

An Extension of Limit Theorem for Markov Chain

YUAN Shujuan¹, LAN Jingtao²

1. College of Light Industry, Hebei Polytechnic University, Tangshan, China, 063000

2. Dept. of Mathematics, Shandong Wanjie Medical College, Zibo, China, 255213

Email: Yuanshujuan1980@163.com

Abstract: The limit theorem for countable non-homogeneous Markov chain was discussed in no few references, where various restrictions were imposed on the Markov chain. In this paper, an extension of limit theorem which holds for arbitrary countable non-homogeneous Markov chain is given.

Keywords: countable non-homogeneous Markov chain; stochastic matrix; strong limit theorem

1 Introduction

Let $\{X_n, n \geq 0\}$ be a Markov chain with state space $S = \{1, 2, \dots\}$, initial distribution

$$q(1), q(2), \dots \quad (1)$$

and transition matrices

$$P_n = (p_n(i, j)), i, j \in S, \quad n = 0, 1, 2, \dots \quad (2)$$

where $p_n(i, j) = P(X_{n+1} = j | X_n = i)$. The strong limit theorem for countable non-homogeneous Markov chains were discussed in no few references [1,2], where various restrictions were imposed on the Markov chains. The purpose of this paper is to present a new strong limit theorem which hold for arbitrary countable non-homogeneous Markov chains by using professor Liu Wen's analytical method [3]. Throughout this paper, we shall deal with the underlying probability space $([0,1], F, p)$, where F is the class of Borel sets in the interval $[0,1]$ and P is the Lebesgue measure. We first give, in the above probability space, a realization of the Markov chain with the initial distribution (1) and the transition matrices (2). Let $q(n_i), i = 1, 2, 3, \dots$ be the positive terms of (1), where $n_1 < n_2 < n_3 < \dots$. Divide the interval $[0,1]$ into countable many (including the finite case) right-semiopen intervals: $d_{x_0}, x_0 = n_1, n_2, n_3, \dots$ that is

$$d_{n_1} = [0, q(n_1)), d_{n_2} = [q(n_1), q(n_1) + q(n_2)), \dots$$

these intervals will be called interval of order 0. In general, suppose $d_{x_0 \dots x_n}$ is an interval of order n , and the positive elements of the x_n -th row of the transition matrix P_n , in their given order are $p_n(x_n, m_i), i = 1, 2, 3, \dots$. By dividing $d_{x_0 \dots x_n}$ into countable many right-semiopen intervals: $d_{x_0 \dots x_n x_{n+1}} (x_{n+1} = m_i, i = 1, 2, 3, \dots)$ at the ratio

$p_n(x_n, m_1) : p_n(x_n, m_2) : p_n(x_n, m_3) : \dots$, the intervals of order $n+1$ are created. Proceeding inductively, the intervals of all orders can be obtained. By the above construction, it is easy to see that

$$p(d_{x_0 \dots x_n}) = q(x_0) \prod_{m=0}^{n-1} p_m(x_m, x_{m+1})$$

For $n \geq 0$, define a random variable $X_n : [0,1] \rightarrow S$ as follow: $X_n(w) = x_n$, if $w \in d_{x_0 \dots x_n}$. Hence we have

$$p(X_0 = x_0, \dots, X_n = x_n) = p(d_{x_0 \dots x_n}) = q(x_0) \cdot \prod_{m=0}^{n-1} p_m(x_m, x_{m+1}).$$

Theorem 1: Let $\{X_n, n \geq 0\}$ be a Markov chain with the initial distribution (1) and The transition matrices (2), let $m \geq 2$ be an integer, $g(x_1, x_2, \dots, x_m)$ is a bounded function defined on S^m and

$$|g(x_1, x_2, \dots, x_m)| \leq 1, \quad \delta_j(\cdot)$$

is the Kronecker function, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=0}^{n-1} \left(g(X_l, X_{l+1}, \dots, X_{l+m-1}) - \sum_{i_0 \in S} \delta_{i_0}(X_l) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1}) \right)$$

a.e. $w \in [0,1]$ (3) **Proof.** Let the collection of interval $[0,1]$ and all intervals of order $n+m-1$ ($n = 0, 1, 2, \dots$) be denoted by A , and $\lambda > 0$ be a constant. Assume $d_{x_0 \dots x_{n+m-1}} (n \geq 1)$ is an interval of order $n+m-1$, and let

$$\mu(d_{x_0 \dots x_{n+m-1}}) = \lambda^{\sum_{i=0}^{n-1} g(X_i, X_{i+1}, \dots, X_{i+m-1})} \prod_{l=0}^{n-1} \prod_{i_0 \in S} \left[\frac{1}{1 + (\lambda - 1) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1})} \right]^{\delta_{i_0}(X_l)} \cdot P(d_{x_0 \dots x_{n+m-1}}) \quad (4)$$

$$\mu(d_{x_0}) = \sum_{x_1} \cdots \sum_{x_{m-1}} \mu(d_{x_0 \cdots x_{m-1}}) \quad \mu([0,1]) = \sum_{x_0} \mu(d_{x_0}) \quad (5)$$

where $\sum_{x_i}$ denote the sum taken over all values of index x_i .

It is easy to prove that μ is an additive set function on A . Hence there exists an increasing function f_λ defined on $[0,1]$ such that for any $d_{x_0 \cdots x_{n+m-1}}$,

$$\mu(d_{x_0 \cdots x_{n+m-1}}) = f_\lambda(d_{x_0 \cdots x_{n+m-1}}^+) - f_\lambda(d_{x_0 \cdots x_{n+m-1}}^-) \quad (6)$$

where $d_{x_0 \cdots x_{n+m-1}}^-$ and $d_{x_0 \cdots x_{n+m-1}}^+$ denote respectively the left and right end points of $d_{x_0 \cdots x_{n+m-1}}$.

Let

$$t_n(\lambda, w) = \frac{\mu(d_{x_0 \cdots x_{n+m-1}})}{p(d_{x_0 \cdots x_{n+m-1}})}, \quad w \in d_{x_0 \cdots x_{n+m-1}}. \quad (7)$$

Let $A(\lambda)$ be the set of points differentiability of f_λ , then by the existence theorem of derivative of monotone function $p(A(\lambda))=1$. Let $w \in A(\lambda)$ and $w \in d_{x_0 \cdots x_{n+m-1}}$ ($n=0,1,2,\dots$).

If $\lim_{n \rightarrow \infty} p(d_{x_0 \cdots x_{n+m-1}}) = d > 0$, we can get

$$\lim_{n \rightarrow \infty} t_n(\lambda, w) = \frac{\lim_{n \rightarrow \infty} \mu(d_{x_0 \cdots x_{n+m-1}})}{d} < +\infty \quad (8)$$

If

$$\lim_{n \rightarrow \infty} p(d_{x_0 \cdots x_{n+m-1}}) = 0,$$

then in view of a property of derivative [4], we get

$$\lim_{n \rightarrow \infty} t_n(\lambda, w) = f'_\lambda(w) < +\infty.$$

Thus $\lim_{n \rightarrow \infty} t_n(\lambda, w)$ is finite, $w \in A(\lambda)$. (9)

By (8) and (9),

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{l=0}^{n-1} \left(g(X_l, X_{l+1}, \dots, X_{l+m-1}) - \sum_{i_0 \in S} \delta_{i_0}(X_l) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1}) \right) \leq \frac{\lambda-1}{\ln \lambda} - 1 \quad (w \in A(\lambda)) \quad (14)$$

Assume

$$\lambda_i > 1 \quad (i=1, 2, \dots), \quad \lambda_i \rightarrow 1 \quad (as \ i \rightarrow \infty),$$

and let

$$A^* = \bigcap_{i=1}^{\infty} A(\lambda_i).$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln t_n(\lambda, w) \leq 0, \quad w \in A(\lambda) \quad (10)$$

$$\ln t_n(\lambda, w) = \sum_{l=0}^{n-1} g(X_l, X_{l+1}, \dots, X_{l+m-1}) \ln \lambda - \sum_{l=0}^{n-1} \sum_{i_0 \in S} \delta_{i_0}(X_l) \cdot \ln[1 + (\lambda-1) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1})]. \quad (11)$$

By (10) and (11), $\limsup_{n \rightarrow \infty} \frac{1}{n}$

$$\left\{ \sum_{l=0}^{n-1} g(X_l, X_{l+1}, \dots, X_{l+m-1}) \ln \lambda - \sum_{l=0}^{n-1} \sum_{i_0 \in S} \delta_{i_0}(X_l) \cdot \ln[1 + (\lambda-1) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1})] \right\} \quad w \in A(\lambda). \quad (12)$$

Letting $\lambda > 1$, dividing the two sides of (20) by $\ln \lambda$, we can get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \left\{ \sum_{l=0}^{n-1} g(X_l, X_{l+1}, \dots, X_{l+m-1}) - \sum_{l=0}^{n-1} \sum_{i_0 \in S} \delta_{i_0}(X_l) \frac{\ln[1 + (\lambda-1) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1})]}{\ln \lambda} \right\} \leq 0 \quad w \in A(\lambda). \quad (13)$$

By (13), the property of superior limit

$$\limsup_{n \rightarrow \infty} (a_n - b_n) \leq 0 \Rightarrow \limsup_{n \rightarrow \infty} (a_n - c_n) \leq \limsup_{n \rightarrow \infty} (b_n - c_n)$$

the inequality

$$0 \leq \ln(1+x) \leq x \quad (x \geq 0)$$

and

$$\sum_{l=0}^{n-1} \sum_{i_0 \in S} \delta_{i_0}(X_l) = n$$

we will get

Since

$$\frac{\lambda_i - 1}{\ln \lambda_i} - 1 \rightarrow 0 \quad (i \rightarrow +\infty),$$

by (14) we can get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{l=0}^{n-1} \left(g(X_l, X_{l+1}, \dots, X_{l+m-1}) - \sum_{i_0 \in S} \delta_{i_0}(X_l) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1}) \right) \leq 0 \quad (w \in A^*) \quad (15)$$

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{l=0}^{n-1} \left(g(X_l, X_{l+1}, \dots, X_{l+m-1}) - \sum_{i_0 \in S} \delta_{i_0}(X_l) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1}) \right) \geq 0 \quad (16)$$

Similarly assume

$$0 < \tau_i < 1 \quad (i = 1, 2, \dots) \tau_i \rightarrow 1 \quad (\text{as } i \rightarrow +\infty)$$

and Then

$$w \in A_* = \bigcap_{i=1}^{\infty} A(\tau_i),$$

Letting

$$A = A^* \cap A_*,$$

by (15) and (16) we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=0}^{n-1} \left(g(X_l, X_{l+1}, \dots, X_{l+m-1}) - \sum_{i_0 \in S} \delta_{i_0}(X_l) \prod_{k=0}^{m-1} p_{l+k}(i_k, i_{k+1}) \right) = 0$$

$$w \in A \quad (17)$$

Since $P(A) = 1$, by (17) holds. Thus the theorem 1 is proved. The result of this paper is to present a new strong

limit theorem which holds for arbitrary countable non-homogeneous Markov chains by using professor Liu wen's analytical method, it is an extension of the theorem in the previous paper (cf. [5]).

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