$$G_{-1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \langle \varphi_1, \varphi_1 \rangle \\ 2\langle \varphi_1, \varphi_2 \rangle \end{pmatrix}$$

Neumann constraint condition is:

$$T: \begin{cases} \langle \varphi_1, \varphi_1 \rangle = 4 \\ \langle \varphi_1, \varphi_2 \rangle = 0 \end{cases} \qquad (3.5)$$

then

$$\begin{cases} q = \frac{1}{4}\langle \Lambda\varphi_1, \varphi_1 \rangle + \frac{1}{4}\langle \Lambda\varphi_1, \varphi_2 \rangle \\ r = \frac{1}{4}\langle \Lambda\varphi_1, \varphi_1 \rangle + \frac{1}{4}\langle \Lambda\varphi_2, \varphi_2 \rangle \end{cases}$$

The Poisson bracket of Smooth Function H and F [4,6] in symplectic space $(R^{2N}, \sum_{k=1}^{N} d\varphi_{2k} \Lambda d\varphi_{1k})$ is defined as followed:

$$\{H, F\} = \sum_{k=1}^{N} \left(\frac{\partial H}{\partial \varphi_{2k}} \frac{\partial F}{\partial \varphi_{1k}} - \frac{\partial H}{\partial \varphi_{1k}} \frac{\partial F}{\partial \varphi_{2k}} \right)$$

Proposition 3.1: (3.1) and (3.2) can be written as a finite-dimensional system:

$$\begin{cases} \varphi_{1x} = \frac{\partial H}{\partial \varphi_2} & \varphi_{2x} = -\frac{\partial H}{\partial \varphi_1} \\ \varphi_{1t_m} = \frac{\partial H_m}{\partial \varphi_2} & \varphi_{2t_m} = -\frac{\partial H_m}{\partial \varphi_1} \end{cases} \qquad (3.6)$$

H and H_m are *Hamilton* functions here.

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$$H = -\frac{1}{4}\langle \Lambda\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_2\rangle - \frac{1}{4}\langle \Lambda\varphi_1,\varphi_2\rangle\langle\varphi_1,\varphi_2\rangle + \langle \Lambda\varphi_1,\varphi_2\rangle + \frac{1}{8}\langle \Lambda\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_1\rangle\frac{1}{8}\langle \Lambda\varphi_2,\varphi_2\rangle\langle\varphi_1, - \langle \Lambda\varphi_1,\varphi_1\rangle \quad (3.7)$$

$$H_m = -\frac{1}{2}\langle \Lambda^{m+1}\varphi_1,\varphi_1\rangle\langle\varphi_2,\varphi_2\rangle - 4\langle \Lambda^{m+1}\varphi_1,\varphi_1\rangle + \langle \Lambda^{m+1}\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_2\rangle + \frac{1}{2}\langle \Lambda^{m+1}\varphi_1,\varphi_1\rangle\langle\varphi_1,\varphi_1\rangle + 4\langle \Lambda^{m+1}\varphi_1,\varphi_2\rangle + \frac{1}{2}\sum_{j-0}^{m+1}\langle \Lambda^{j}\varphi_2,\varphi_2\rangle\langle \Lambda^{m-j+1}\varphi_1,\varphi_1\rangle - \frac{1}{2}\sum_{j-0}^{m+1}\langle \Lambda^{j}\varphi_2,\varphi_1\rangle\langle \Lambda^{m-j+1}\varphi_2,\varphi_1\rangle \quad (3.8)$$

The generator E_k is as followed:

$$E_k = \frac{1}{2}\lambda_k\Gamma_k - \varphi_{1k}^{2}\langle\varphi_1,\varphi_2\rangle + 4\varphi_{1k}\varphi_{2k} + \frac{1}{2}\varphi_{1k}^{2}\langle\varphi_1,\varphi_1\rangle - 4\varphi_{1k}^{2} - \frac{1}{2}\varphi_{1k}^{2}\langle\varphi_2,\varphi_2\rangle$$

$$(\text{Let}\ \Gamma_k = \sum_{\substack{j=1,\\ j\neq k}}^{N} \frac{(\varphi_{1k}\varphi_{2j} - \varphi_{2k}\varphi_{1j})^2}{\lambda_k - \lambda_j}),$$

Proposition3.2:

(1) $\{E_k, k = 1, 2, \ldots N\}$ is involutive system, so that $\{E_k, E_j\} = 0, \forall k, j = 1, 2, \ldots, N$, and $\{dE_k\}$ has nothing relation with the gradient;

(2)
$$H_\lambda = \sum_{k=1}^{n}\frac{E_k}{\lambda-\lambda_k} = \sum_{m=0}^{\infty}\lambda^{-m-1}G_m[2]$$
$$G_m = \sum_{k=1}^{n}\lambda_k^{m}\frac{E_k}{\lambda-\lambda_k} \quad H_m = G_{m+1}$$

Among them,

$$H_m = \sum_{k=1}^{n}\lambda_k^{m+1}E_k, m=1,2,\cdots,$$ [3]

Proposition3.3:

Let $f_1 = \langle\varphi_1,\varphi_1\rangle - 4$, $f_2 = \langle\varphi_1,\varphi_2\rangle$, then we have follow solutions on Γ:

(1) $\{H, f_j\} = 0, j = 1,2$

(2) $\{f_j, E_k\} = 0, j = 1,2 \quad k = 1,2,\cdots n$

(3) $\det\{f_i, f_j\} \neq 0, \quad i, j = 1,2$

(4) $\{H, E_k\} = 0, k = 1, 2, \ldots N.$

(5) $\{H, H_m\} = 0, m = 0, 1, 2, \ldots$

(6) $\{H_n, H_m\} = 0, m, n = 0, 1, 2, \ldots$

(7) $(R^{2N}, \sum_{k=1}^{N} d\varphi_{1k}\Lambda d\varphi_{2k}, H), (R^{2N}, \sum_{k=1}^{N} d\varphi_{1k}\Lambda d\varphi_{2k}, H_m)$

is the completely integrable system in the Liouville sense.

Proposition 3.4: If φ_1, φ_2 are involutive solution of the Hamilton regular system [5,7], so

$$\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\langle\varphi_1,\varphi_2\rangle - \frac{1}{2}\langle\varphi_1,\varphi_1\rangle \\ \frac{1}{4}\langle\varphi_1,\varphi_2\rangle + \frac{1}{4}\langle\varphi_1,\varphi_1\rangle \end{pmatrix}$$

is the solution of evolution equation hierarchy (2.4).

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