Almost Sure Limit Theorem for the Product of Partial Sums for j^*-Mixing Sequences

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Abstract: In this paper, under some suitable conditions, we get almost sure limit theorem for the product of partial sums for j^*-mixing sequences.

Keywords: almost sure limit theorem; strictly stationary; partial sums; j^*-mixing; j-mixing

1 Introduction

Let $\{X_n\}$ be a strictly stationary sequences defined on a common probability spaces $(\Omega, \Re, P)$. $\Re_a^b$ denotes s –algebra of events generated by $X_a, \mathbf{L}, X_b$ $(a \le b)$. We shall say that $\{X_n\}$ satisfies the uniformly mixing conditions in both directions of times $(j^*\text{-mixing})$. Also, recall that $\{X_n\}$ is uniformly mixing conditions $(j\text{-mixing})$. Their definition is usual, so we omit them here. Since $j(n) \le j^*(n)$, it means that if $\{X_n\}$ is j^*-mixing, then it is also j-mixing.

The purpose of the current note is to obtain an almost sure limiting result for the logarithmic averages under the above definition. As is well known, the central limit theorems of the products of partial sum for i.i.d. variables was initiated by some papers. This point was first obtained by Arnold and Villaseor[1], who considered the following form of the CLT for a sequence (Y_n) of i.i.d. exponential random variables with the mean equal to one, as $n \to \infty$,

$$\frac{\sum_{k=1}^{n} \log S_k - n\log(n) + n}{\sqrt{2n}} \xrightarrow{D} N .$$

Here and in the sequel, $S_n = Y_1 + \mathbf{L} + Y_n$, $n = 1, 2, \mathbf{L}$, and N is a standard normal random variable. Their proof is heavily based on a very special property of exponential (gamma) distributions: namely that there is independence of ratios of subsequent partial sums and the last sum. It uses also Resnick's result[2] on weak limits for records. In particular, Rempala and Wesolowski[3] have proved the asymptotic behavior of a product of partial sums holds for any sequence of independent and identically distributed positive random variables. This result was swiftly extended by Qi[8], who has shown that whenever (Y_n) is in the domain of attraction of a stable law L with index $a \in (1, 2]$ then there exists a numerical sequence A_n (which for $a = 2$ can be taken as $s\sqrt{n}$) such that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\, m^n}\right)^{\frac{m}{A_n}} \xrightarrow{D} e^{\left(\Gamma(a+1)^{\frac{1}{a}}\right)L},$$

where $\Gamma(a+1) = \int_0^\infty x^a e^{-x} dx$.

In the past decade, several authors researched the so-called almost sure central limit theorem (ASCLT) on the basis of the results of Brosamler[4] and Schatte[9] for partial sums of random variables. Recently, the following ASCLT of product $\prod_{j=1}^{n} S_j$ for i.i.d. sequence have been obtained by Khurelbaatar and Rempala[5].

Theorem A: Let (Y_n) be a sequence of i.i.d. positive random variables with $EY_1 = m > 0$ and $Var(Y_1) = s^2$. Denote $g = s/m$ the coefficient of variation. Then for any real x,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\left(\frac{\prod_{k=1}^{n} S_k}{n!\, m^n}\right)^{\frac{1}{g\sqrt{n}}} \le x\right\} = F(x),$$

where $F(\cdot)$ is the distribution function of the random variable $e^{\sqrt{2}N}$.

Inspired by the above results, in this note we study the almost sure limit theorem for the product of partial sums for j^*-mixing sequences. The following is our main result.

Theorem 1: Let $\{X_n\}$ be a strictly stationary j^*-mixing sequence of positive random variables with $EY_1 = m > 0$ and $Var(Y_1) = s^2$. Denote $g = s/m$ the coefficient of variation. Under the following conditions

(H_1) for $d > 2$, $j^*(n) = O\left(n^{-2}(\log n)^{-2d}\right)$;

(H_2) $s_1^2 := 1 + 2\sum_{j=2}^{\infty} Cov\left(\frac{X_1 - m}{s}, \frac{X_j - m}{s}\right) > 0$.

We have, for any real x,

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\left(\frac{\Pi_{j=1}^{k}S_j}{k!k^n}\right)^{\frac{1}{gs_1\sqrt{k}}}\le x\right\}=F(x),$$

where $s_1^2=1+2\Sigma_{j=2}^{\infty}Cov(\frac{X_1-m}{s},\frac{X_j-m}{s})$, $I(\cdot)$ denotes indicator function and $F(\cdot)$ is the distribution function of the random variable $e^{\sqrt{2}N}$.

Remark 1: It is obvious that Theorem B is a special case of Theorem 1.

Remark 2: $s^2<\infty$ is obvious from condition (H1) and the following lemma 2.2.

2 Proof

In the sequel we shall use the following notation.

Let $b_{k,n}=\sum_{i=k}^{n}\frac{1}{i}$, $k\le n$ with $b_{k,n}=0$, $k>n$. Let $\tilde{S}_n=\sum_{k=1}^{n}Y_k$ and $S_{n,n}=\sum_{k=1}^{n}b_{k,n}Y_k$,

where $Y_k=(X_k-m)/s$, $k\ge 1$. Let $s_n^2=Var(S_{n,n})$, and we write << for the inequality $\le$ up to some universal constant. The proof of Theorem 1 is chiefly based on the following lemmas.

Lemma 2.1[3] $s_{n,n}^2=2n-b_{1,n}$.

Lemma 2.2[6] Let $\{X_n\}$ be a strictly stationary j^*-mixing sequences, $X\in L_p(\Re_1^k)$, $Y\in L_q(\Re_{k+n}^{\infty})$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then

$$|EXY-EXEY|\le 2(j^*(n))^{\frac{1}{p}}\|X\|_p\|X\|_q,$$

$$\text{where } \|X\|_p=(E(X)^p)^{\frac{1}{p}}.$$

Lemma 2.3[6] Let $\{X_n\}$ be a strictly stationary j^*-mixing sequences, for $1\le r<2$, $E|X_1|^r<\infty$, satisfy $\sum_{n=1}^{\infty}(j^*(2^n))^{\frac{1}{2}}<\infty$. Then

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-EX_i)=o\left(n^{-\left(1-\frac{1}{r}\right)}\right)a.s..$$

Lemma 2.4[11] Under the assumptions of Theorem 1, we have

$$\frac{s_n^2}{2n}\to s_1^2,\quad as\ n\to\infty.$$

Lemma 2.5[7] Let $\{X_n\}$ be a strictly stationary j^*-mixing sequences, $E(X_1)=0$, $Var(X_1)<\infty$, $S_n=\sum_{k=1}^{n}X_k$. Assume that $\lim_{n\to\infty}j^*(n)<1$, and $s_n^2=Var(S_n)\to\infty$, then as $n\to\infty$,

$$\frac{S_n}{s_n}\xrightarrow{D}N.$$

Lemma 2.6[5] Under the conditions of Theorem 1, for any real x,

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{S_{k,k}}{\sqrt{2k}s_1}\le x\right\}=\Phi(x).$$

Now, we begin to prove Theorem 1.

Proof of Theorem 1. Let $C_i=\frac{S_i}{mi}$, $i\ge 1$, we have

$$\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}(C_i-1)=\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}\left(\frac{S_i}{mi}-1\right)$$
$$=\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}b_{i,k}Y_i=\frac{S_{k,k}}{\sqrt{2k}s_1}.$$

Thus, for any real x, Lemma 2.6 is equivalent to

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}\log C_i\le x\right\}=\Phi(x). \quad (1)$$

By Lemma 2.3, for enough large i, for some $\frac{4}{3}<r<2$, we have $|C_i-1|<<i^{\frac{1}{r}-1}$. We will expand the logarithm:

$$\log(1+x)=x+O(x^2)\qquad \text{for } |x|<\frac{1}{2}.$$

Thus

$$\left|\sum_{i=1}^{k}\log C_i-\sum_{i=1}^{k}(C_i-1)\right|<<\sum_{i=1}^{k}(C_i-1)^2<<k^{\frac{2}{r}-1}.$$

Hence for arbitrary small $e>0$, there is $N:=N(e)$ such that for every $k>N$.

$$I\left\{\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}(C_i-1)\le x-e\right\}$$
$$\le I\left\{\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}\log C_i\le x\right\}$$
$$\le I\left\{\frac{1}{g\sqrt{2k}s_1}\sum_{i=1}^{k}(C_i-1)\le x+e\right\}$$

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Then (1) holds true, so we complete the proof of Theorem 1.

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