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Generalized Stochastic Processes: The Portfolio Model

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ABSTRACT

Using the portfolio model, we introduce a general stochastic process that is not necessarily a diffusion/jump process and the random variable is not necessarily normally distributed.

Keywords: Stochastic Process; Investment; Portfolio

1. Introduction

The literature on stochastic processes (especially in finance) relied mainly on Levy processes such as Wiener process, Poisson process, and the Variance-Gamma process. Examples include Madan and Seneta [1], Focardo and Fabozzi [2], among many others. Much of the literature assumes a Wiener process (Brownian motion), which implies normally distributed and independent stationary increments. The Brownian motion is extensively used in stochastic finance especially in investment models (see, for example, Alghalith [3]).

However, these assumptions of diffusion/jump process and Gaussian/Poisson distribution (or any specific probability distribution) can be relaxed. That is, we can introduce a general stochastic process that is more general than the Levy process without losing significant analytical convenience. Consequently, this paper offers three major contributions. First, it relaxes the assumption of a diffusion/jump process. Secondly, it relaxes the Gaussian/Poisson distribution or any specific probability distribution. Thirdly, it provides solutions without reliance on the existing duality or variational methods. Moreover, we introduce a general model that can be applied to any specific topic.

2. The Model

In general, a continuous stochastic process $\{X_s\}_{t\le s\le T}$ can be written as a function of a control variable, state variables and a random variable as the following (the first two integrals can be zero)

$$X_T^{\pi} = f\left(\int_t^T \pi_s \mathrm{d}s, \int_t^T \boldsymbol{\theta}_s\left(Y_s\right)\mathrm{d}s, \int_t^T \xi_s \mathrm{d}s\right), t \le T, \tag{1}$$

where π_s is the control variable, $\boldsymbol{\theta}_s$ is a vector of state variables or coefficients, Y_s is a stochastic factor, and ξ_s is a random variable (not necessarily a Brownian motion) and thus the assumption of normal distribution (or any specific probability distribution) is not required. Moreover, in contrast to Levy processes, f is not necessarily a linear (diffusion) function. In addition, we assume X_s is admissible and progressively measurable, where $\{\mathcal{F}_s\}_{t\le s\le T}$ is the filtration.

The objective is to maximize the expected utility of X_T^{π} with respect to π_t

$$\max_{\pi} E\left[u\left(X_T^{\pi}\right)\middle|\mathcal{F}_t\right],$$

where u is a differentiable, bounded and concave utility function. Using the method of Alghalith [4], the solution yields

$$E\left[u'\left(X_T^{\pi}\right)f_{\pi}\middle|\mathcal{F}_t\right] = 0, \tag{2}$$

where the subscript denote the derivatives.

Consider this *exact* Taylor polynomial (and suppressing the notations) (Equation (3))

Taking expectations of both sides yields (Equation (4))

$$u'(.)\,f_{\pi} = u'f_{\pi} + (T-t)\left[\left(u''f_{\pi}^2 + u'f_{\pi\pi}\right)\pi_t + \sum_i\left(u''f_{\pi}f_{\theta_i} + u'f_{\pi_{\theta_i}}\right)\theta_i + \left(u''f_{\pi}f_{\xi} + u'f_{\pi\xi}\right)\xi\right]. \tag{3}$$

$$E_t u'(.)\,f_{\pi} = E_t u'f_{\pi} + (T-t)\left\{\left(u''f_{\pi}^2 + u'f_{\pi\pi}\right)\pi_t^* + \sum_i\left(u''f_{\pi}f_{\theta_i} + u'f_{\pi_{\theta_i}}\right)\theta_i + \left(u''f_{\pi}f_{\xi} + u'f_{\pi\xi}\right)\xi\right\} = 0. \tag{4}$$

Thus,

$$\pi_t^* = -\frac{E_t\left[u'f_\pi + (T-t)\left\{\sum_i\left(u''f_\pi f_{\theta_i} + u'f_{\pi\theta_i}\right)\theta_i + \left(u''f_\pi f_\xi + u'f_{\pi\xi}\right)\xi\right\}\right]}{(T-t)E_t\left(u''f_\pi^2 + u'f_{\pi\pi}\right)}. \tag{5}$$

3. Example—The Investment/Consumption Model

It is well-known that the stock price S_s is a function of the expected return μ, the volatility σ, and a random variable ξ

$$S_T = S\left(\int_t^T \mu(Y_s)\,\mathrm{d}s, \int_t^T \sigma(Y_s)\,\mathrm{d}s, \int_t^T \xi_s\,\mathrm{d}s\right), \tag{6}$$

where Y_s is stochastic economic factor. However, ξ is not necessarily normally distributed and $S(.)$ is not necessarily a linear function. Consequently, the wealth function is given by

$$X_T^{\pi,c} = X\left(\int_t^T \pi_s\,\mathrm{d}s, \int_t^T c_s\,\mathrm{d}s, x, \int_t^T r_s(Y_s)\,\mathrm{d}s, S_T\right), \tag{7}$$

where π_s is the portfolio process, c_s is the consumption process, x is the initial wealth, r_s is the risk-free rate of return. Thus,

$$X_T^{\pi,c} = X\left(x, \int_t^T \pi_s\,\mathrm{d}s, \int_t^T c_s\,\mathrm{d}s, \int_t^T \mu_s(Y_s), \int_t^T r_s(Y_s)\,\mathrm{d}s, \int_t^T \sigma(Y_s)\,\mathrm{d}s, \int_t^T \xi_s\,\mathrm{d}s\right). \tag{8}$$

The objective is to maximize the expected utility of wealth and consumption with respect to the portfolio and consumption

$$\max_{\pi,c} E\left[u_1\left(X_T^{\pi,c}\right) + \int_t^T u_2(c_s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

The solutions are

$$E_t u_1'\left(X_T^{\pi,c}\right)X_\pi = 0, \tag{9}$$

$$E_t\left[u_1'\left(X_T^{\pi,c}\right)X_c + u_2'\left(c_t^*\right)\right] = 0. \tag{10}$$

Using an *exact* Taylor expansion (and suppressing the notations), we obtain

$$\begin{aligned} u'(.)X_\pi = u'X_\pi + (T-t)\Big\{&\left(u''X_\pi^2 + u'X_{\pi\pi}\right)\pi_t^* + \left(u''X_\pi f_c + u'X_{\pi c}\right)c_t^* + \left(u''X_\pi f_\mu + u'X_{\pi\mu}\right)\mu \\ &+ \left(u''X_\pi f_r + u'X_{\pi r}\right)r + \left(u''X_\pi f_\sigma + u'X_{\pi\sigma}\right)\sigma + \left(u''X_\pi f_\xi + u'X_{\pi\xi}\right)\xi\Big\}. \end{aligned} \tag{11}$$

Thus,

$$\begin{aligned} E_t u(.)X_\pi = E_t\Big[u'X_\pi + (T-t)\Big\{&\left(u''X_\pi^2 + u'X_{\pi\pi}\right)\pi_t^* + \left(u''X_\pi f_c + u'X_{\pi c}\right)c_t^* + \left(u''X_\pi f_\mu + u'X_{\pi\mu}\right)\mu \\ &+ \left(u''X_\pi f_r + u'X_{\pi r}\right)r + \left(u''X_\pi f_\sigma + u'X_{\pi\sigma}\right)\sigma + \left(u''X_\pi f_\xi + u'X_{\pi\xi}\right)\xi\Big\}\Big] \end{aligned} \tag{12}$$

Therefore we can obtain expressions for the optimal portfolio and consumption

$$\pi_t^* = -\frac{\begin{aligned} E_t\Big[u'X_\pi + (T-t)\Big\{&\left(u''X_\pi f_c + u'X_{\pi c}\right)c_t^* + \left(u''X_\pi f_\mu + u'X_{\pi\mu}\right)\mu \\ &+ \left(u''X_\pi f_r + u'X_{\pi r}\right)r + \left(u''X_\pi f_\sigma + u'X_{\pi\sigma}\right)\sigma + \left(u''X_\pi f_\xi + u'X_{\pi\xi}\right)\xi\Big\}\Big] \end{aligned}}{(T-t)E_t\left[u''X_\pi^2 + u'X_{\pi\pi}\right]} \tag{13}$$

$$c_t^* = -\frac{\begin{aligned} E_t\Big[u'X_\pi + (T-t)\Big\{&\left(u''X_\pi f_\mu + u'X_{\pi\mu}\right)\mu + \left(u''X_\pi f_r + u'X_{\pi r}\right)r \\ &+ \left(u''X_\pi f_\sigma + u'X_{\pi\sigma}\right)\sigma + \left(u''X_\pi f_\xi + u'X_{\pi\xi}\right)\xi\Big\}\Big] \end{aligned}}{(T-t)E_t\left[\left(u''X_\pi f_c + u'X_{\pi c}\right)\right]} \tag{14}$$

We can obtain explicit solutions under specific forms of the utility function. For example, under mean-variance (quadratic) preference, we can obtain explicit solutions since u'' is constant and u' is linear. It is worth noting that even with Levy process general explicit solutions were not provided by the literature; thus, the assumption of a Levy process does not offer a significant analytical convenience.

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